

# The Paradox of the Expected Time until the Next Earthquake

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**Abstract** We show analytically that the answer to the question, (can it be that) “The longer it has been since the last earthquake, the longer the expected time till the next?” depends crucially on the statistics of the fluctuations in the interval times between earthquakes. The periodic, uniform, semi-Gaussian, Rayleigh, and truncated statistical distributions of interval times, as well as the Weibull distributions with exponent greater than 1, all have decreasing expected time to the next earthquake with increasing time since the last one, for long times since the last earthquake; the lognormal and power-law distributions and the Weibull distributions with exponents smaller than 1 have increasing times to the next earthquake as the elapsed time since the last increases, for long elapsed times. There is an identifiable crossover between these models, which is gauged by the rate of fall-off of the long-term tail of the distribution in comparison with an exponential fall-off. The response to the question for short elapsed times is also evaluated. The lognormal and power-law distributions give one response for short elapsed times and the opposite for long elapsed times. Even the sampling of a finite number of intervals from a Poisson distribution will lead to an increasing estimate of time to the next earthquake for increasing elapsed time since the last one.

## Introduction

While small earthquakes after removal of aftershocks have a Poissonian distribution (Gardner and Knopoff, 1974), intermediate and large earthquakes in a given region are clustered in time (Kagan and Knopoff, 1976; Lee and Brilinger, 1979; Vere-Jones and Ozaki, 1982; Grant and Sieh, 1994; Kagan and Jackson, 1991, 1994; Kagan, 1983; Knopoff *et al.*, 1996); “clustering” is taken to mean that the earthquakes do not have a purely Poissonian, memoryless distribution of time intervals. According to this interpretation, periodic earthquakes are the extreme limit of clustering and can be predicted exactly. More generally, if there is temporal clustering, the estimate of the probability of occurrence of a future earthquake in a given time interval is improved if there is a knowledge of the times of previous events, since clustering implies a memory. To express this property quantitatively, we relate the elapsed time since the last earthquake in a region to the conditional probability of occurrence of the next earthquake within a given time interval from the present. Davis *et al.* (1989) have posed a version of this problem in the form of the following question (hereafter referred to as Q.):

(can it be that) “The longer it has been since the last earthquake, the longer the expected time till the next?”

The observation of Davis *et al.* for the log-normal distribution was that the answer to Q. is positive. Ward and Goes

(1993) and Goes and Ward (1994) showed numerically that, in the case of the Weibull distribution, the response to Q. can be either yes or no, depending on the exponent in the distribution. The positive responses would seem to be counterintuitive, since it is to be expected that an earthquake should be more likely to occur with increasing time in response to an inexorable tectonic loading that brings a fault ever closer to its finite threshold of fracture.

The intuitive interpretation is of course consistent with simple relaxation oscillator models of the earthquake process, such as the slip- or time-predictable models. But these models should be reconsidered if the stress field is altered on a given fault segment due to redistribution derived from earthquakes on nearby fault elements; these interactions can cause fluctuations in the stress field, with consequent fluctuations in the interval times. Knopoff (1996) has proposed that the fluctuations in the interval times between great earthquakes on the San Andreas Fault (Sieh *et al.*, 1989) may be associated with stress interactions between the San Andreas Fault and other nearby faults.

Below, we give a rigorous statistical framework for the derivation of a quantitative response to Q. Statistical estimates of recurrence times will be found to be very sensitive to assumptions about statistical distributions. Our results confirm, quantify, and extend the numerical analyses of Davis *et al.* (1989), Ward and Goes (1993), and Goes and Ward (1994) by providing an analytic basis for the problem.

### The Time to the Next Earthquake

Let  $p(t)$  be the probability density of the time intervals between earthquakes. If the time (now) since the last earthquake is  $t$ , what is the probability density function  $P(t')$  that we must wait an additional time  $t'$  until the next earthquake? From Bayes' theorem for conditional probabilities, cited in elementary statistics textbooks, the probability that an event  $A$ , given the knowledge of an event  $B$ , is simply the quotient of the probability of the event  $A$  without constraint and the probability of event  $B$ :

$$P(A|B) = \frac{P(A)}{P(B)}. \tag{1}$$

Applied to this problem,  $P(A) = p(t + t')$ , which is the probability that the next earthquake will occur at time  $t'$  from now, and  $P(B) = \int_t^\infty p(s)ds$ , which is the probability that no earthquake has occurred up to now. Thus,

$$P(t') = \frac{p(t + t')}{\int_t^\infty p(s)ds}, \tag{2}$$

which is normalized.

We calculate the expected time until the next earthquake  $\langle t' \rangle$  as a function of the time since the last one.

(A.) *The answer to Q. is given by the sign of  $d\langle t' \rangle/dt$ , if  $\langle t \rangle$  exists.*

From equation (2), the average expected time to the next earthquake is

$$\langle t' \rangle = \frac{\int_0^\infty t' p(t + t') dt'}{\int_t^\infty p(u) du}. \tag{3}$$

By a simple change of variable,

$$\langle t' \rangle = \frac{\int_t^\infty (u - t) p(u) du}{\int_t^\infty p(u) du}. \tag{4}$$

We integrate the numerator of (4) by parts and get

$$\langle t' \rangle = \frac{\int_t^\infty ds \int_s^\infty p(u) du}{\int_t^\infty p(u) du}. \tag{5}$$

The denominator and numerator of (5) are the familiar first cumulative integral and the less familiar second cumulative integral of  $p(u)$ . For simplicity, we write (5) as

$$\langle t' \rangle = - \frac{f(t)}{f'(t)}, \tag{6}$$

where  $f''(t) = p(t)$ ; that is,  $f(t)$  is the second cumulative integral of  $p(u)$ . Thus,

$$\frac{d\langle t' \rangle}{dt} > 0 \quad \text{if} \quad f(t)f''(t) - [f'(t)]^2 > 0. \tag{7a}$$

Equivalently,

$$\frac{d\langle t' \rangle}{dt} = - \frac{g''(t)}{[g'(t)]^2} > 0 \quad \text{if} \quad g''(t) < 0, \tag{7b}$$

where we have set  $f(t) = e^{-g(t)}$ . The signs are appropriately reversed in the case  $g''(t) > 0$ . Equation (7b) especially favors an appreciation of the behavior at large values of elapsed time  $t$ .

