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Probability theory and stochastic processes for physicists

Problem 12: Gott's law

In 1993, J. Richard Gott proposed a method to predict the future duration of a phenomenon (such as the existence of an institution or a civilization) based only on the knowledge of how long it has been in progress, by envoking a temporal Copernican principle[1]. One formulation of Gott's law reads

$$\mathrm{Prob}[t_f > Yt_p] = \frac{1}{1+Y}, \tag{1}$$

where t_p and t_f denote the past and future duration of the phenomenon in question.

a.) Use (1) to estimate the probability that (i) you will reach the age of 150 years and (ii) that the Universities of Cologne (established in 1388) and Düsseldorf (established in 1965) will still accept students in the year 2500.

b.) Gott's approach was criticized by Carlton Caves[2], who showed that a proper Bayesian analysis of the problem leads to the relation (1) only if the past duration is *unknown* and the only available information consists of the fact that the phenomenon is currently in progress. Therefore (1) cannot, strictly speaking, be used to predict t_f from t_p.

To derive Caves' result, we start by writing down an expression for the joint probability density $f(t_p, t_f)$. Obviously this should be proportional to $f_\Delta(t_p + t_f)$, where $f_\Delta(t)$ denotes the (unknown) probability density of lifetimes. Find the correct normalization factor, and carry out the appropriate integration to see that (1) results, independent of the choice of f_Δ.

c.) In the lectures it was shown that the probability of future duration t_f, *conditioned* on the past duration t_p, is generally given by

$$\mathrm{Prob}[t_f > t | t_p] = \frac{1 - F_\Delta(t + t_p)}{1 - F_\Delta(t_p)}, \tag{2}$$

where $F_\Delta(t)$ is the distribution function corresponding to the density $f_\Delta(t)$. How would f_Δ have to be chosen for (2) to reduce to (1)?

Problem 13: Index of dispersion of a renewal process

It was shown in the lectures that the index of dispersion of a renewal process (defined as the ratio of the variance of the number of events to the mean) is equal to the square of the

[1] J.R. Gott III, *Implications of the Copernican principle for our future prospects*, Nature **363**, 315 (1993).

[2] C.M. Caves, *Predicting future duration from present age: a critical assessment*, Cont. Phys. **41**, 143 (2000)

coefficient of variation of the underlying waiting time distribution (defined as the ratio of the standard deviation to the mean). Using this relation, compute the index of dispersion I_β for a renewal process defined through a waiting time probability density of the generalized exponential form

$$f_\beta(t) = \mathcal{N}_\beta \exp[-t^\beta], \quad t \geq 0, \tag{3}$$

where $\beta > 0$ and $\mathcal{N}_\beta$ is a normalization factor. Verify that I_β decreases monotonically with increasing β. Since $\beta = 1$ corresponds to the exponential case with $I_1 = 1$, this implies that waiting time distributions with tails that are heavier than exponential ($\beta < 1$) have super-Poissonian fluctuations ($I_\beta > 1$), while distributions with lighter tails ($\beta > 1$) display sub-Poissonian fluctuations ($I_\beta < 1$).

Problem 14: Photon statistics

According to quantum optics, the number N of photons observed at a detector during a time interval T has a Poisson distribution with mean IT, where I is the intensity of the light field. For a coherent (laser) light field the intensity is constant, and the arrival of photons is described by shot noise. For a classical, incoherent light field, however, the intensity itself is a random variable with an exponential probability density,

$$f(I) = \frac{1}{\bar{I}} \exp(-I/\bar{I}) \tag{4}$$

where $\bar{I}$ denotes the mean intensity.

a.) Show that the probability to observe N photons in a time interval T for a classical light field is given by

$$P_N = \frac{\bar{N}^N}{(\bar{N}+1)^{N+1}}, \tag{5}$$

where $\bar{N} = \bar{I}T$. Derive (5) by averaging the Poisson distribution with respect to the intensity distribution (4); alternatively, this result can be obtained from the Bose statistics of photons.

b.) Compute the index of dispersion of the photon number from (5). How does this quantity behave for $t \to \infty$?

c.) Using the results for waiting time distributions of a general point process derived in the lectures, compute the distributions $f_\Delta(t)$ and $f_\theta(t)$ for classical light. Which of the quantities θ and Δ has a finite expectation value?