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Probability theory and stochastic processes for physicists

Problem 15: 1/f-noise from power law waiting times

It was shown in Problem 3 that an exponential distribution of activation energies with mean E_0 leads to a waiting time distribution

$$\psi(\tau) = \alpha(\tau_0/\tau)^{\alpha+1} \tag{1}$$

for a diffusing particle, where $\tau \geq \tau_0$ and $\alpha = k_B T/E_0$. Following the calculation presented in the lectures for a uniform distribution of activation energies, compute the averaged power spectrum $\bar{S}(\omega)$ for a system of independent telegraph processes whose characteristic time τ_c is distributed according to (1). Show that

a.) $\bar{S}(\omega) \sim \omega^{-2}$ for $\omega \gg 1/\tau_0$.

b.) $\bar{S}(\omega) \to$ const. for $\omega \ll 1/\tau_0$, when $\alpha > 1$.

c.) $\bar{S}(\omega) \sim \omega^{-(1-\alpha)}$ for $\omega \ll 1/\tau_0$, when $\alpha < 1$.

Problem 16: Poisson process and Wiener process

a.) Consider shot noise with intensity $\rho = 1$, and let $N_{[0,t]}$ denote the number of events in the time interval $[0, t]$. The Poisson process is defined by $Y(t) = N_{[0,t]}$. Write down the transition probability $P_{1|1}(N_2, t_2|N_1, t_1)$ for this process (as also given in the lectures) and verify by explicit computation that it satisfies the Chapman-Kolmogorov equation.

b.) Next show that the autocorrelation function of the Poisson process is given by the expression

$$\kappa(t_1, t_2) = \min[t_1, t_2]. \tag{2}$$

Hint: Write down the variance of $Y(t_2) - Y(t_1)$, using the fact that this quantity is Poisson-distributed.

c.) The Wiener process is defined by the transition probability

$$P_{1|1}(y_2, t_2|y_1, t_1) = \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left[-\frac{(y_2 - y_1)^2}{2(t_2 - t_1)}\right]$$

with $t_2 > t_1 > 0$ and the initial condition $P_1(y, 0) = \delta(y)$. Show that the autocorrelation function for the Wiener process is identical to the expression (2) for the Poisson process.

Problem 17: The telegraph process

The *telegraph process* is defined by

$$Y(t) = (-1)^{N_{[0,t]}}$$

where $N_{[0,t]}$ was introduced in Problem 16. We want to prove the relations

$$\langle Y(t) \rangle = e^{-2t} \tag{3}$$

$$\langle Y(t_1)Y(t_2) \rangle = e^{-2|t_1 - t_2|}, \tag{4}$$

and show that the telegraph process is Markovian.

a.) To prove (3) show first that the probability $P_g(t)$ to have an even number of events[1] in the interval $[0, t]$ is given by

$$P_g(t) = e^{-t} \cosh(t)$$

and express the distribution function $P_1(y, t)$ through P_g. Then we obviously have that $\langle Y(t) \rangle = P_1(1, t) - P_1(-1, t)$.

b.) To derive (4) express the conditional probability $P_{1|1}(y_2, t_2|y_1, t_1)$ by P_g and determine the two-point distribution function $P_2(y_1, t_1; y_2, t_2)$. This yields $\langle Y(t_1)Y(t_2) \rangle$ by an argument similar to part a.).

c.) From part b.) we have obtained the expression

$$P_{1|1}(y_2, t_2|y_1, t_1) = \frac{1}{2}[1 + e^{-2|t_2-t_1|}] \, \delta_{y_1,y_2} + \frac{1}{2}[1 - e^{-2|t_2-t_1|}] \, \delta_{y_1,-y_2} \tag{5}$$

for the transition probability of the telegraph process. Verify that (5) satisfies the Chapman-Kolmogorov equation.

Aufgabe 18: The Ehrenfest urn model

Consider N identical balls which are distributed among two urns. In one time step one of the N balls is selected at random and transferred from the urn in which it is found to the other urn. The state of the system is described by the number $n = 0, 1, 2, ..., N$ of balls in urn 1 (the number of balls in urn 2 is then $N - n$).

a.) Show that this process defines a finite Markov chain with transition matrix

$$T_1(n|m) = \frac{m}{N} \, \delta_{n+1,m} + \frac{N-m}{N} \, \delta_{n-1,m}.$$

b.) Show that the binomial distribution

$$P_s(n) = 2^{-N} \binom{N}{n}$$

is a stationary distribution of the model.

c.) A Markov chain is called *reversible*, if the matrix $T_1(n|m)P_s(m)$ is symmetric. Show that the urn model is reversible.

[1] Zero is even.