If  $p(t)$  is finite at  $t = 0$ , we can find yet a third version of (5), which is useful for small  $t$ . A straightforward expansion for small  $t$  shows that

$$\lim_{t \rightarrow 0^+} \frac{d\langle t' \rangle}{dt} = p(0)\Delta - 1, \tag{7c}$$

where

$$\Delta = \int_0^\infty ds \int_s^\infty p(u) du = \langle t \rangle.$$

The result of the integration follows directly from (5);  $\langle t \rangle$  is the average (unconditional) time of recurrence between two earthquakes. Let  $\tau \equiv 1/p(0)$ , where  $\tau$  is the estimate of the waiting time until the next earthquake made immediately after the occurrence of the preceding earthquake. We call  $\tau$  the instantaneous estimate of  $t'$ . Thus,  $d\langle t' \rangle/dt$  can be rewritten in the simple form

$$\lim_{t \rightarrow 0^+} \frac{d\langle t' \rangle}{dt} = \frac{\langle t \rangle}{\tau} - 1. \tag{7d}$$

- If the instantaneous estimate  $\tau$  of the waiting time is smaller than the average waiting time  $\langle t \rangle$ , the time to the next earthquake increases with increasing time since the last one for small  $t$ : this reflects the fact that the average waiting time  $\langle t \rangle$  is formed by contributions from the distribution over all time, and a value of  $\langle t \rangle$  larger than  $\tau$  indicates contributions from the distribution that are larger than  $\tau$  at nonzero times; in this case,  $\lim_{t \rightarrow 0^+} d\langle t' \rangle/dt > 0$ .
- If  $\langle t \rangle < \tau$ , the reverse is true, shorter and shorter timescales are sampled on the average as time increases, and the time to the next earthquake decreases with increasing time since the last one for small  $t$ .

In particular, if  $p(0) = 0$ , then  $\lim_{t \rightarrow 0^+} d\langle t' \rangle/dt = -1$ , and the time to the next earthquake decreases with increasing time since the last one for small  $t$ ; if, however,  $p(0) = \infty$ , then  $\lim_{t \rightarrow 0^+} d\langle t' \rangle/dt = \infty$ , and the time to the next earthquake increases with increasing time since the last one for small  $t$ .

The generalization of (7c) to times other than  $t = 0$  is the criterion (7a), when the mean  $\langle t \rangle$  exists. When it does not exist, as, for example, when the tails of the distributions decay slower than  $t^{-2}$ , we must compare  $P(t')$  as given by (2), with  $p(t')$ .

### Exponential Distribution

In order to develop some intuition, we first consider the exponential distribution, which is the familiar case of Poissonian statistics,

$$p(t) = \frac{e^{-t/t_0}}{t_0}, \quad t \geq 0,$$

where  $t_0$  is the mean interval time between earthquakes. From equation (2),

$$P(t') = \frac{e^{-t'/t_0}}{t_0}. \tag{8}$$

Not unexpectedly, the estimate of the time of occurrence of the next earthquake does not depend on the elapsed time: the average time from now to the future earthquake is  $t_0$ , no matter what the value of  $t$ . This case is memoryless; indeed, it is the only distribution that has no memory. The expected time to the next earthquake is  $\langle t' \rangle = t_0$ ; there is no need to invoke the machinery of (7) to derive  $d\langle t' \rangle/dt = 0$ . The Poisson distribution is the unique case  $g''(t) = 0$ , which gives the same result.

The exponential distribution is the fixed point of the transformation  $p(t) \rightarrow P(t')$ ; that is, it is the solution to the functional equation

$$P(t') = p(t'). \tag{9}$$

To verify that (8) is the solution to (9), differentiate (2) with respect to  $t$  and substitute in (9). We get

$$-p(t)p(t') = \frac{dp(t + t')}{dt}. \tag{10}$$

Take the Laplace transform of (10) with respect to  $t'$ , with  $1/t_0$  the transform variable. The result (8) follows. Thus the exponential distribution is the fixed point of (2).

We restate these results: Except for the Poisson distribution, all statistical distributions must have an average time from now to the future earthquake that depends on the time since the last earthquake. If the long-time tail of the function  $f(t)$ , defined as the integral of the integral of the distribution  $p(t)$ , falls off at a rate that is faster than exponential, the expected time to the next earthquake is reduced, the longer the elapsed time since the last, and vice versa. The Poisson distribution is the crossover between the two states. The exponential case has neither a positive nor a negative response

to  $Q$ , since the time since the last earthquake has no influence on the time of the next.

### Other Conditional Distributions

We calculate the expected time to the next earthquake for several examples of statistical distributions  $p(t)$  with memory. We illustrate the results in Figure 1 by displaying the average time to the future earthquake  $t'$  plotted against the time since the last earthquake  $t$  for selected distributions  $p(t)$ ; whether the values are greater or less than 1 gives the answers to  $Q$ . The details of the calculations are given in the Appendix.

