A Scaling Approach to Quantum Gases*

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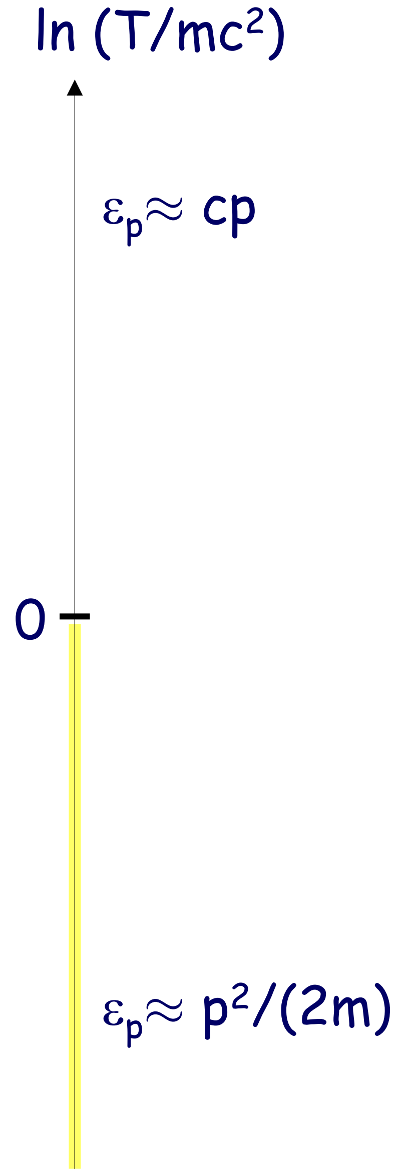
- Ideal classical gas
- Ideal quantum gas at low T : $\hbar \neq 0$
- Fermions and Bosons
- Ideal quantum gas at high T
- Conclusions

Ideal classical gas

- non-interacting particles of rest mass m
- $\varepsilon_p = (m^2c^4+c^2p^2)^{1/2} - mc^2$
- fixed particle number $N=V/v$, $v=a^3$
- temperature T



pressure $p \sim T^\alpha m^\beta c^\gamma a^\delta \rightarrow$

dimensional analysis:

$$M L^{-1} t^{-2} \sim (M L^2 t^{-2})^\alpha M^\beta (L/t)^\gamma L^\delta$$

$$1 = \alpha+\beta, \quad -1 = 2\alpha+\gamma+\delta, \quad -2 = -2\alpha-\gamma$$

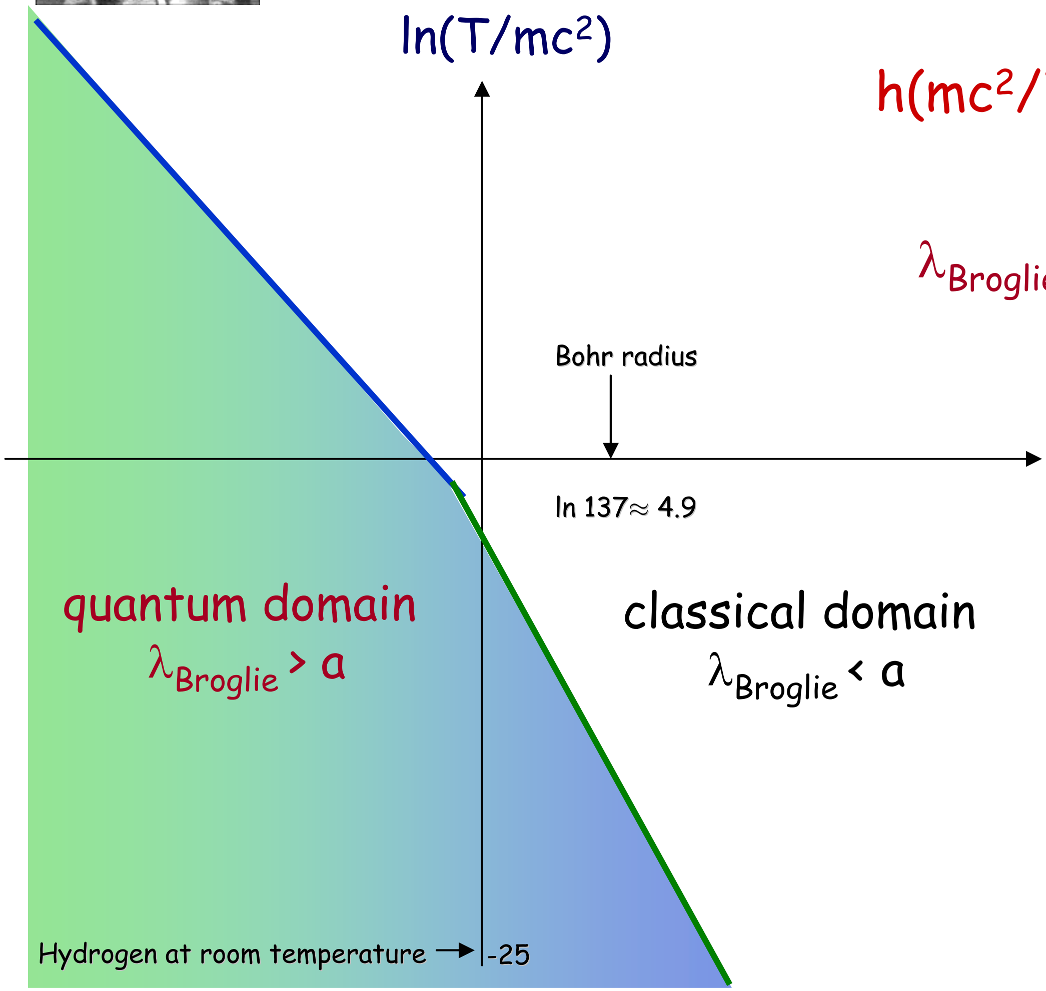
$$\boxed{\Rightarrow p \sim T^\alpha (mc^2)^{1-\alpha}/a^3 \Rightarrow \quad p=NT/V}$$

