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8. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 18: Roughening in the Edwards-Wilkinson equation with different initial conditions

In the lectures the mean squared displacement of the interface position in the one-dimensional Edwards-Wilkinson equation was computed, with the asymptotic result

$$G(0, t - t') = \langle [h(x, t) - h(x, t')]^2 \rangle = \frac{4}{\sqrt{\pi}} \frac{D}{\sqrt{\nu}} \sqrt{|t - t'|}. \tag{1}$$

Both t and t' were assumed to be in the stationary state, $t, t' \to \infty$ at finite $t - t'$.

Here we want to consider instead the early time regime starting from a flat initial condition, $h(x, 0) \equiv 0$. Following the steps outlined in the lectures, compute the variance of the height $\langle h(x, t)^2 \rangle$ as a function of time and show that the result is of the same form as (1), but with a different numerical prefactor. Interpret your finding.

Exercise 19: Stationary roughness of finite one-dimensional interfaces

In this exercise we consider one-dimensional interfaces of Edwards-Wilkinson type supported on a finite 'substrate' of length L with periodic boundary conditions. This implies that the allowed wave numbers of the spatial Fourier coefficients of the height fluctuations are $k_n = \frac{2\pi}{L} n, \quad n \in \mathbb{Z}$, and a function $f(x)$ is represented through its Fourier coefficients $\hat{f}_n \equiv \hat{f}(k_n)$ according to

$$f(x) = \frac{1}{L} \sum_{n \in \mathbb{Z}} e^{-ik_n z} \hat{f}_n.$$

a) Show that the correlation function of the Fourier coefficients of spatio-temporal white noise $\zeta(x, t)$ is given by

$$\langle \hat{\zeta}_n(t) \hat{\zeta}_{n'}(t') \rangle = LD \delta_{n,-n'} \delta(t - t'). \tag{2}$$

b) Use the result (2) to derive the amplitude $\langle |\hat{h}_n|^2 \rangle$ of the Fourier coefficients of the height fluctuations in steady state, $t \to \infty$. By summing over n, show that the stationary variance of the interface fluctuations is given by

$$W^2(L) \equiv \langle [h - \bar{h}]^2 \rangle = \frac{D}{6\nu} L$$

where $\bar{h} = L^{-1} \int_0^L dx \, h(x)$ is the spatial average of the height. *Hint:* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Problem 20: From the Wiener path integral to directed polymers

The Wiener process $w(t)$ is defined by the *transition probability* $P(w, t|w_0, t_0)$ for the path to be at w at time t, given that it was at w_0 at time t_0, which satisfies the diffusion equation

$$\frac{\partial}{\partial t} P(w, t|w_0, t_0) = \frac{1}{2}\frac{\partial^2}{\partial w^2} P(w, t|w_0, t_0), \tag{3}$$

with the initial condition $P(w, t_0|w_0, t_0) = \delta(w - w_0)$.

a.) Solve Eq. (3) and show that

$$\langle w(t)\rangle = w_0, \quad \langle (w(t) - w_0)^2\rangle = t - t_0. \tag{4}$$

b.) Since the Wiener process is a Markov process, the joint n-point probability density for the path to visit the positions w_k at times t_k, $k = 1, ..., n$ is given by

$$P(w_n, t_n; w_{n-1}, t_{n-1}; \ldots ; w_1, t_1; w_0, t_0) = \left(\prod_{i=0}^{n-1} P(w_{i+1}, t_{i+1}|w_i, t_i)\right) p(w_0, t_0), \tag{5}$$

where $p(w_0, t_0)$ denotes the probability density of the initial condition. Show that as $n \to \infty$ and $\max(t_{i+1} - t_i) \to 0$ with $t_n = t$ fixed, the transition probability can be formally written as

$$P(w, t|w_0, t_0) = \int_{w(t_0)=w_0}^{w(t)=w} \mathcal{D}w \exp\left(-\frac{1}{2}\int_{t_0}^{t}\left(\frac{dw(t')}{dt'}\right)^2 dt'\right) \tag{6}$$

where the quantity $\mathcal{D}w$ (to be identified by the calculation) represents the path measure.

c.) In b.), we used the solution of the Eq. (3) to derive Eq. (6). Actually, we can arrive at the same conclusion without solving the equation. Let $t_k = \epsilon k + t_0$ $(k = 0, \ldots, n)$ with $\epsilon = (t - t_0)/n$ (n is very large). First show that

$$P(w_{k+1}, t_{k+1}|w_k, t_k) = \left(1 + \frac{\epsilon}{2}\frac{\partial^2}{\partial w_{k+1}^2} + O(\epsilon^2)\right)\delta(w_{k+1} - w_k) = \tag{7}$$

$$\frac{1}{2\pi}\int dq_k \exp\left(iq_k(w_{k+1} - w_k) - \frac{\epsilon}{2}q_k^2\right) + O(\epsilon^2).$$

Then integrating out the q_k's and using the Markov property in Eq. (5), derive Eq. (6).

d.) Following the methodology of c.), write down the formal solution of the diffusion equation with a multiplicative noise $\zeta(w, t)$:

$$\frac{\partial}{\partial t} P(w, t|w_0, t_0) = \frac{1}{2}\frac{\partial^2}{\partial w^2} P(w, t|w_0, t_0) + P(w, t|w_0, t_0)\zeta(w, t). \tag{8}$$

The solution can be interpreted as the partition function of a directed polymer in a random enviroment. Further, the Cole-Hopf transformation, $h(w, t) = \ln [P(w, t|w_0, t_0)]$, relates solutions of Eq. (8) to solutions of another well known problem. Which one ?