The analytical results are summarized in Table 1. The times in the table are scaled by a characteristic time  $t_0$  for a given distribution; the precise definition of  $t_0$  for each distribution is given in the Appendix. In general,  $t_0$  is of the order of the mean time between earthquakes, if it is not so exactly. In most cases, we can give an answer A. that is valid over the entire range of elapsed times since the last earth-

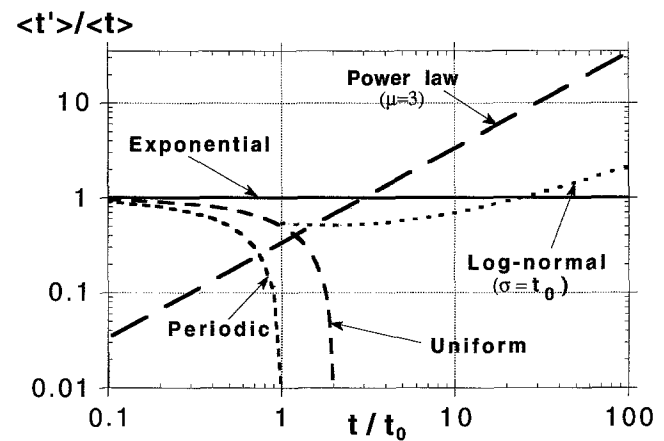


Figure 1. Expected time of the next earthquake as a function of the elapsed time since the last one. The response to  $Q$ . is given by the value of the curve with respect to 1.

Table 1  
Response to  $Q$ .

Distribution	Short Times	Long Times
Exponential	0	0
Periodic	-	-
Uniform	-	-
Gaussian	-	-
Semi-Gaussian	-	-
Lognormal	-	+
$\frac{1}{4\sqrt{x}}e^{-\sqrt{x}}$	-	+
Weibull ( $m > 1$ )	-	-
Rayleigh ( $m = 2$ )	-	-
Weibull ( $m < 1$ )	+	+
Power law	-	+
Truncated power law	+	-

quake. In some cases, we can only give the answer for the limits of short and long time since the last earthquake. The table is arranged to favor the limiting responses, even though we may have the complete solution. In the case of the exponential distribution, the response is neutral, as we have already discussed. In the cases of the periodic and uniform distributions, the response is only meaningful for times up to  $t_0$ ; in the case of the Gaussian distribution, the response is meaningful for long elapsed times, as we discuss in the Appendix. In the cases of the Weibull distribution with  $m > 1$  and the semi-Gaussian distribution, the response can be proved to be negative for both short and long elapsed times and can be inferred to be negative for all elapsed times. For the Weibull distribution with  $m < 1$  and the power-law distribution, the response is positive for both short and long elapsed times and can be inferred to be positive for all elapsed times. For the lognormal, power-law, and truncated power-law distributions, there is one response for short elapsed times and the opposite response for long elapsed times, with the implications of a crossover and hence neutral response at an intervening time scaled by  $t_0$ ; the lognormal and truncated power-law distributions have opposite responses to each other in the long and short time regimes.

Since the truncated and ordinary power-law distributions give opposite results for long elapsed times, it follows that the answers to Q. are unstable with respect to the presence or absence of a cutoff in the distributions. It is by definition problematical that a presumed existence of a cutoff can be identified from a finite set of observations of interval times: there is no guarantee that a presumed cutoff will not disappear with a future observation of a longer interval between earthquakes. Thus a positive response to the question for long elapsed times since the last earthquake is only conjectural; that is, it is only as strong as one knows the distribution to times longer than have been observed, which is impossible. Of course, the distribution can always be postulated *a priori*, as in the numerical examples of Davis *et al.* (1989) and Ward and Goes (1993), but the postulate does not ensure that it represents nature.

### Estimate of $t_0$

Suppose that we do not know *a priori* the characteristic time  $t_0$  of time intervals between successive earthquakes. We then have to estimate it from a finite suite of observations of interearthquake time intervals. Assume that  $(n - 1)$  observations of time intervals  $t_1, t_2, \dots, t_{n-1}$  are made precisely; we ignore here the additional problem of the uncertainties in the time intervals that occur for historical earthquakes; this can be treated by standard statistical methods (Sieh *et al.*, 1989). Suppose that the time since the last event is  $t$ . Then, in the case of the Poisson distribution  $p(t) = e^{-t/t_0}/t_0$ , the standard maximum likelihood method gives the estimate of  $t_0$  as the value that maximizes

$$\frac{1}{t_0^n} e^{-t/t_0} - (t + \sum_{j=1}^{n-1} t_j)/t_0^n;$$

that is,

$$t_0 = \frac{1}{n} \left( t + \sum_{j=1}^{n-1} t_j \right). \quad (11)$$

Thus, even for the Poissonian case, the use of the information that no event has occurred since  $t$  gives an estimate of the average recurrence time  $t_0$  for the next event that increases with  $t$ ! The Poisson distribution is memoryless only if its parameter  $t_0$  is known *a priori*.

The calculation (11) can be generalized for the other distributions discussed above. Our previous results must thus be reconsidered if the parameters of the distributions are themselves not known precisely but are estimated using presently available information. This does not pose any difficulty in principle, but must be addressed case by case. This simple calculation demonstrates the sensitivity of the “prediction” to the assumptions concerning what is really known and what is only inferred from the data.

### Summary

These observations can be summarized as follows:

- The Poisson or exponential distribution is memoryless and the expected time until the next event is independent of previous observations and of the elapsed time since the last earthquake. The exponential thus acts as a fixed point in the space of distributions of the transformation (2) and sits at the boundary between the positive and negative classes of memory, that is, at the boundary between positive and negative responses to Q. Any statistics of the fluctuations of recurrence times that is different from Poissonian entails the explicit assumption of a memory.
- Any distribution that falls off at large time intervals at a faster rate than an exponential, such as the periodic, quasiperiodic, uniform, and semi-Gaussian distributions, and the Weibull distribution with  $m > 1$ , has the property “the longer it has been since the last earthquake, the shorter the expected time until the next.” The truncated power-law distribution for times close to the cutoff time also has this property.
- Any distribution that falls off at large time intervals at a slower rate than an exponential, such as the Weibull distribution with  $m < 1$ , the unbounded lognormal and power-law distributions, and the truncated versions of these distributions for times remote from the cutoff, has the property “the longer it has been since the last earthquake, the longer the expected time till the next.”
- All distributions that have an instantaneous expectation time interval between earthquakes *smaller* than the average waiting time between earthquakes have the property of an *increasing* time to the next earthquake for an increas-

ing time since the last one, for short times since the last one. This includes the cases  $p(0) = \infty$ .

