

# A Scaling Approach to Quantum Gases\*

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- Ideal classical gas
- Ideal quantum gas at low  $T$ :  $\hbar \neq 0$
- Fermions and Bosons
- Ideal quantum gas at high  $T$
- Conclusions

# Ideal classical gas

- non-interacting particles of rest mass  $m$
- $\varepsilon_p = (m^2c^4 + c^2p^2)^{1/2} - mc^2$
- fixed particle number  $N=V/v$ ,  $v=a^3$
- temperature  $T$

pressure

$$p \sim T^\alpha m^\beta c^\gamma a^\delta \rightarrow$$

dimensional analysis:

$$M L^{-1} t^{-2} \sim (M L^2 t^{-2})^\alpha M^\beta (L/t)^\gamma L^\delta$$

$$1 = \alpha + \beta, \quad -1 = 2\alpha + \gamma + \delta, \quad -2 = -2\alpha - \gamma$$

$$\Rightarrow p \sim T^\alpha (mc^2)^{1-\alpha} / a^3 \Rightarrow p = NT/V$$

$\ln(T/mc^2)$

$$\varepsilon_p \approx cp$$

0

$$\varepsilon_p \approx p^2/(2m)$$

classical theory incomplete:  $F(T,V,N)/N = f(T,v) \sim T \ln(v/v_0)$

# Ideal quantum gases



Max Planck 1900: fundamental quantum of action  $h$

→ new dimensionless parameter

$$h(mc^2/T)^\alpha / (mca) \equiv \lambda_{\text{Broglie}}/a$$

$$\lambda_{\text{Broglie}} = h/p \quad |_{\varepsilon_p=T} \approx \begin{cases} hc/T & (\alpha=1) \\ h/(mT)^{1/2} & (\alpha=1/2) \end{cases}$$

$$\lambda_{\text{Compton}} = h/mc$$

$\ln(T/mc^2)$

Bohr radius

$\ln 137 \approx 4.9$

$\ln(a/\lambda_{\text{Compton}})$

quantum domain

$$\lambda_{\text{Broglie}} > a$$

classical domain

$$\lambda_{\text{Broglie}} < a$$

$$p = TN/V \cdot \Phi(\lambda_{\text{Broglie}}/a)$$

$$\Phi(x \rightarrow 0) = 1$$

$$\Phi(x \rightarrow \infty) = ?$$

Hydrogen at room temperature → -25

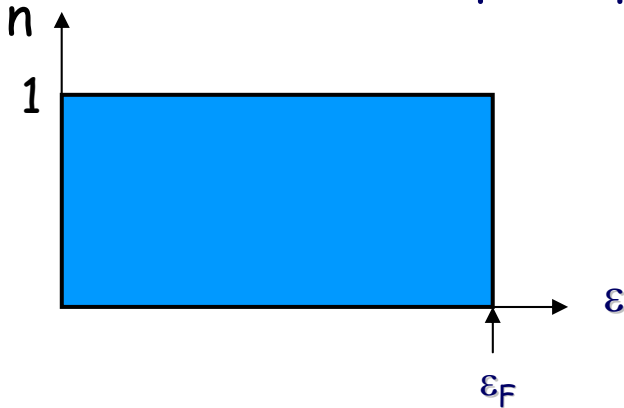
# Ideal quantum gases - Fermions



Pauli 1925...: particles have spin

$$\left\{ \begin{array}{ll} S=(n+1/2)\hbar & \text{Fermions} \\ S=n\hbar & \text{Bosons} \end{array} \right.$$

Fermions: Pauli principle  $\rightarrow$  non-zero pressure even at  $T=0$



$$p = T/a^3 \Phi(\lambda_{\text{Broglie}}/a)$$

(i)  $\epsilon_F = \hbar^2/ma^2 \ll mc^2 \rightarrow \lambda_{\text{Broglie}} \sim h/(mT)^{1/2}$

$$\rightarrow p \sim h^2/(m^2a^5)$$

(ii)  $\hbar^2/ma^2 \gg mc^2 \rightarrow a < \hbar/mc = \lambda_{\text{Compton}}$

