UNIVERSITÄT ZU KÖLN

Prof. Dr. Joachim Krug
Dr. Ivan Szendro
Institut für Theoretische Physik
Sommersemester 2012

9. Exercise sheet to the lecture "Statistical Physics Far from Equilibrium"

Exercise 21: One-dimensional interfaces with surface diffusion

In this excercise we consider a linear stochastic PDE describing (among other things) the roughening of an atomic step on a crystalline surface in the case when mass transport occurs only *along* the step (no attachment or detachment of atoms). The area under the height function $h(x,t)$ is then conserved, which implies that the right hand side of the equation has to be the divergence of a one-dimensional current. The equation turns out to be of fourth order, and reads

$$\frac{\partial h}{\partial t} = -\kappa \frac{\partial^4 h}{\partial x^4} - \frac{\partial}{\partial x}\zeta, \tag{1}$$

where $\kappa > 0$ (why?) and $\zeta(x,t)$ is spatio-temporal white noise of strength D, $\langle \zeta(x,t)\zeta(x',t')\rangle = D\delta(x-x')\delta(t-t')$.

a.) Determine the scaling exponents α and z for the equation (1) using the rescaling analysis introduced in the lectures.

b.) You should have found from part a.) that the height variance grows (from a flat initial condition) as $\langle h(x,t)^2\rangle \sim t^{2\beta}$ with $\beta = \alpha/z = 1/8$. To determine also the prefactor, solve the equation using Fourier transformation and compute $\langle h(x,t)^2\rangle$ along the lines of Exercise 18.
Hint: The integral that appears in the calculation can be expressed in terms of the Γ-function.

c.) Repeat parts a.) and b.) for the corresponding equation with *non-conserved* noise[1],

$$\frac{\partial h}{\partial t} = -\kappa \frac{\partial^4 h}{\partial x^4} + \zeta. \tag{2}$$

d.) For equation (2), compute the stationary height variance $W^2(L)$ in a *finite* system of length L, using the approach of Exercise 19. *Hint:* $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Exercise 22: Dimensionality dependence of interface roughness

Here we consider the properties of stationary Edwards-Wilkinson interfaces in general substrate dimension d.

[1] D.E. Wolf and J. Villain, Europhys. Lett. **13** (1990) 389.

a.) Following the approach of Exercise 19, write down an expression for the stationary height variance W^2 for a two-dimensional Edwards-Wilkinson interface on a square-shaped substrate domain of size $L \times L$ with periodic boundary conditions. Verify that the resulting expression is infinite.

b.) The divergence observed in part a.) reflects a breakdown of the continuum approximation at short length scales (similar to the ultraviolet catastrophe in black body radiation). To cure it, we have to introduce a short length cutoff a (e.g. the lattice constant of a crystal or the size of a grain of sand) and restrict the summation over Fourier coefficients to wave numbers $|\vec{k}| \leq \frac{\pi}{a}$. For computational convenience, you may replace the sum by an integral and integrate over the annulus $\frac{2\pi}{L} \leq |\vec{k}| \leq \frac{\pi}{a}$ in the two-dimensional $\vec{k}$-plane. You should find a result of the form

$$W^2(L) \sim \frac{D}{\nu} \ln(L/2a). \tag{3}$$

c.) To visualize the extremely slow increase of the logarithm in (3), consider the surface of a pile of sand grains[2] of size $a = 1$ mm. By what factor does the *surface width* W increase when the lateral size L of the pile is increased from 1 m to 150×10^6 km (the distance to the sun)?

d.) Finally, analyze the stationary Edwards-Wilkinson interface in substrate dimensions $d > 2$. How does the surface width W behave as $L \to \infty$?

---

[2]S.F. Edwards and D.R. Wilkinson, Proc. R. Soc. London A **381** (1982) 17.