Weakly interacting Bose gas in disordered environment

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Outline

Introduction

Interacting Bose gas in weak random potential

Strong disorder

Bosons in lower dimensions

Bosons in traps
**Introduction**

**BEC:** finite part of atoms in the state with minimal energy.

**Examples:** Superfluid $^4$He, laser cooled atoms in a trap*

**Disorder:** Superfluid He in porous media (J.D. Reppy et al '92)
Cold atoms in speckle potential (R.G. Hulet et al. '08)

**Breakdown of superfluidity at strong disorder**

*Other examples: excitons in semiconductors, BEC of spin waves*
Phase Coherence and Superfluid-Insulator Transition in a Disordered Bose-Einstein Condensate

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Hulet et al

\( t_{\text{TOF}} = 0 \text{ ms} \)

(a)

(c)

(e)

(g)

\( t_{\text{TOF}} = 8 \text{ ms} \)

(b)

(d)

(f)

(h)
Bosons in disordered environment:

Review: Weichman, Mod. Phys. Lett. '08

\[ T_c(n) \]

\[ n_c \]

\[ \text{normal fluid} \]

\[ \text{superfluid} \]

\[ \text{Bose glass} \]

Weichman+Fisher'86

Fisher, Weichman, Grinstein+Fisher '89 → \[ z=d \]

Halperin, Lee+ Ma'86

Giamarchi+Schulz '87 (d=1)

Shklovskii+Müller '08

Shklovskii '08

Babichenko² '08

\[ \rho_s \approx \rho_0[1 - c_1 \sqrt{n_c/n}] \]

Bogoliubov theory

Huang+Meng '92

Cross-Pitaevskii equation

\[ \rho_0 \]

\[ c_1 \]
Fermions

- Insulator
- Metal
- Anderson localization
- Disorder

\[ n \approx L_c^{-3} \]

Bosons

- Bose glass
- Superfluid
- Disorder
- Collapsed, non-ergodic

Interaction
No Disorder

- Disorder
- Superfluid
- Bose glass

Diagram with axes:
- Disorder
- Interaction

Region labeled "Bose glass" in the diagram.
(a) BEC: Ideal bosons $T=0$

$$\mathcal{H} = -\frac{\hbar^2}{2m} \int d^d x \hat{\Psi}^\dagger(x) \nabla^2 \hat{\Psi}(x), \quad \hat{\Psi} = \sum_q e^{i q x} \hat{a}_q$$

$$N = nV = \int d^d x \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \rangle$$

$$\epsilon_q = \frac{\hbar^2}{2m} q^2$$

$\mu=0$

all particles in ground state
**BEC : Interacting bosons**

\[ \mathcal{H} = \frac{\hbar^2}{2m} \int d^3x \hat{\Psi}^\dagger \left[ -\nabla^2 + 4\pi a \hat{\Psi}^\dagger \hat{\Psi} \right] \hat{\Psi} \]

\[ \hat{\Psi}^\dagger \hat{\Psi} = \hat{n} \]

Scattering length

Bogoliubov transform \( \Rightarrow \)

\[ \epsilon_q^2 = \frac{\hbar^2 q^2}{m} \left( \frac{\hbar^2}{m\xi^2} + \frac{\hbar^2 q^2}{4m} \right) \]

\[ \mu = \frac{\hbar^2}{m\xi^2} \left( 1 + \frac{c}{n^{1/3}\xi} \right) \]

Lee & Yang '57

healing length \( \xi = \frac{1}{\sqrt{4\pi an}} \)
Weak Disorder

- Bose glass
- superfluid
- collapsed, non-ergodic

disorder

interaction
(d) Weakly repulsive bosons in a weak random potential

\[ \mathcal{H} = \int d^3 x \Psi^\dagger \left( -\frac{\hbar^2}{2m} \nabla^2 + U(x) + \frac{2\pi \hbar^2 a}{m} \Psi^\dagger \Psi \right) \Psi, \]

\[ \langle U(x)U(x') \rangle = \kappa^2 \delta(x - x') \]

Huang & Meng

\[ L_c = \frac{\hbar^4}{m^2 \kappa^2} \]

mean free path

\[ \Delta \mu = \Delta \epsilon \approx \int_0^{\xi^{-1}} \frac{d^d q}{\hbar^3} \frac{\kappa^2}{c_s q} \sim \frac{\hbar^2}{m \xi^2} \left( \frac{\xi}{L_c} \right)^{4-d}, \quad d > 1 \]
Strong Disorder, no Interaction