- All distributions that have an instantaneous expectation time interval between earthquakes *larger* than the average waiting time between earthquakes have the property of a *decreasing* time to the next earthquake for an increasing time since the last one, for short times since the last one. This includes the cases  $p(0) = 0$ .
- Caution should be exercised in the use of statistics of fluctuations of interval times deduced from data sets that describe only the distributions for short time intervals between earthquakes. This is because of the strong dependence of our result for long times, and in some cases for short times as well, on the properties of the tails of the distributions as well as on the values of the parameters of the distributions.
- The estimate of the time until the next earthquake depends on a precise estimate of the tail of  $p(t)$  and is unstable with respect to presently available data for the recurrence of large earthquakes. Even a finite sampling of the Poisson distribution will lead to an estimate of the time to the next earthquake that increases with increased time since the last one.

Thus the positive response of Davis *et al.* (1989) to Q., “the longer it has been since the last earthquake, the longer the expected time till the next,” is shown to arise from the use of a distribution that decays slower than an exponential and that is unbounded for large time intervals; the result is valid for other distributions as well. If a slowly decaying law itself undergoes a transition at even longer intervals to a more rapidly decaying law, as in the extreme case of a distribution with a cutoff, one can expect that eventually the next earthquake will become more and more probable. The response to the question (Q.) is also related, in part, to the finiteness of the number of observations of time intervals between earthquakes that gives the estimate of the distribution  $p(t)$ ; the extrapolation of the estimate of the distribution to its asymptote for very large time intervals is exceedingly dangerous, since this procedure is likely to be based on few, if any, observations. The results of this exercise suggest that caution be used in the extrapolation of statistics deduced from short timescale data sets to long timescales.

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### Appendix

We calculate the expected time to the next earthquake for several examples of statistical distributions  $p(t)$  with memory.

#### Periodic Distribution

The simplest of the distributions with memory is the periodic distribution,

$$p(t) = \delta(t_0 - t).$$

By inspection, we have

$$P(t') = \delta(t_0 - t' - t), \quad \langle t' \rangle = t_0 - t.$$

Thus,  $d\langle t' \rangle / dt = -1$  without invoking the generalized machinery. In this simplest of cases, the expected time of the forthcoming earthquake decreases as the elapsed time since the preceding earthquake increases. Extension to quasiperiodic cases can be made.

### Uniform Distribution

The uniform distribution is

$$p(t) = \frac{1}{2t_0} \quad 0 \leq t \leq 2t_0,$$

where  $2t_0$  is the maximum interval between earthquakes and  $t_0$  is the mean. From equation (2), we get

$$P(t') = \frac{1}{2t_0 - t} \quad 0 \leq t' \leq 2t_0 - t. \quad (A1)$$

$P(t')$  is independent of  $t'$ ; that is, it is itself a uniform distribution, but its value is dependent on  $t$ . The probability that an earthquake will occur at any time in the future up to  $t_0$  increases as the time since the last earthquake increases and becomes infinite as  $t \rightarrow 2t_0$ , which simply expresses the intuitive result that the event will occur with certainty before  $2t_0$ . It is easy to see that the average time to the future earthquake from the present is  $\langle t' \rangle = \frac{1}{2}(2t_0 - t)$ . The negative value of  $d\langle t' \rangle/dt$  gives the answer (A.): the expected time to the next earthquake decreases with increasing time since the last earthquake. In Figure 1, we show the average time to the future earthquake plotted against the time since the last earthquake; both coordinates are normalized by the mean time between earthquakes,  $t_0$ . The linear relationship between  $\langle t' \rangle$  and  $t$  is strongly curved on the log-log plot.

In Figure A1, we display the probability that the next earthquake will occur at time  $t'$  from now. We show the unconditional probability, that is, the probability as though we knew the distribution of intervals  $p(t)$  but did not know the time of the last earthquake. We also show the (conditional) probability of an earthquake in the future knowing that the last earthquake took place at time  $1.33t_0$  in the past. In the latter case, no earthquake can occur after  $0.67t_0$  from now; according to (A1), the probability of occurrence of the future earthquake is higher by a factor of 3 than the unconditional probability and is independent of the time of the future earthquake.

### (Semi)-Gaussian Distribution

The Gaussian distribution is

$$p(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(t-t_0)^2/2\sigma^2},$$

where  $t_0$  is both the mean and the most probable time interval of earthquake recurrence; the standard deviation is  $\sigma$ . The Gaussian distribution has a finite probability that the next earthquake will occur before the preceding one. The drawback is minor if  $\sigma \ll t_0$ , a condition that describes a nearly periodic distribution; we have considered the periodic case above. To restrict the problem to cases of positive  $t$ , we could

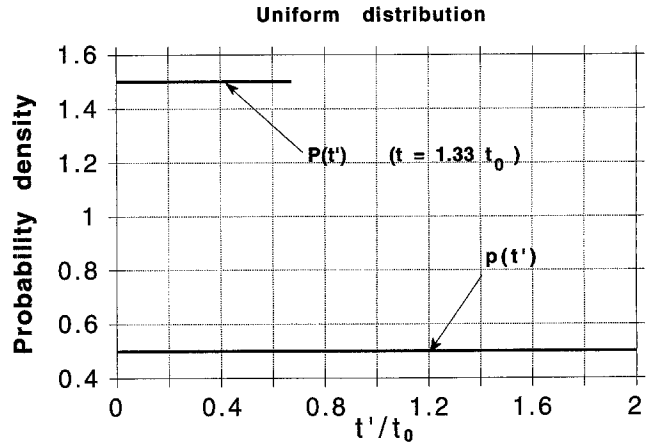


Figure A1. Uniform distribution:  $p(t')$  and  $P(t')$  for  $t = (4/3)t_0$ .

truncate the distribution at  $t = 0$ ; however, this leads to messy mathematics; for large times, the drawback is minor.

