UNIVERSITÄT ZU KÖLN

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Probability theory and stochastic processes for physicists

Problem 8: Generating functions

a.) Compute the generating function for the uniform distribution on the interval $[0, 1]$, with pdf

$$f(x) = \begin{cases} 1 & : \quad 0 \leq x \leq 1 \\ 0 & : \quad \text{else}, \end{cases} \tag{1}$$

and use it to determine the moments of the distribution.

b.) Compute the generating function and the cumulants of the Poisson distribution

$$f(x) = \sum_{n=0}^{\infty} p_n \delta(x-n) \quad \text{with} \quad p_n = \frac{e^{-a} a^n}{n!}, \quad a > 0.$$

Problem 9: Partition ratios for Lévy random variables [1]

We consider N i.i.d. RV's X_i with Pareto density

$$f(x) = \alpha x^{-\alpha+1}, \quad x \geq 1 \tag{2}$$

and define the partition ratios

$$W_N^i = \frac{X_i}{\sum_{j=1}^N X_j}. \tag{3}$$

We want to compute the moments $\langle Y_k \rangle = \langle \sum_{i=1}^N (W_N^i)^k \rangle$ for $\alpha < 1$ and large N.

a.) Using the properties of the Γ-function, show that

$$\langle Y_k \rangle = \frac{N}{\Gamma(k)} \int_0^\infty dt\, t^{k-1} \langle e^{-tx} \rangle_X^{N-1} \langle x^k e^{-tx} \rangle_X, \tag{4}$$

where $\langle \cdot \rangle_X$ refers to an average with respect to the pdf $f(x)$.

b.) Next show that for $k > \alpha$ and $t \to 0$

$$\langle x^k e^{-tx} \rangle_X \approx \alpha t^{-(k-\alpha)} \Gamma(k-\alpha) \tag{5}$$

and verify that, for $\alpha < 1$ and $t \to 0$,

$$\langle e^{-tx} \rangle_X \approx 1 - t^\alpha \alpha(-\Gamma(-\alpha)) \tag{6}$$

by taking derivates with respect to t and comparing to (5).

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[1] Based on B. Derrida, in *On Three Levels*, ed. by M. Fannes et al. (1994).

c.) Insert (5) and (6) in (4) to show that

$$\lim_{N\to\infty} \langle Y_k \rangle = \frac{\Gamma(k-\alpha)}{\Gamma(k)\Gamma(1-\alpha)}. \tag{7}$$

To convert this into bounds on moments of the maximal participation ratio $W_N^{\max}$, prove the general inequality

$$\sum_{i=1}^{N} (W_N^i)^k \leq (W_N^{\max})^{k-1}. \tag{8}$$

d.) Finally, evaluate (4) for a pdf for which all moments exist, and show that in this case

$$\langle Y_k \rangle \approx \frac{1}{N^{k-1}} \frac{\langle x^k \rangle}{\langle x \rangle^k}. \tag{9}$$

Problem 10: Theory of records I

Here we provide an alternative proof of the expression for the record rate and the stochastic independence of record events presented in the lectures.

a.) Given N i.i.d. RV's $X_1, ..., X_N$, the probability that all are less than x is given by $F(x)^N$. The probability p_n that $X_n = \max\{X_1, X_2, ..., X_n\}$ can therefore be written as

$$p_n = \int dx f(x) F(x)^{n-1}.$$

Verify that $p_n = \frac{1}{n}$ independent of F.

b.) Now write down a similar expression for the probability p_{ij} that both event i and event j is a record, and show by explicit computation that $p_{ij} = p_i p_j = \frac{1}{ij}$.

Problem 11: Theory of records II

Consider the sequence $\{R_k : k = 1, 2, ...\}$ of record times for identically and continuously distributed random variables. We have seen that the ratios R_k/R_{k+1} become asymptotically independent uniform random variables. Use this result to determine the asymptotic distribution of the ratios Δ_k/Δ_{k+1} of *inter-record times*

$$\Delta_k = R_{k+1} - R_k$$

and compare it to the distribution of R_k/R_{k+1}. What is the asymptotic probability that $\Delta_{k+1} < \Delta_k$?