classical theory incomplete: $F(T,V,N)/N=f(T,v)\sim T \ln (v/v_0)$

Ideal quantum gases

Max Planck 1900: fundamental quantum of action h

⇒ new dimensionless parameter



$$h(mc^2/T)^{\alpha}/(mca) \equiv \lambda_{Broglie}/a$$

$$\lambda_{Broglie} = h/p\,\big|_{\varepsilon_p = T} \approx \begin{cases} hc/T & (\alpha=1) \\ h/(mT)^{1/2} & (\alpha=1/2) \end{cases}$$

$$\lambda_{Compton} = h/mc$$

$$p = TN/V \cdot \Phi(\lambda_{Broglie}/a)$$

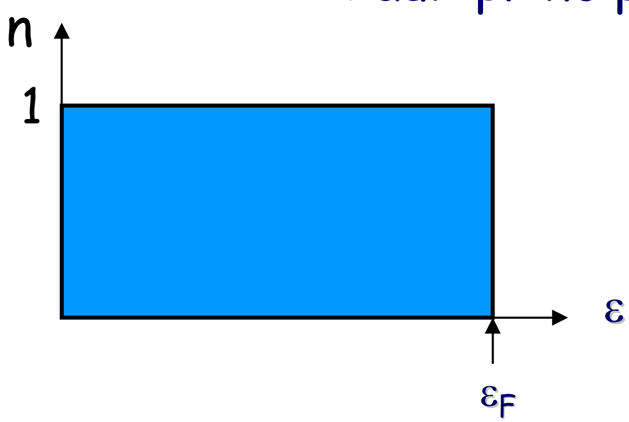
$$\Phi(x \to 0) = 1$$

$$\Phi(x \to \infty) = ?$$

Ideal quantum gases - Fermions

Pauli 1925...: particles have spin $\begin{cases} S=(n+1/2)\hbar & \text{Fermions} \\ S=n\hbar & \text{Bosons} \end{cases}$