For the simpler problem  $t \gg t_0$ , we use the approximation

$$p(t) = \frac{2}{\pi t_0} e^{-(t/t_0)^2/\pi}, \quad t \geq 0,$$

which is the semi-Gaussian distribution; that is, it is a Gaussian centered at  $t = 0$ ; the mean time interval of earthquake recurrence is  $t_0$ , and the most probable time for recurrence is of course zero for this distribution. Formally, equation (2) yields

$$P(t') = \frac{2}{\pi t_0} \frac{e^{-(t+t')^2/\pi t_0^2}}{\operatorname{erfc}\left(\frac{t}{\tau t_0}\right)}, \quad (A2)$$

where  $\operatorname{erfc}(x)$  is the usual complementary error function.

If the elapsed time since the last earthquake is very large,  $t \gg t_0$ , we can use the first term of the asymptotic expansion of the second cumulative integral that is

$$f(t) \sim \frac{e^{-t^2/\pi t_0^2}}{t^2}.$$

Then  $g(t) \sim t^2 + O(\log t)$ ,  $g''(t) > 0$  for large  $t$ , and from (7b),

$$\frac{d\langle t' \rangle}{dt} < 0$$

For short times, we use (7c) with  $p(0) = 2/(\pi t_0)$  and  $\Delta = t_0$  and get

$$\frac{d\langle t' \rangle}{dt} \approx \frac{2}{\pi} - 1 = -0.363.$$

Thus in both the short and long time limits, the expected time until the next earthquake decreases as the time since the last one increases.

### Lognormal Distribution

The lognormal distribution is

$$\begin{aligned}
 p(t) &= \frac{1}{\sqrt{2\pi\sigma}} \frac{1}{t} e^{-(\log t/t_0)^2/2\sigma^2} \\
 &= \frac{1}{\sqrt{2\pi\sigma t_0}} e^{-1/2\sigma^2(\log t/t_0 + \sigma^2)^2 + \sigma^2/2},
 \end{aligned}
 \tag{A3}$$

which is similar in shape to the Rayleigh distribution (see below) near  $t = 0$  but has a much more slowly decaying tail for large times. In this case,  $t_0$  is the median time; the mean time is  $\langle t \rangle = t_0 e^{\sigma^2/2}$ ; the most probable time is  $t_0 e^{-\sigma^2}$ . From (A3), we can write

$$g(t) = \log\left(\frac{t}{t_0}\right) + \sigma^2 + O[\log \log\left(\frac{t}{t_0}\right)],$$

whence

$$g''(t) \approx \frac{2}{t^2} \left(1 - \sigma^2 - \log \frac{t}{t_0}\right).$$

For  $t \gg t_0$ ,  $g'' < 0$ , and hence  $d\langle t' \rangle / dt > 0$ . For  $t \ll t_0$ ,  $g'' > 0$ , and  $d\langle t' \rangle / dt < 0$ ; alternatively, we note that  $p(0) = 0$ , and hence from (7c),  $d\langle t' \rangle / dt < 0$ , which is the same result. Thus the lognormal distribution has a crossover in response to Q.

We express these results graphically. For the case  $\sigma = t_0$ , we display  $P(t')$  for times  $t = 2t_0$  and  $t = 5t_0$  (Fig. A2) together with the unconditional lognormal distribution  $p(t)$ . From Figure A2a, we see that  $P(t')$  is significantly smaller than  $p(t')$  for times comparable to the elapsed time  $t$ , but  $P(t')$  is, as expected, larger than  $p(t')$  at large times (see extension of Fig. A2a to long times in Fig. A2b). This is a small effect for  $t = 2t_0$  but is much stronger for  $t = 5t_0$  and all the more so if  $t$  increases even more. Thus, numerically as well as analytically, for early elapsed times  $t$  that are comparable to the peak of the distribution, the longer the elapsed time since the last event, the shorter the time until the next event; but for large elapsed times since the last earthquake, the longer the time since the last event, the longer the time until the next one. In the lognormal case, it is correct to state that "the longer it has been since the last earthquake, the longer the expected time until the next" but only for elapsed times greater than times of the order of the characteristic time. The lognormal distribution is an example of a case that has one answer to Q. for short times since the last earthquake and the opposite answer for long times. Of course, the crossover takes place at elapsed times that are of the order of the characteristic time  $t_0$ . Note that the proba-

