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Sommersemester 2012

6. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 14: The Katz-Lebowitz-Spohn model [1]

In this exercise, we would like to construct a lattice gas model whose stationary distribution is given by the Boltzmann distribution

$$P^*(\eta) = \frac{\mathrm{e}^{-\beta H}}{Z}, \qquad \text{with the Hamiltonian} \qquad H = -J\sum_{j=1}^{L} \eta_j \eta_{j+1}.$$

Periodic boundary conditions are assumed, $\eta_{L+1} = \eta_1$. To archive this goal, we must make an appropriate choice for the transition rates $\alpha(\eta)[i \to i+1]$ (jumps to the right) and $\gamma(\eta)[i \to i-1]$ (jumps to the left). Note that the Hamiltonian implies nearest neighbor interactions. Therefore, the rates must also depend on the neighbors of the particle before and after the jump.

a) Name all the possible jump configurations taking into account all the possible neighborhoods before and after the jump. How many different rates does this imply?

b) To control the degree of asymmetry we switch on an external field, E, such that for two neighboring sites the energy of the right hand one is lowered by E. Thus, E measures how strongly jumps to the left are suppressed with respect to jumps to the right. A physical example would be an electric field $\mathcal{E}$, where E would stand for the combination $q\mathcal{E}a$, with charge q and lattice spacing a. Demanding that detailed balance should still hold locally in the presence of the field yields relations between the γ's and the α's.

c) Let us for now restrict ourselves to the totally antisymmetric case $E \to \infty$, i.e. particles can only jump to the right. We want to identify conditions upon the different α's that must be fulfilled so that $P^*(\eta)$ is invariant under the dynamics. To that purpose, consider a ring consisting of $L_{\min}$ sites and $N_{\min}$ particles, where $L_{\min}$ and $N_{\min}$ are the minimal number of sites and particles necessary to construct all possible jump configurations. Evaluate the flow into, and out of, all specific configurations η under the condition that the η are distributed according to $P^*(\eta)$. Demanding stationarity yields two conditions. Which are these?

d) Now return to the case of finite E. Verify that the conditions obtained in c) are still sufficient to ensure the stationarity of P^* when using the γ's obtained in b).

[1] S. Katz, J. Lebowitz and H. Spohn, J. Stat. Phys., 34 (1984), p. 497

Exercise 15: The zero range process

In the zero range process[2] (ZRP) an unlimited number $n_i = 0, 1, 2, ...$ of particles can occupy each site $i = 1, .., N$ of the one-dimensional lattice with periodic boundary conditions (a ring). A particle at site i jumps to the right ($i \to i+1$) with probability q and to the left ($i \to i-1$) with probability $1-q$ at a rate which is a function $\gamma(n_i)$ of the number of particles at the site of origin with $\gamma(0) = 0$. There is no dependence on the occupancy of the target site (= *zero range interaction*). The ZRP has the remarkable property that the stationary distribution is a *product measure* for a broad class of functions $\gamma(n)$, i.e. the stationary weight of a configuration $\{n_1, ..., n_N\}$ is of the form

$$\mathbb{P}[n_1, ..., n_N] \sim \prod_{i=1}^{N} f(n_i). \tag{1}$$

a.) In the symmetric case $q = 1/2$ use the condition of detailed balance to show that

$$f(n) \sim \alpha^n \prod_{k=1}^{n} \gamma(k)^{-1} \tag{2}$$

where α is a constant to be fixed by normalization. Under what conditions on the rate function $\gamma(n)$ is (2) in fact normalizable?

b.) Using the concept of pairwise balance introduced in the lectures, show that the product measure (1, 2) remains stationary also when $q \neq 1/2$.

c.) We now show that the asymmetric exclusion process is equivalent to a special case of the ZRP. To this end we consider the ASEP with N particles on a ring of L sites. We label the particles by an index $i = 1, ..., N$, and denote by n_i the *number of vacant sites in front of particle* i. Show that in this representation the ASEP dynamics is of ZRP form and identify the corresponding rate function $\gamma(n)$. What is the meaning of the ASEP lattice size L in this context? Use the general result (2) to write down the stationary distribution of the 'gaps' n_i, and show that the same expression follows from the stationary Bernoulli measure of the ASEP.

[2] F. Spitzer, Adv. Math. **5**, 246 (1970); M.R. Evans, T. Hanney, J. Phys. A **38**, R195 (2005).