Fermions: Pauli principle $\rightarrow$ non-zero pressure even at T=0



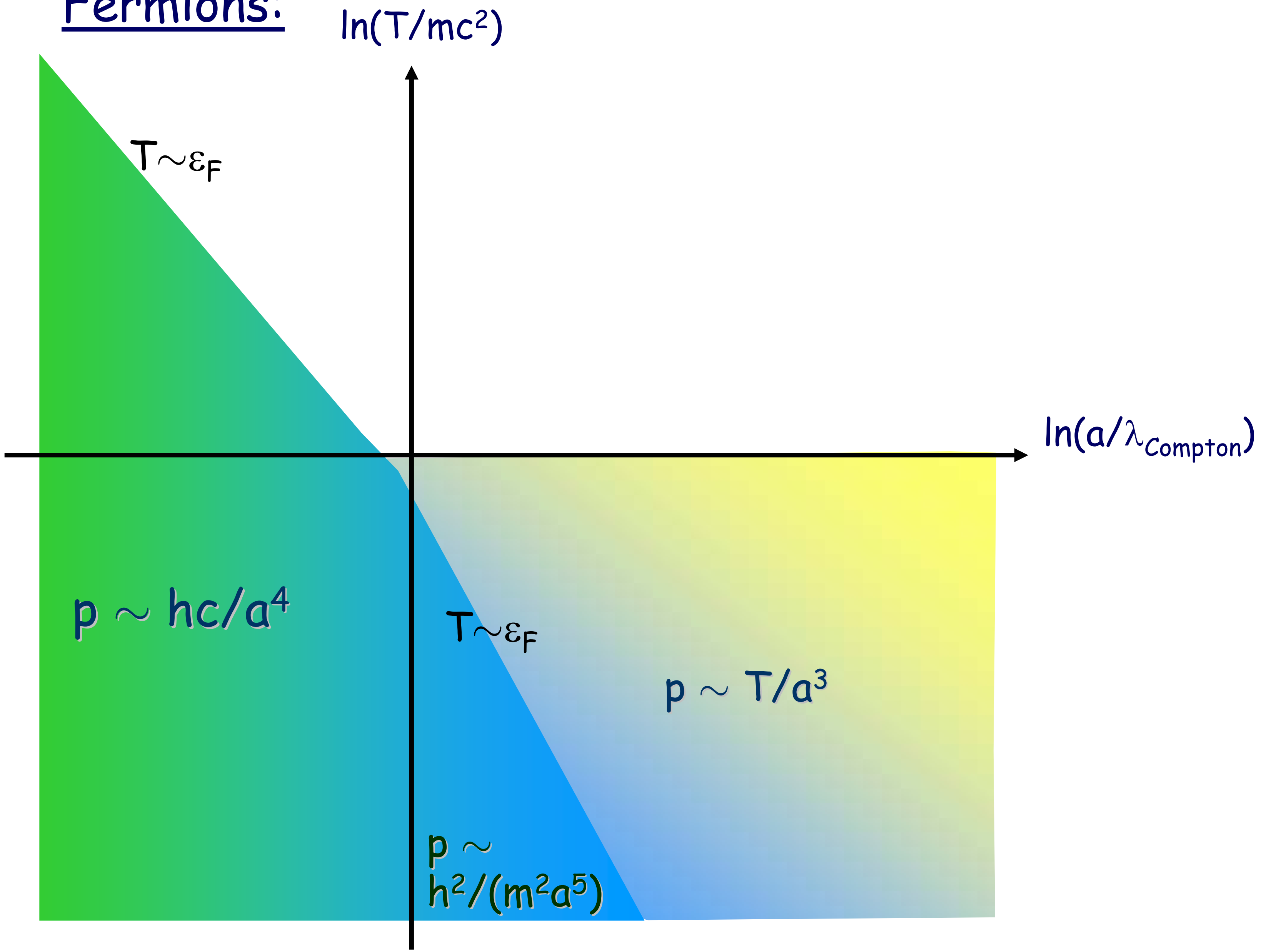
$p=T/a^3\,\Phi\,(\lambda_{Broglie}/a)$

(i) $\varepsilon_F=\hbar^2/ma^2 \ll mc^2 \Rightarrow \lambda_{Broglie} \sim h/(mT)^{1/2}$ $\boxed{\Rightarrow p \sim h^2/(m^2a^5)}$

(ii) $\hbar^2/ma^2 \gg mc^2 \Rightarrow a < \hbar/mc=\lambda_{Compton} \Rightarrow \varepsilon_F=\hbar c/a \gg mc^2$

$\Rightarrow \lambda_{Broglie} \sim ch/T$ $\boxed{\Rightarrow p \sim hc/a^4}$

Fermions:



$\ln(T/mc^2)$
$T \sim \varepsilon_F$
$\ln(a/\lambda_{Compton})$
$p \sim hc/a^4$
$T \sim \varepsilon_F$
$p \sim T/a^3$
$p \sim$
$h^2/(m^2 a^5)$

Ideal quantum gases - Bosons

Pauli 1925...: particles have spin $\begin{cases} S=(n+1/2)\hbar & \text{Fermions} \\ S=n\hbar & \text{Bosons} \end{cases}$

All bosons can occupy ground state

Bosons:

$\rightarrow$ pressure does not depend on volume $V=Na^3$

$p=T/a^3\, f(\lambda_{Broglie}/a) \quad \rightarrow \quad \Phi(x) \sim x^{-3}$

$$\boxed{\Rightarrow \quad p \sim T^{5/2} m^{3/2}/h^3}$$

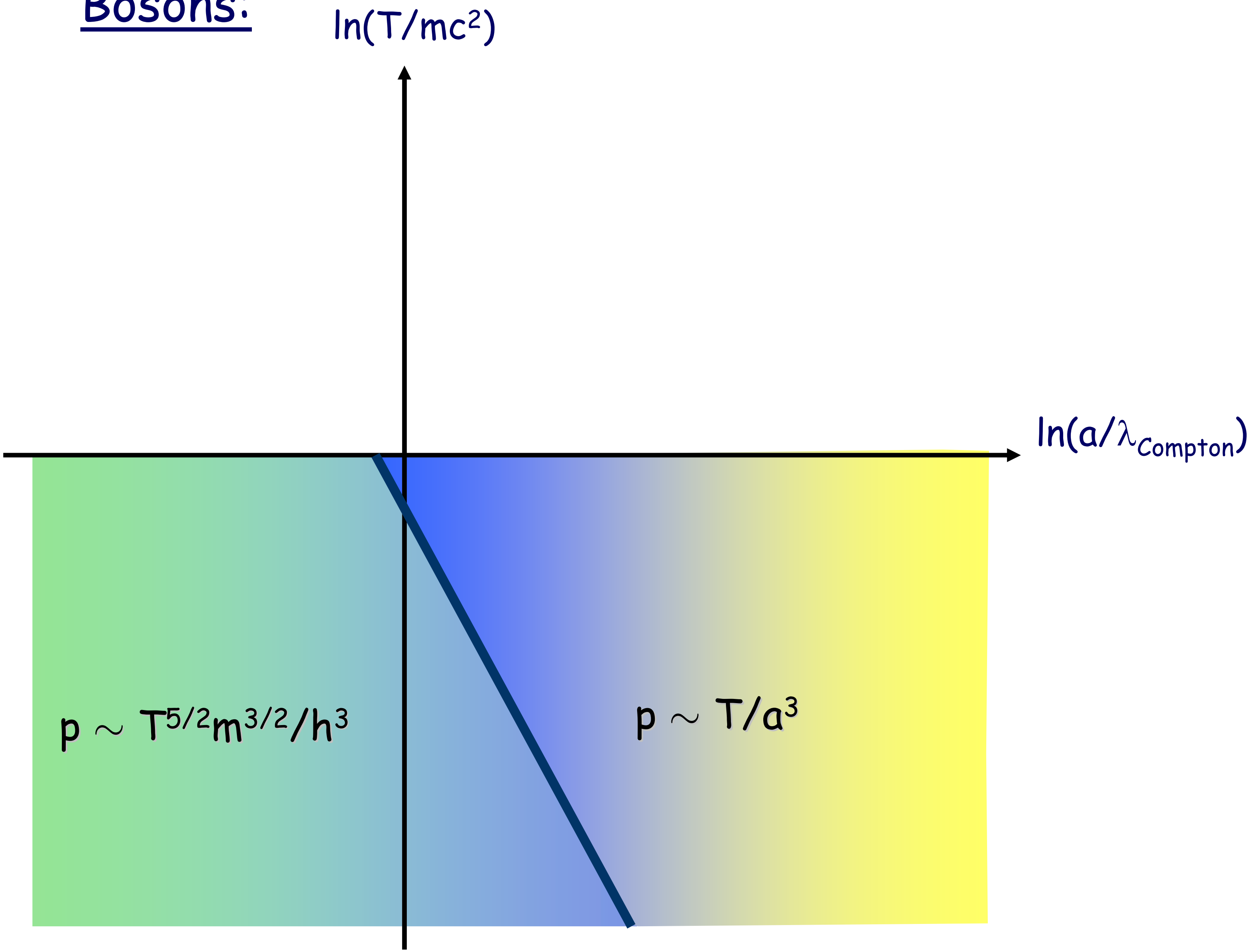
Since $p \sim T/\lambda_{Broglie}{}^3 < p_{klass} = T/a^3$

$\Rightarrow$ a fraction of $1-(a/\lambda_{Broglie})^3 = 1-(T/T_c)^{3/2}$

does not contribute to the pressure $\quad T_c \sim h^2/ma^2$

$\boxed{\Rightarrow \quad N_0=N(1-(T/T_c)^{3/2})}$ particles in the ground state

Bosons:



Ideal quantum gases – high temperatures :

$T \gg mc^2$ (fermions $T > \max(hc/a = \varepsilon_F, mc^2)$)