Bose glass

collapsed, non-ergodic

superfluid

disorder

interaction
Ideal 3d Bose gas in random potential

\[ \frac{\hbar^2}{2m} \nabla^2 \psi + (E - U(x))\psi = 0 \quad \langle U(x)U(x') \rangle = \kappa^2 \delta(x-x') \]

\[ L_c = \frac{\hbar^4}{m^2 \kappa^2} \]

\[ E_c = \frac{\hbar^2}{2mL_c^2} \] energy scale of localized states

T=0: All particles in ground state \( E_0 \approx -E_c \ln^2(L_0/L_c) \)
Single particle density of states DOS $E \to -\infty$

$$\nu(E, V) = \frac{1}{V} \int \delta(E - E[U(x)]) dW[U(x), V]$$

Consider potential fluctuation of depth $U$ and width $R$

probability $W[U] \sim \exp\left[\frac{-U^2 R^d}{2 \kappa^2}\right]$

$\to$ localized state of energy $E \sim \frac{\hbar^2}{2mR^2} + U$

Contribution of DOS at energy $E \sim \max W[E - \frac{\hbar^2}{2mR^2}]$

Maximize $W$ with respect to $R$ $\Rightarrow$ $R = L_c(E_c/|E|)^{1/2}$

$\to$ $\nu \sim \exp\left\{\frac{-|E|}{E_c(4-d)/2}\right\}$
Ideal Bose gas in random potential

DOS for $E \ll -E_c$ dominated by wells of width $R \sim \hbar/\sqrt{m|E|} \ll L_c$

$$\nu(E) = \frac{1}{V} \langle \delta(E - E[U(x)]) \rangle \sim |E|^{3/2} e^{-\sqrt{|E|/E_c}}$$

I.M. Lifshitz '66, Zittartz and Langer '66, Halperin and Lax,'66 Cardy '78
Ideal Bose gas in random potential

Spatial density $n_w(R)$ of wells with radius $R \ll L_c$ ($E \ll \hbar^2/(2mR^2) \ll E_c$):

$$n_w(R) = \int_{-\infty}^{-\frac{\hbar^2}{2mR^2}} dE \, \nu(E) \sim \frac{L_c}{R^4} e^{-L_c/R}$$

Tunneling amplitude $t(R)$ between wells with radius $R$:

$$t(R) = \exp \left( -\frac{1}{\hbar} \int |p| dl \right)$$

$$\frac{1}{\hbar} \int |p| dl \approx n_w^{-1/3}/R \sim e^{L_c/3R}$$

$$t(R) \sim e^{-\left(\frac{R}{L_c} e^{L_c/R}\right)^{1/3}}$$
Strong Disorder + Interaction

Bose glass

superfluid

collapsed, non-ergodic

disorder

interaction
Weakly repulsive bosons in a random potential

\[ \mathcal{H} = \int d^3x \Psi^\dagger \left( -\frac{\hbar^2}{2m} \nabla^2 + U(x) + \frac{2\pi \hbar^2 a}{m} \Psi^\dagger \Psi \right) \Psi \]

Assume that all potential wells with radii up to \( R \) are filled:

\( \Rightarrow \) number of particles per well of size \( R \) : \( N_w(R) = n/n_w(R) \gg 1 \)

\( \Rightarrow \) repulsion energy per particle: \( E_g(R) \approx g N_w/R^3 \sim g n e^{L_c/R} \)

\( \Rightarrow \) total energy per particle: \( \mu(R) = -\hbar^2/(2mR^2) + E_g(R) \)
Weakly repulsive bosons in a random potential

⇒ number of particles per well of size \( R \): 
\[ N_w(R) = n/n_w(R) \gg 1 \]

⇒ repulsion energy per particle: 
\[ E_g(R) \approx g N_w/R^3 \sim g n e^{L_c/R} \]

⇒ total energy per particle: 
\[ \mu(R) = -\hbar^2/(2mR^2) + E_g(R) \]

Minimization over \( R \): 
\[ R(n)=L_c/\ln(n_c/n), \]

\[ n \ll n_c \approx 1/(3L_c^2a) \]

(non-interacting Fermions: Ioffe-Regel \( a \rightarrow L_c \))

\[ \mu(n) = -\frac{\hbar^2}{2mR^2(n)} = -\frac{1}{2} E_c(\ln \frac{n_c}{n})^2 \]

\[ \frac{n_c}{n} = \frac{\xi^2}{L_c^2} \]
Variable hopping conductivity:

Absence of interaction: probability that two localized states have the same energy is zero.

