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Probability theory and stochastic processes for physicists
Solutions to selected problems

Solution 4. Stable laws and renormalization

Since the generating function of the transformed density is

$$\int e^{iky} b f_\Sigma(by) dy = \int dx f(x) e^{i(k/b)x} \int dy b e^{i(k/b)(by-x)} f(by-x) = G_X(k/b)^2, \tag{1}$$

the stability condition becomes

$$G_X(k) = G_X(k/b)^2. \tag{2}$$

Since the generating functions of the Gaussian and the Cauchy distributions are

$$G_G(k) = e^{-k^2/2}, \quad G_C(k) = e^{-\pi|k|}, \tag{3}$$

respectively, the Gauss (Cauchy) distribution is shown to be stable if $b = \sqrt{2}$ (2) is used. In general, the Lévy distribution with the generating function $e^{-C|k|^\mu}$ is stable, if $b = 2^{1/\mu}$ is used.

Solution 5. Zipf's law for random texts [1]

The frequency of a given word with size ℓ is

$$x = q_s q^\ell \to \ell = \frac{\ln x - \ln q_s}{\ln q}, \tag{4}$$

and there are $N(\ell) = m^\ell$ such words. To obtain the number of words $N(x)$ with frequency x we need to perform a variable transformation from the probability density of ℓ [which is proportional to $N(\ell)$] to that of x. By the usual rule for transformation of random variables, we have

$$N(x) \sim |\frac{d\ell}{dx}| m^\ell = \frac{1}{x|\ln q|} \exp\left(\frac{\ln m}{\ln q} \ln\left(\frac{x}{q_s}\right)\right) \sim \frac{1}{x}\left(\frac{x}{q_s}\right)^{\ln m/\ln q} \sim x^{-\alpha}, \tag{5}$$

with $\alpha = 1 - \ln m/\ln q$. When m is very large and $q_s \ll 1$ (recall that $q = (1-q_s)/m$),

$$\alpha = 1 + \frac{\ln m}{\ln m - \ln(1-q_s)} \approx 2 - \frac{q_s}{\ln m} \approx 2. \tag{6}$$

[1] This derivation follows M.E.J. Newman, Cont. Phys. **46**, 323 (2005). In fact the validity of Zipf's law for random texts is a (rather controversial) topic of current research, see e.g. S. Bernhardsson et al., J. Stat. Mech. P07013 (2011). Both papers are available on the course webpage.