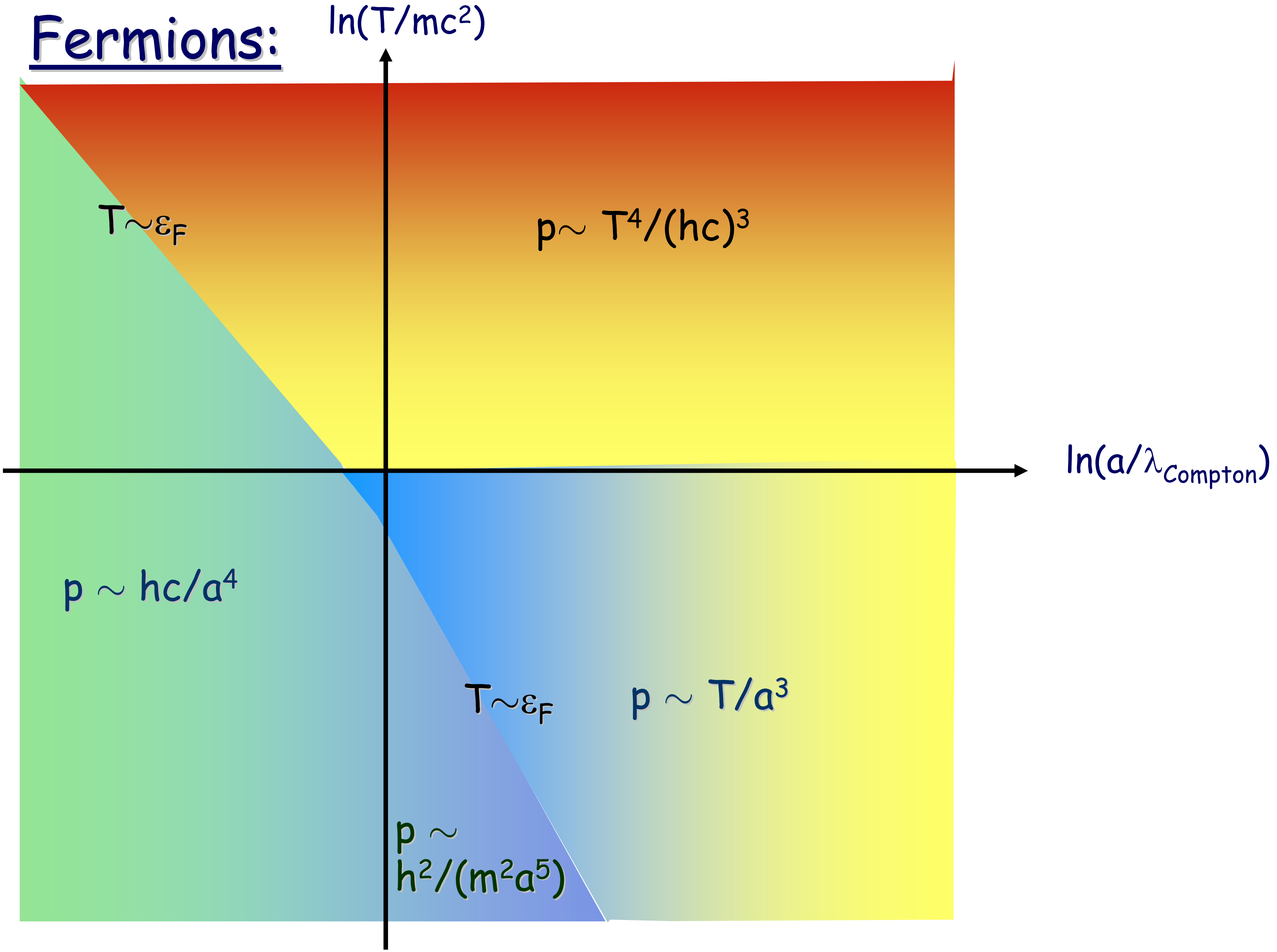
➡ Particle-antiparticle generation

a only good parameter if conservation law exist (charge, baryon number etc.)

➡ Pressure depends only on T, h, c

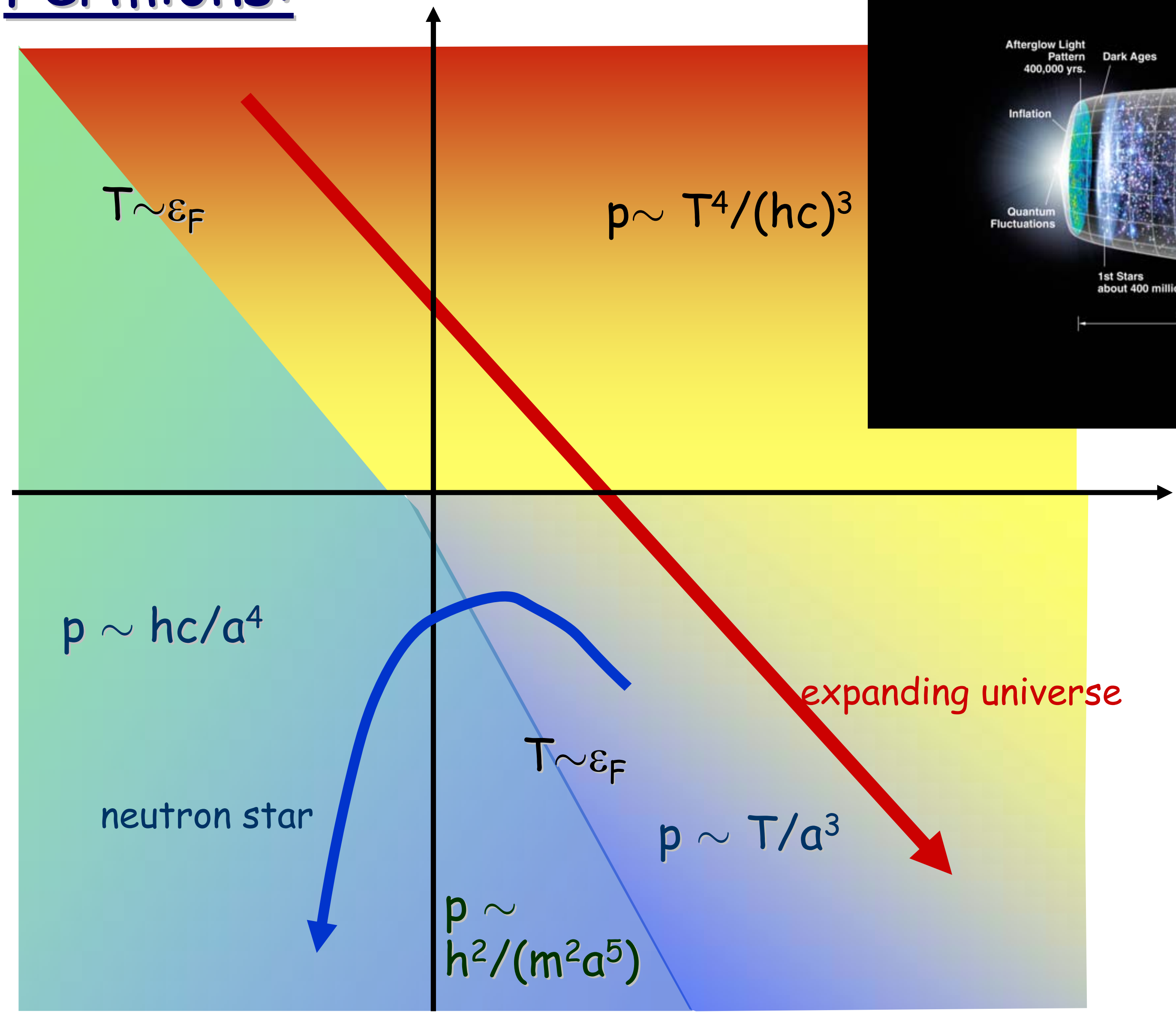
$$\boxed{\Rightarrow\ p \sim T^4/(hc)^3 \sim T/\lambda_{Broglie}{}^3}$$