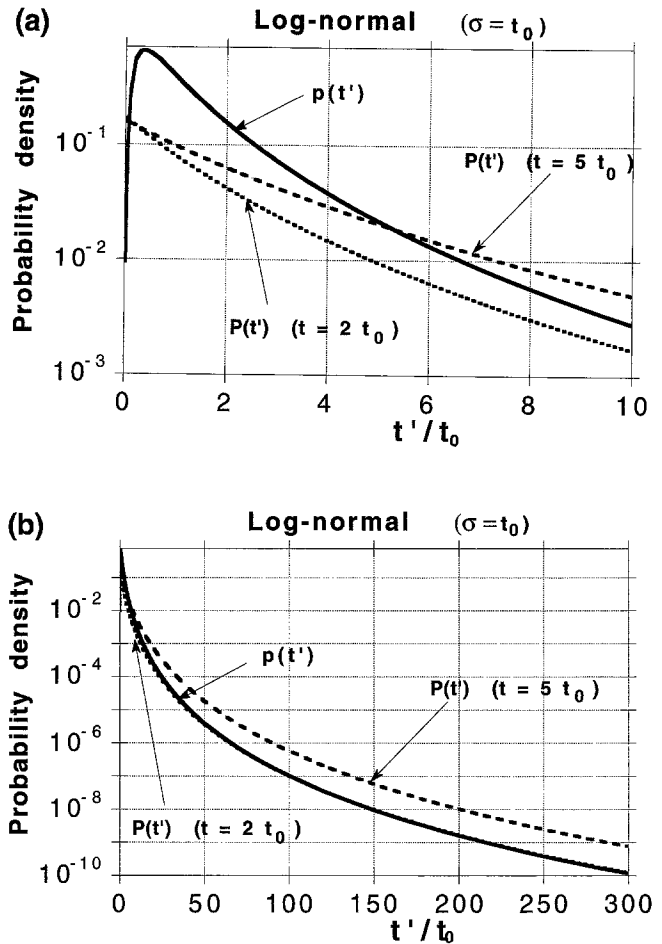


Figure A2. Lognormal distribution with  $\sigma = t_0$ :  $p(t')$  and  $P(t')$  for  $t = 2t_0$  and  $t = 5t_0$ : (a) short times and (b) long times.

bilities for long times in the future are, as might be expected, very small. The lognormal case is an example of a distribution with a tail that decays slower than an exponential.

To illustrate more concretely the properties of a system that has a crossover response to Q., we concoct the distribution

$$p(t) = \frac{1}{4t_0} \sqrt{\frac{t}{t_0}} e^{-\sqrt{t/t_0}},$$

which probably has no redeeming virtue in nature but has the property that it has easily calculable integrals. It is evident that this distribution has both a long-time tail that decays slower than exponential, and the property  $p(0) = 0$ , as in the case of lognormal distribution. Thus we are guaranteed that there is a crossover in the response to Q. More precisely, the criterion function (7a) is  $f(t)f''(t) - [f'(t)]^2 = \frac{1}{4}(x^3 + 4x^2 + 4x - 4)e^{-2x}$ , where  $x = \sqrt{t/t_0}$ . The criterion has a crossover in sign between  $x$  small and large at  $t/t_0 = 0.3532$ . The lognormal distribution has similar properties.

### Weibull Distribution

The Weibull distribution is

$$p(t) = mt_0^{-m} t^{m-1} e^{-(t/t_0)^m}, \quad 0 < t < \infty, \quad m > 0,$$

having a most probable value  $(m - 1)^{1/m}t_0$ , a mean  $\alpha(m)t_0$  where  $\alpha(m) = \int_0^\infty e^{-t^m} dt$ , and median  $(\log 2)^{1/m}t_0$ . Values of  $m$  smaller than 1 correspond to  $p(t)$  decaying slower than an exponential for large  $t$  and give the so-called stretched exponential distributions, while values of  $m$  larger than 1 lead to a decay that is faster than exponential. For large  $m$ ,  $p(t)$  approaches a delta function centered on  $t_0$ , that is, to the periodic distribution we have considered above. Ward and Goes (1993) and Goes and Ward (1994) have studied the degree of earthquake clustering as a function of  $m$ ; in their notation,  $\nu = 1/m$ . Equation (2) yields

$$P(t') = mt_0^{-m} (t + t')^{m-1} e^{-(t+t')^m - t^m}/t_0^m. \quad (A4)$$

The first term of the asymptotic series for  $f(t)$ , which is the second cumulative integral of  $p(t)$ , is

$$f(t) \sim \frac{e^{-(t/t_0)^m}}{\left(\frac{t}{t_0}\right)^{3(m-1)}} = e^{-(t/t_0)^m - (3m-1)\log t/t_0}.$$

Evidently,  $g(t) \sim t^m + O(\log t)$ , and hence  $g''(t) \sim m(m - 1)t^{(m-2)}$ . Thus if  $0 < m < 1$ , then  $g''(t) < 0$ , and  $d\langle t' \rangle/dt > 0$  for large  $t$ , while if  $m > 1$ , then  $g''(t) > 0$ , and  $d\langle t' \rangle/dt < 0$  for large  $t$ . For small  $t$ ,  $p(0) = 0$ ,  $m > 1$ , and it follows from (7c) that  $d\langle t' \rangle/dt < 0$ . For cases  $m < 1$ ,  $p(0) = \infty$ , and from (7c),  $d\langle t' \rangle/dt > 0$ . Thus there is a reversal in response between the cases  $m < 1$  and  $m > 1$ , in agreement with the result of Ward and Goes.

In Figure A3a, we exhibit the interesting subcase of the Weibull distribution with  $m = 2$ , which is appropriate for rectified Gaussian noise and is known as the Rayleigh distribution. The Rayleigh distribution has a tail with similar properties to that of the Gaussian and decays faster than an exponential. We show  $P(t')$  for times  $t = t_0$  and  $t = 2t_0$ . It is clear that  $P(t')$  has a progressively shrinking width to the origin as  $t$  increases; that is, the expected time decreases as the waiting time  $t$  increases. The answer A. is negative for both short times and long times, the latter property evidently connected with the rapid fall-off in  $p(t)$  for large  $t$ .

The opposite situation is found in the case  $m < 1$ ; in Figure A3b, we plot the case  $m = 1/2$  and show  $P(t')$  for times  $t = t_0$ ,  $t = 2t_0$ , and  $t = 10t_0$ , as well as  $p(t)$ . It is clear that  $P(t')$  lies well above  $p(t')$  at long times  $t' > t_0$  and all the more so as  $t$  increases. Thus the longer we wait, the longer the time to the next event, in this case.