Switch on interaction: energy levels split by amount $g n_p$. If $n \ll n_c$ wave function is still localized.

$\Rightarrow T=0$ conductivity (response to external force) in Bose-glass is still zero.

Tunneling probability between wells of distance $L$ is $\sim \exp\{-2L/R\}$

$\Rightarrow$ hopping probability $P(T) \sim \exp\{-2L/R-\Delta E/T\}$

$\Delta E \propto (E) L^3 \approx 1$, use relation $R(n)$ and maximize $P(T)$ with respect to hopping distance $L$$\Rightarrow$

$$\sigma(T) \sim e^{-C[E_c n_c/(T n)]^{1/4}}$$
Preliminary conclusions

⇒ At $n \ll n_c$ Bose gas decays into fragments, particle density in fragments each of density $n_c \sim 1/(aL_c^2)$

⇒ tunneling exponentially suppressed: $t(n) \sim e^{-c(n_c/n)^{1/3}}$

⇒ particle number in fragments $N_w = L_c / \left[3a(\ln \frac{n_c}{n})^3 \right]$ well defined

⇒ phase uncertain, no phase coherence ⇒ no superfluidity

⇒ finite compressibility $\frac{n}{E_c} \ln \left( \frac{n_c}{n} \right)$ "Bose glass"

⇒ $\hat{H}_{\text{eff}} = \sum_j C_j (\hat{N}_j - \langle N_j \rangle)^2 - \sum_{i,j} t_{ij} \cos(\hat{\phi}_i - \hat{\phi}_j)$

⇒ charged bosons VRH $\sigma(T) \sim e^{-C[E_c n_c / (Tn)]^{1/4}}$

For $n \approx n_c$ i.e. fragments merge → transition to superfluid
Correlated disorder

\[ \langle U(x)U(x') \rangle = \frac{U_0^2}{b^3} e^{-|x-x'|/b} \]

⇒ 2 length scales \( b, B = (\hbar^2/(mU_0))^{1/2} \)

\( b \ll B \Rightarrow \) uncorrelated disorder

\[ \nu(E) \sim |E|^3 \exp\left(-E^2/2U_0^2\right) \]

Keldysh & Proshko '63
Kane '63
Shklovskii and Efros '70
John & Stephen '84

\[ \mu(b, n) \approx -U_0 \sqrt{2 \ln\left(\frac{n_c}{n}\right)} \]

\[ n \ll n_c \sim 1/(B^2a) \]

\[ n_w(E) = b^{-3} \exp\{E^2/2U_0^2\} \]
Generalization to $d<3$ dimensions

What is different?

DOS,

$a \rightarrow \alpha_d d^{-2} = a \cdot r^{-d-3}$

$\xi, L_c, E_c$

$n/n_c \sim n/L_c^2 \alpha_d d^{-2}$
Bose gas in one dimensions

Superfluid

Bose glass

\( K \sim 1/(n a_1) \)

\( K \sim 1-8n a_1 \)

\( K \sim 1/(n a_1) \)

\( K \sim 1/(n a_1) \)

\( k_{\text{LL parameter}} \)

\( U \)

\( \Delta \)

\( \Delta_{\text{max}} \)

Bose glass

Bose glass

superfluid

Mott insulator

Giamarchi & Schulz '87

this work

(Disorder)

free fermions

Lieb '65

(Disorder)
Bosons in traps
Ideal quantum gas in a harmonic trap

- **Oscillator length**: $\ell = (\hbar/m\omega)^{1/2}$ ($\approx 1000\text{nm}$), $\hbar\omega \approx nK$

- **Bosons**: $T=0$: all particles in ground state of size $\ell$

- $T_c$: $\lambda_T^3 n \sim \lambda_T^3 N/R^3 \approx 1$

- $T_{c0} \sim \hbar\omega N^{1/3}$

- $\lambda_T = (\hbar^2 / Tm)^{1/2}$

- $m\omega^2 R^2 \approx T$  
  $N \approx 10^3 \ldots 10^8$

- **Fermions**: $\varepsilon_F \sim T_{c0}$

- $R_F \approx r_\omega N^{1/6}$

**Diagram Overview**

- The diagram illustrates the potential $U(x)$ and the oscillation length $\ell$.