Fermions:

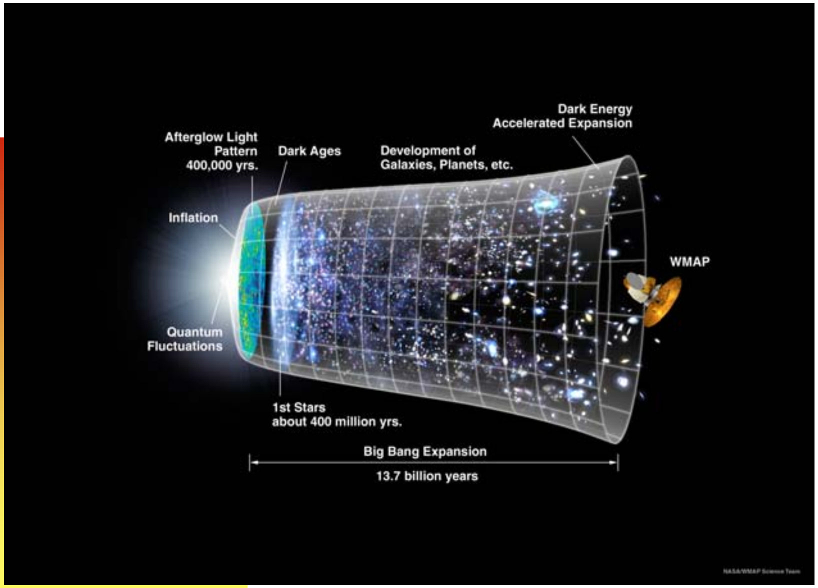


$\ln(T/mc^2)$
$\ln(a/\lambda_{Compton})$
$T \sim \varepsilon_F$
$p \sim T^4/(hc)^3$
$p \sim hc/a^4$
$T \sim \varepsilon_F$
$p \sim T/a^3$
$p \sim$
$h^2/(m^2 a^5)$

Fermions:

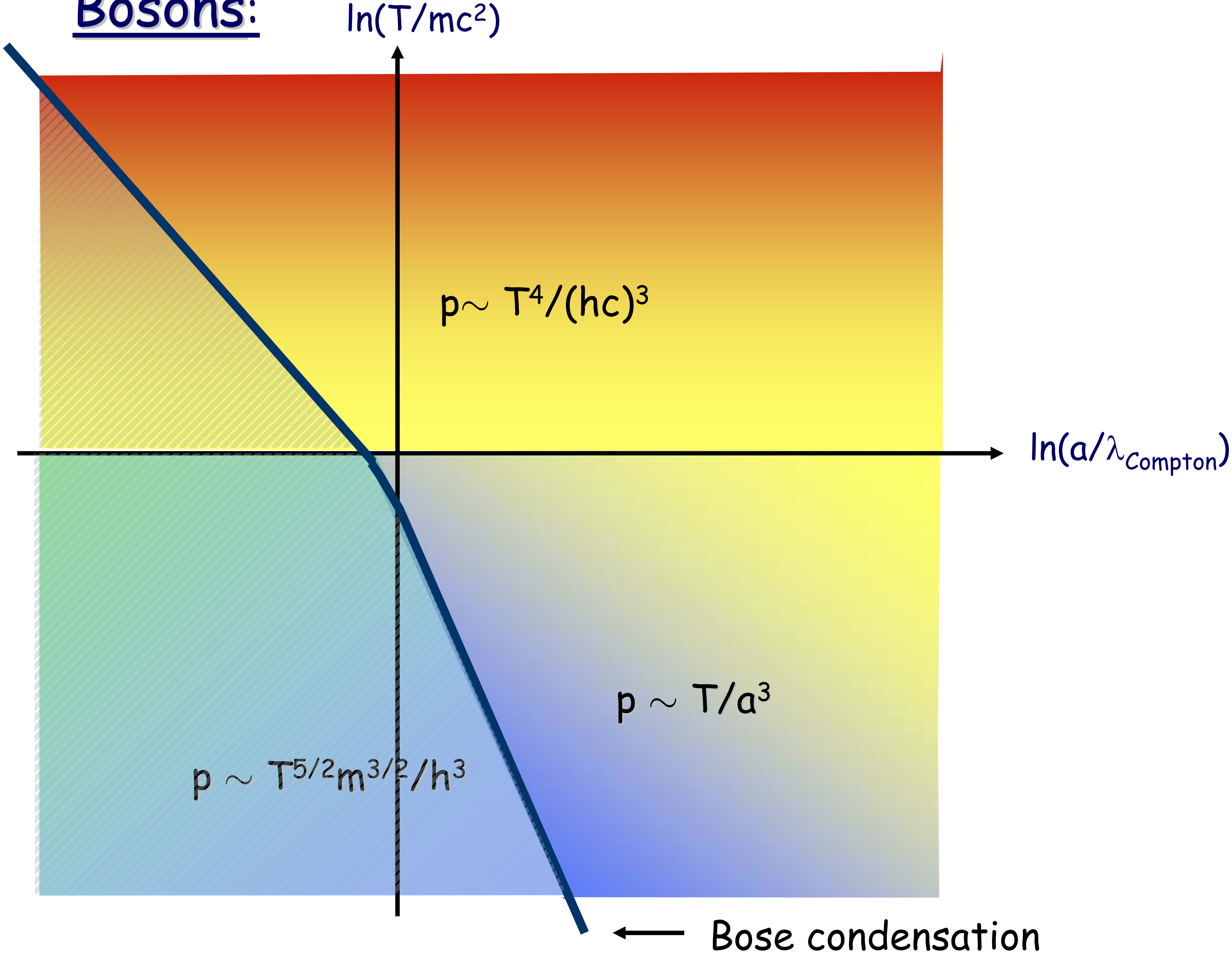


$\ln(T/mc^2)$
$T \sim \varepsilon_F$
$p \sim T^4/(hc)^3$
$\ln(a/\lambda_{Compton})$
$p \sim hc/a^4$
expanding universe
$T \sim \varepsilon_F$
neutron star
$p \sim T/a^3$
$p \sim$
$h^2/(m^2 a^5)$

