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Probability theory and stochastic processes for physicists

Problem 1: The log-normal distribution

A random variable X is log-normally distributed if $\ln X$ has a Gaussian distribution. Similar to the Gaussian distribution which retains its shape under addition of RV's, the log-normal distribution is invariant under multiplication, and thus arises in applications where random effects are multiplied.

a.) Using the rules for the transformation of probability density functions, show that the pdf of the log-normal distribution is of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{x} \exp\left[-\frac{\ln^2(x/x_0)}{2\sigma^2}\right]. \tag{1}$$

Compute the mean $\langle X \rangle$ and the most likely value $X_{\max}$, and show that $\langle X \rangle \gg X_{\max}$ when σ is large.

b.) In applications, it is often difficult to distinguish the log-normal distribution from a power law. To see why this is so, rewrite (1) as a power law with an x-dependent exponent $\alpha(x)$, and plot (1) in double-logarithmic scales.

c.) Show that the moments of the modified log-normal distribution

$$f_a(x) = \frac{1}{\sqrt{2\pi}} x^{-1} e^{-\frac{1}{2}(\ln x)^2} [1 + a \sin(2\pi \ln x)], \quad -1 \leq a \leq 1, \tag{2}$$

are independent of a. This illustrates the fact that the moments of a distribution do not generally suffice to completely characterize it.

Problem 2: The Weierstrass walk

The Weierstrass walk on the real axis is defined by $Z_W(N) = \sum_{i=1}^{N} X_i$, where the X_i are i.i.d. RV with the pdf

$$f(x) = \frac{a-1}{2a} \sum_{m=0}^{\infty} a^{-m} [\delta(x - b^m) + \delta(x + b^m)],$$

where $a, b > 1$; in words: The walk performs jumps with equal probabilities to the left and to the right, and the probability for a jump of length b^m is proportional to a^{-m}.

a.) Compute the characteristic function $G_X(k)$ of the X_i, and show that it satisfies the relation

$$G_X(k) = \frac{1}{a} G_X(bk) + \frac{a-1}{a} \cos k. \tag{3}$$

b.) Show that $Z_W(N)$ satisfies the central limit theorem only if $a > b^2$, while for $a \leq b^2$ one obtains a Lévy-law with index $\alpha = \ln a / \ln b$. This follows by investigating the behavior of $G_X(k)$ near $k = 0$ using the relation (3).

Problem 3: Subdiffusive transport in a disordered medium

Consider a particle performing a random walk on a crystal lattice. With each site $\mathbf{r}$ we associate a binding energy $E_\mathbf{r} > 0$, which is an i.i.d. exponential random variable with mean E_0, i.e. the pdf is

$$f(E) = E_0^{-1} e^{-E/E_0}.$$

The jumps between neighboring sites are thermally activated, hence the waiting time $\tau_\mathbf{r}$ at site $\mathbf{r}$ is given by the *Arrhenius law*

$$\tau_\mathbf{r} = \tau_0 e^{E_\mathbf{r}/k_B T},$$

where k_B is the Boltzmann constant, T is the temperature and τ_0 denotes a microscopic time scale, e.g. an inverse phonon frequency.

Determine the waiting time distribution $\psi(\tau)$, and show that the motion of the particle becomes subdiffusive below a critical temperature T_c.