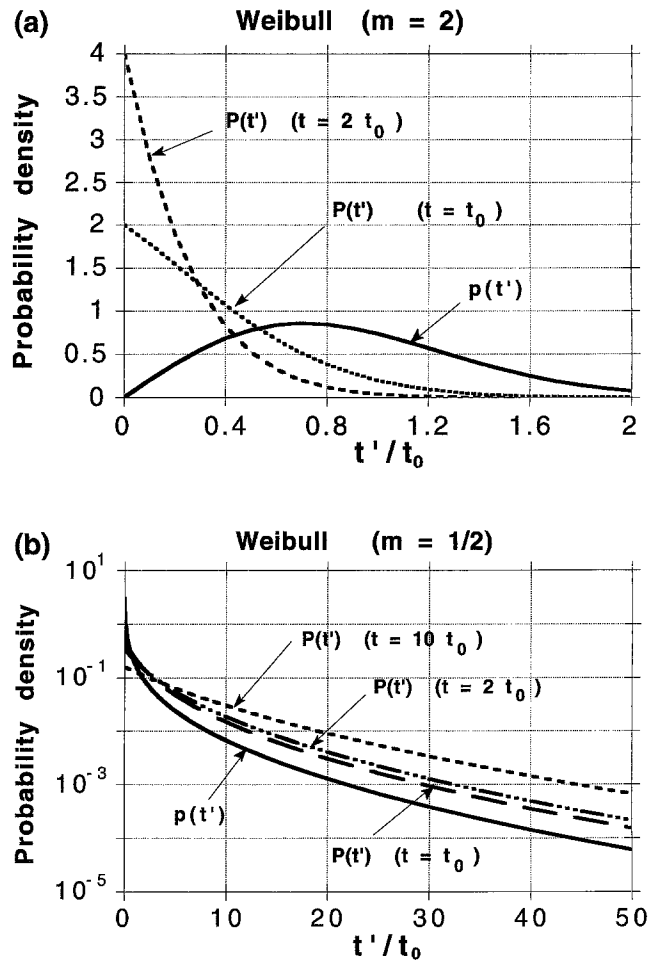


Figure A3. Weibull distribution. (a)  $m = 2$  (Rayleigh distribution), corresponding to a tail decaying faster than an exponential.  $P(t')$  is shown for  $t = t_0$  and  $t = 2t_0$  together with  $p(t')$ . (b)  $m = 1/2$  corresponding to a tail decaying slower than an exponential.  $P(t')$  is shown for  $t = t_0$ ,  $t = 2t_0$ , and  $t = 10t_0$ , together with  $p(t')$ .

### Power-Law Distribution

The unconditional power-law distribution is

$$p(t) = 0, \quad 0 < t < t_0, \quad (A5)$$

$$p(t) = \frac{\mu}{t_0} \left(\frac{t}{t_0}\right)^{-(1+\mu)}, \quad t_0 \leq t \leq \infty, \quad 0 < \mu < \infty.$$

The characteristic timescale  $t_0$  is proportional to the mean  $(\mu/(\mu - 1))t_0$  for  $\mu > 1$  and the median  $t_0 2^{1/\mu}$ ; for  $\mu < 1$ , the mean is infinite. This is an example of a distribution with a waiting time and has been used in the case  $\mu = 1/2$  in short-term earthquake prediction calculations by Kagan and Knopoff (1981, 1987).

For  $\mu > 1$ , we evaluate  $d\langle t' \rangle/dt$  for this case by applying the criterion function (7a), which gives  $(t_0/t)^{2\mu} 1/(\mu - 1)$ . Thus  $d\langle t' \rangle/dt > 0$  for all  $\mu > 1$  and all  $t > t_0$ .



For  $\mu \leq 1$ ,  $\langle t \rangle$  diverges, and the criterion (7a) cannot be used. In this case, we have to examine the conditional distribution (2) directly and compare it with the unconditional distribution. This general method is also applicable to the case  $\mu > 1$ . Substitution of (A5) in equation (2) yields

$$P(t') = \frac{\mu}{t} \left(1 + \frac{t'}{t}\right)^{-(1+\mu)}. \quad (\text{A6})$$

Formula (A6) is almost the same as the unconditional distribution, except for the additional 1 in the parentheses; the two expressions are identical in the limit  $t' \gg t$  except for obvious scaling factors. Thus, in this limit, the distribution  $P(t')$  is also a power law with a characteristic scale given by the waiting time  $t$ , instead of  $t_0$  for the unconditioned  $p(t')$ . Thus the longer it has been since the last earthquake, the longer the expected time until the next, for all cases  $\mu > 1$ .

Figure A4a shows  $P(t')$  for  $t = 10t_0$  and  $t = 100t_0$  together with  $p(t)$  for an exponent  $\mu = 3$  (for this choice,  $p(t)$  possesses a finite mean and variance). We observe the asymptotic power-law behavior of  $P(t')$  at times  $t' > t$  with amplitude significantly larger than  $p(t')$ , showing the enhanced probability for large conditional waiting times. Figure A4b shows  $P(t')$  for  $t = 10t_0$  and  $t = 100t_0$  together with  $p(t)$  for the threshold case of exponent  $\mu = 1$ ; in this case, the mean and variance are not defined. This is an illustration of a power law with a very weighty tail. The behavior of  $P(t')$  is qualitatively similar to the previous case.

### Truncated Power-Law Distribution

Except for the uniform and periodic distributions, we have considered thus far only distributions of fluctuations in interval times that extend to infinity. In these cases of distributions with long-time tails, there is a finite but small probability that a second earthquake will occur after a very long time interval after the first. If the distributions describe the seismicity of a region, rather than that of an individual fault, the very long time intervals imply very large accumulations of deformational energy and hence very large fracture sizes. To avoid the problems of earthquake sizes greater than the size of a given region, we consider a cutoff in the distributions  $p(t)$  (Knopoff, 1996). To demonstrate the influence of a cutoff, we restrict the previous case to

$$p(t) = \frac{1}{1 - \left(\frac{t_{\max}}{t_0}\right)^{-\mu}} \frac{\mu}{t_0} \left(\frac{t}{t_0}\right)^{-(1+\mu)}, \quad t_0 \leq t \leq t_{\max},$$

$$= 0, \quad 0 < t < t_0, \quad (\text{A7})$$

which is normalized. Substitution in equation (2) yields

$$P(t') = \frac{\mu}{(t + t')^{1+\mu}} \frac{1}{t^{-\mu} - t_{\max}^{-\mu}}. \quad (\text{A8})$$