- The graph shows the typical behavior of bosons and fermions in a harmonic trap, highlighting the relationship between the oscillation length and the thermal properties of the system.
Bosons in traps (uncorrelated disorder)

oscillator length $\ell = (\hbar/m\omega)^{1/2}$ ($\approx 1000\text{nm}$), $\hbar\omega \approx nK$

$$\mu(R) = -\frac{\hbar^2}{2mR^2} + E_{\text{int}}(R) + \frac{\hbar^2}{2m} \frac{R^2}{\ell^4}$$

$$\Gamma = \frac{\ell^6}{3NaL_c}$$

$$n(r) = n_c \left( \frac{n}{n_c} \right)^{\sqrt{1 + r^2/r_F^2}}$$

interaction

$\ln(Na/\ell)$ $\Gamma = 1$

Thomas-Fermi $R \sim (Na\ell^4)^{1/5}$

fragmented

harmonic $R \approx \ell$

non-ergodic

disorder

$\ln(\ell/L_c)$ $\ln(\ell/L_c)$

$\frac{\ell^2}{L_c} \ln \left( \frac{\ell^6}{NaL_c^5} \right)$

fragmented state
Bosons in traps (correlated disorder, $d=3$)

\[ \Gamma = \frac{\ell^6}{N\alpha B^5} \]

\[ L \sim \frac{\ell^2}{B} \left( \ln \frac{\ell^2}{\alpha B} \right)^{1/4} \]

\[ \ln(N\alpha/\ell) \]

Thomas-Fermi \hspace{1cm} \text{harmonic}

\[ R \sim (N\ell^4)^{1/5} \hspace{1cm} R \approx \ell \]

fragmented \hspace{1cm} \text{non-ergodic}
**Bose gas in 1 dimensions: parabolic trap**

\[ \ln(N\ell/a_1) \quad L \sim \frac{\ell^2}{L_1} \left( \ln(\frac{\ell^2 a_1}{NL_1^3}) \right)^{1/3} \]

**Uncorrelated disorder**
- **Thomas-Fermi**
  \[ R \sim (N\ell^2/a_1)^{1/3} \]
- **harmonic**
  \[ R \sim \ell \]

\[ L \sim \frac{\ell^2}{L_1} \left( \ln(\frac{\ell^2 a_1}{NL_1^3}) \right)^{1/3} \]

**Correlated disorder**
- **Thomas-Fermi**
  \[ R \sim \left( \frac{N\ell^4}{a_1} \right)^{1/3} \]
- **harmonic**
  \[ R \approx \ell \]

\[ L \sim \frac{\ell^2}{B} \left( \ln\left( \frac{\ell^2 a_1}{NB^3} \right) \right)^{1/4} \]
Prediction which could be tested

1. Cloud size as function of these parameters in fragmented state?

2. Cross-over from non-ergodic to ergodic state at critical $N_c = L_c/3a$, $N_c = b^3/(3aB^2)$, number of particles in fragments?

3. Time of flight spectroscopy, $\Delta p \sim \hbar/R$, fragmented state should be visible.

4. Ground state reachable? According to our estimates ($L_c \approx 1 \mu m$) relaxation time $\approx 0.06s$. Easier in lower dimension.

Changeable parameter: $N$, $\omega$, $a$, $U_0$, $b$
Conclusions

- Semi-quantitative analysis of the phase states of a weakly interacting strongly diluted Bose gas in a random Gaussian potential.

- The system is characterized by the mean free path $L_c$ and the scattering length $a$ (or $a$, $U_0$ and $B$ for correlated disorder).

- At particle density $n \ll n_c \approx 1/(aL_c^2)$ the Bose particles occupy deep potential wells and exponentially weakly tunnel to other wells. The number of particles in each well is defined, but phases are uncertain.

- At average particle density $n \approx n_c$ the transition to the superfluid proceeds.

- In a trap the oscillator length $l$ appears as a new length scale. Four different regimes are found, depending on the mutual strength of $L_c$, $aN$ and $l$, respectively.

- All results can be extended to lower dimensions and to correlated disorder.