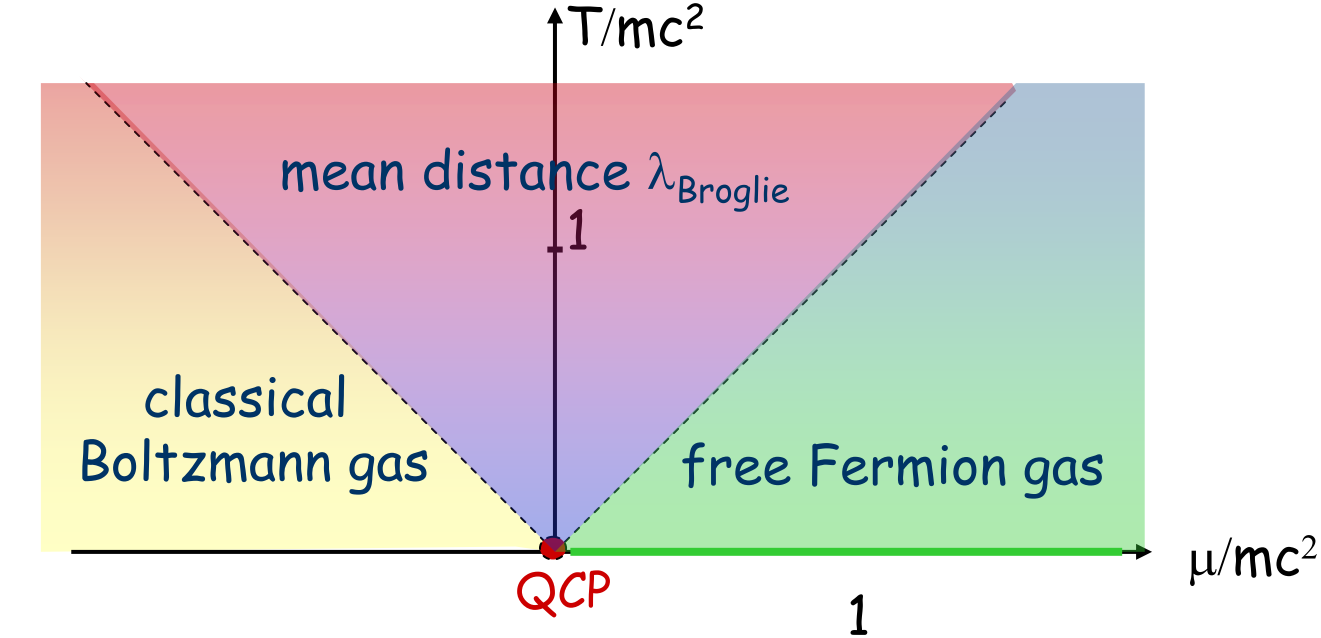


Dark Energy
Accelerated Expansion
Afterglow Light
Pattern
400,000 yrs.
Dark Ages
Development of
Galaxies, Planets, etc.
Inflation
WMAP
Quantum
Fluctuations
1st Stars
about 400 million yrs.
Big Bang Expansion
13.7 billion years

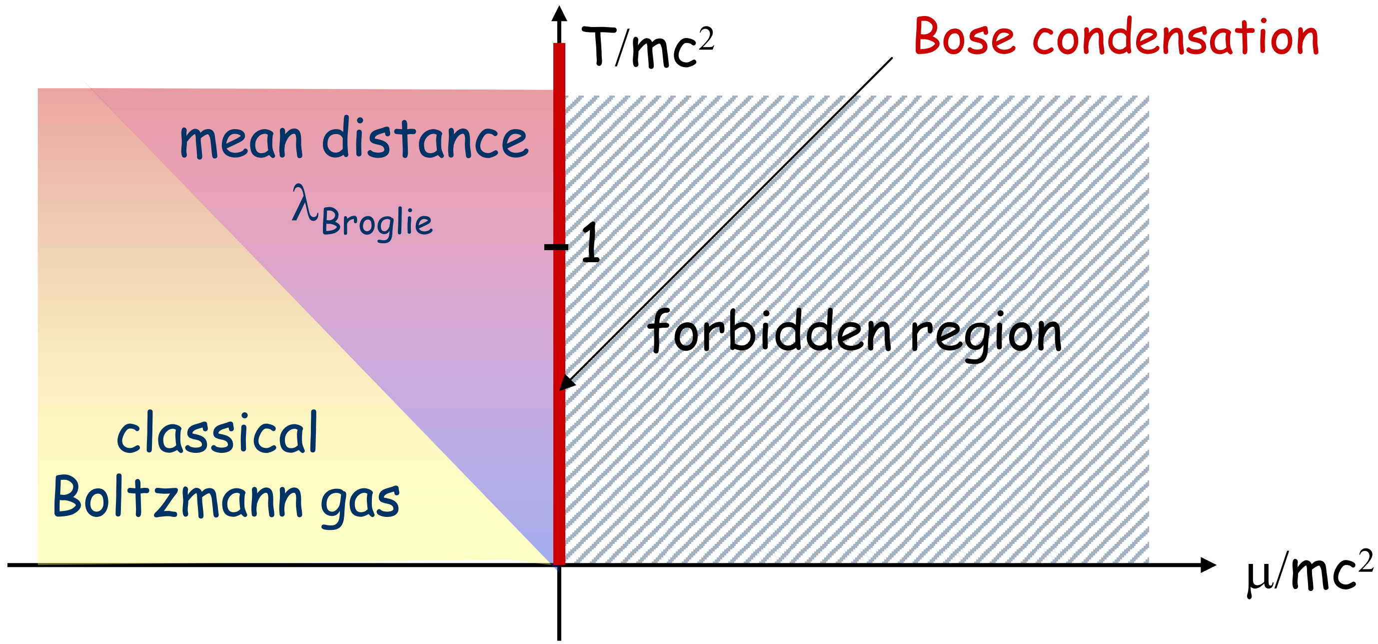
Bosons:



$\ln(T/mc^2)$
$p \sim T^4/(hc)^3$
$\ln(a/\lambda_{Compton})$
$p \sim T/a^3$
$p \sim T^{5/2} m^{3/2}/h^3$
Bose condensation



T/mc²
mean distance λBroglie
1
classical
Boltzmann gas
free Fermion gas
μ/mc²
QCP
1



T/mc²
Bose condensation
mean distance
λBroglie
1
forbidden region
classical
Boltzmann gas
μ/mc²

Conlusions:

Landau & Lifshitz Vol. 5   Statistical Physics

(45.5)
(56.8)
(56.9)
(56.15)
(57.7)
(58.6)
(61.4)
(61.7)
(62.4)
(62.6)
(62.9)
(63.16)
(63.15)
(63.16)
(63.17)
(101.2)
(106.5)
+superfluids