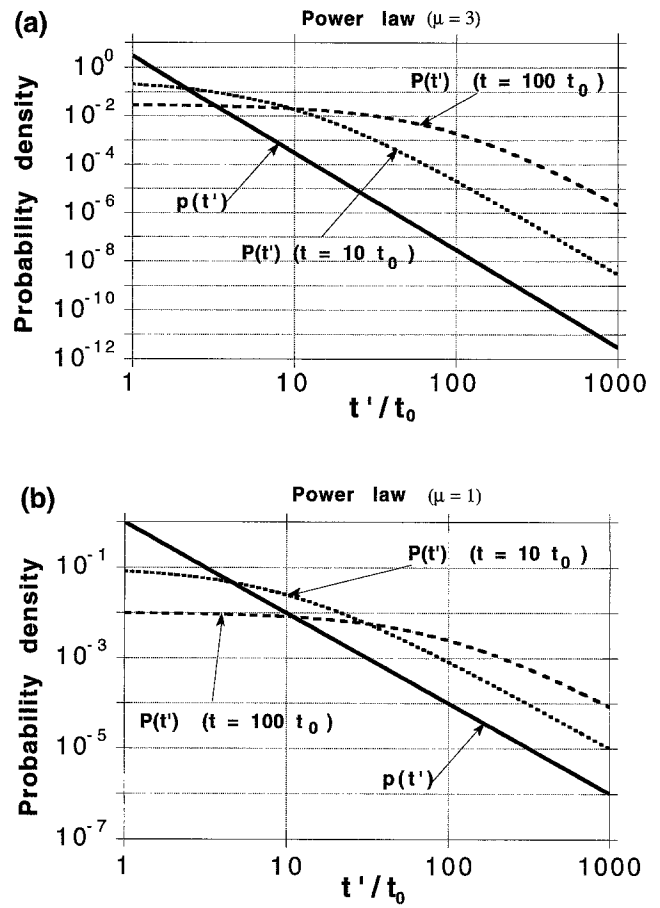


Figure A4. Power-law distribution. (a)  $\mu = 3$ ;  $p(t)$  possesses a finite mean and variance.  $P(t')$  is shown for  $t = 10t_0$  and  $t = 100t_0$  together with  $p(t')$ . (b)  $\mu = 1$ ; the mean and variance are not defined.  $P(t')$  is shown for  $t = 10t_0$  and  $t = 100t_0$  together with  $p(t')$ .

For  $t \ll t_{\max}$ , the second factor of (A8) is  $t^\mu$ , which is the same as letting  $t_{\max} \rightarrow \infty$ . Thus we recover the previous case of the simple power law without truncation.

The interesting regime is found when  $t$  is not very small compared to  $t_{\max}$ . Consider the case  $t \rightarrow t_{\max}$ . From equation (A8), we get

$$P(t') \approx \frac{1}{t_{\max} - t}, \quad (\text{A9})$$

which is independent of  $t'$  and becomes very large as  $t \rightarrow t_{\max}$ . This case is identical to that of the uniform distribution (A1) above. Thus it is not unexpected that the longer we wait, the shorter will be the expected time until the next event. Without truncation, the result is reversed. There is a crossover between the truncated and untruncated cases, as illustrated in Figure A5. In the figure, we take  $\mu = 3$  as in Figure A4a,  $t_{\max} = 100t_0$ , and show  $P(t')$  for  $t = 10t_0, 90t_0$ , and  $98t_0$ . For  $t = 10t_0$ ,  $P(t')$  is found to be much larger than  $p(t')$  in the tail, as in the previous untruncated case. For  $t = 90t_0$ ,  $P(t')$  is defined only for  $0 \leq t' \leq 10t_0$ . In agree-

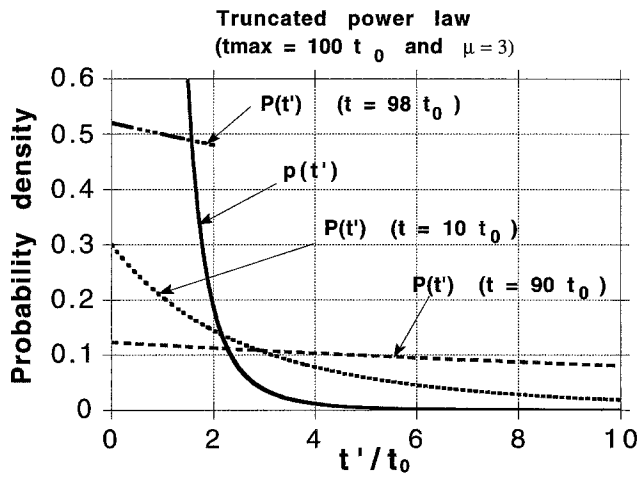


Figure A5. Truncated power-law distribution with  $\mu = 3$  and  $t_{\max} = 100t_0$ .  $P(t')$  is shown for  $t = 10t_0$ ,  $90t_0$ , and  $98t_0$ .

ment with (A8), we see that  $P(t')$  becomes almost constant and close to  $1/10t_0$ . For  $t = 98t_0$ ,  $P(t')$  is defined only for  $0 \leq t' \leq 2t_0$  and is close to  $1/2t_0$ . This illustrates the crossover from a longer expected time when  $t$  is small to a shorter expected time as  $t$  approaches  $t_{\max}$ .

Since the truncated power-law distribution is very close to a uniform distribution for times near  $t_{\max}$ , we expect that the truncated lognormal distribution and truncated Weibull distributions with  $\mu < 1$  will also have a shorter time until the next event, the longer we have been waiting for an earthquake to happen; thus we expect a crossover in the response to  $Q$ , between these truncated and untruncated cases as well.

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