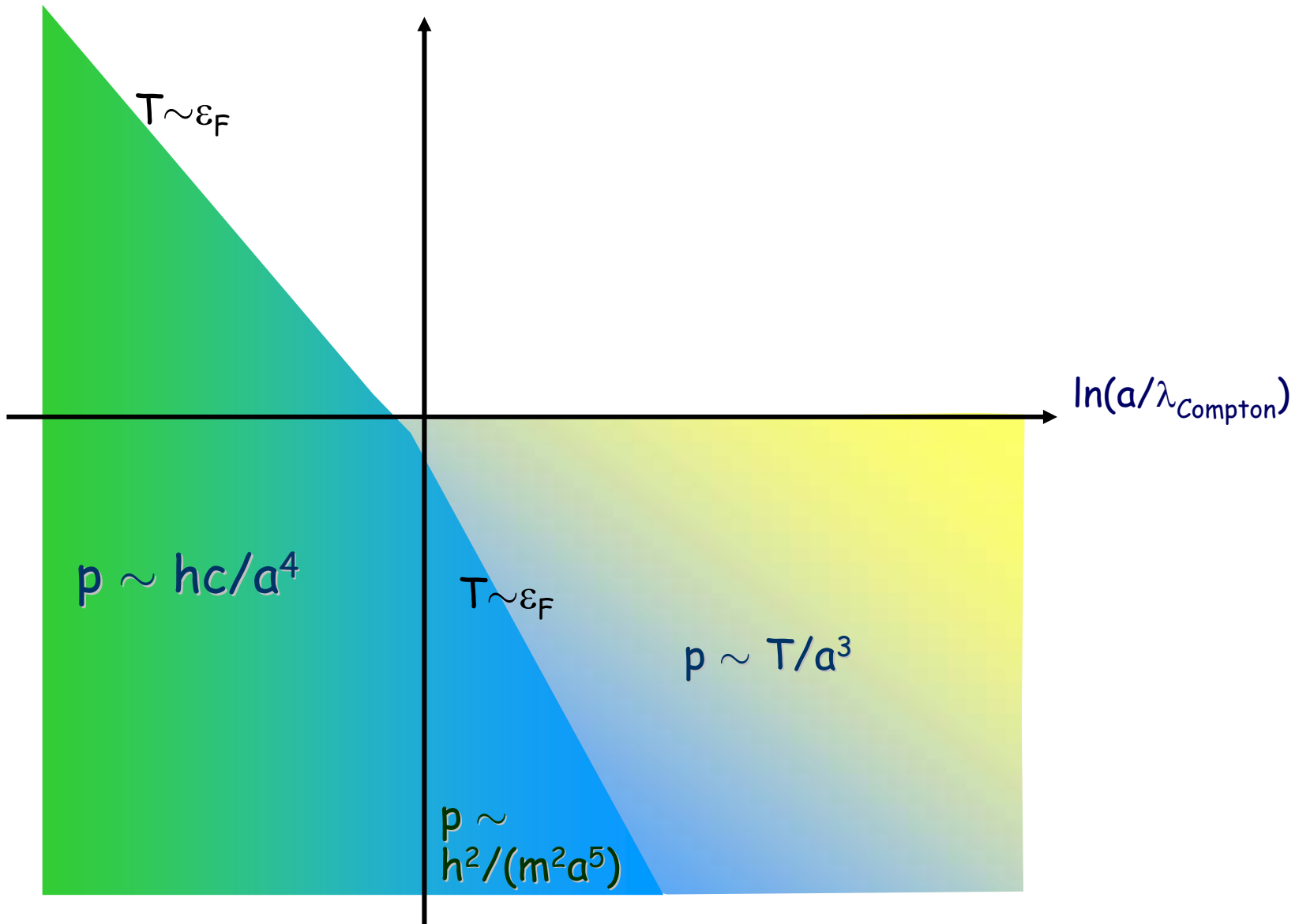
$$\rightarrow \epsilon_F = \hbar c/a \gg mc^2$$

$$\rightarrow \lambda_{\text{Broglie}} \sim ch/T$$

$$\rightarrow p \sim hc/a^4$$

# Fermions:

$\ln(T/mc^2)$



# Ideal quantum gases - Bosons



Pauli 1925...: particles have spin

$$S = (n + 1/2)\hbar \quad \text{Fermions}$$

$$S = n\hbar \quad \text{Bosons}$$

All bosons can occupy ground state

Bosons:

→ pressure does not depend on volume  $V = Na^3$

$$p = T/a^3 f(\lambda_{\text{Broglie}}/a) \rightarrow \Phi(x) \sim x^{-3}$$

$$\rightarrow p \sim T^{5/2} m^{3/2} / h^3$$

Since  $p \sim T / \lambda_{\text{Broglie}}^3 < p_{\text{klass}} = T/a^3$

→ a fraction of  $1 - (a/\lambda_{\text{Broglie}})^3 = 1 - (T/T_c)^{3/2}$

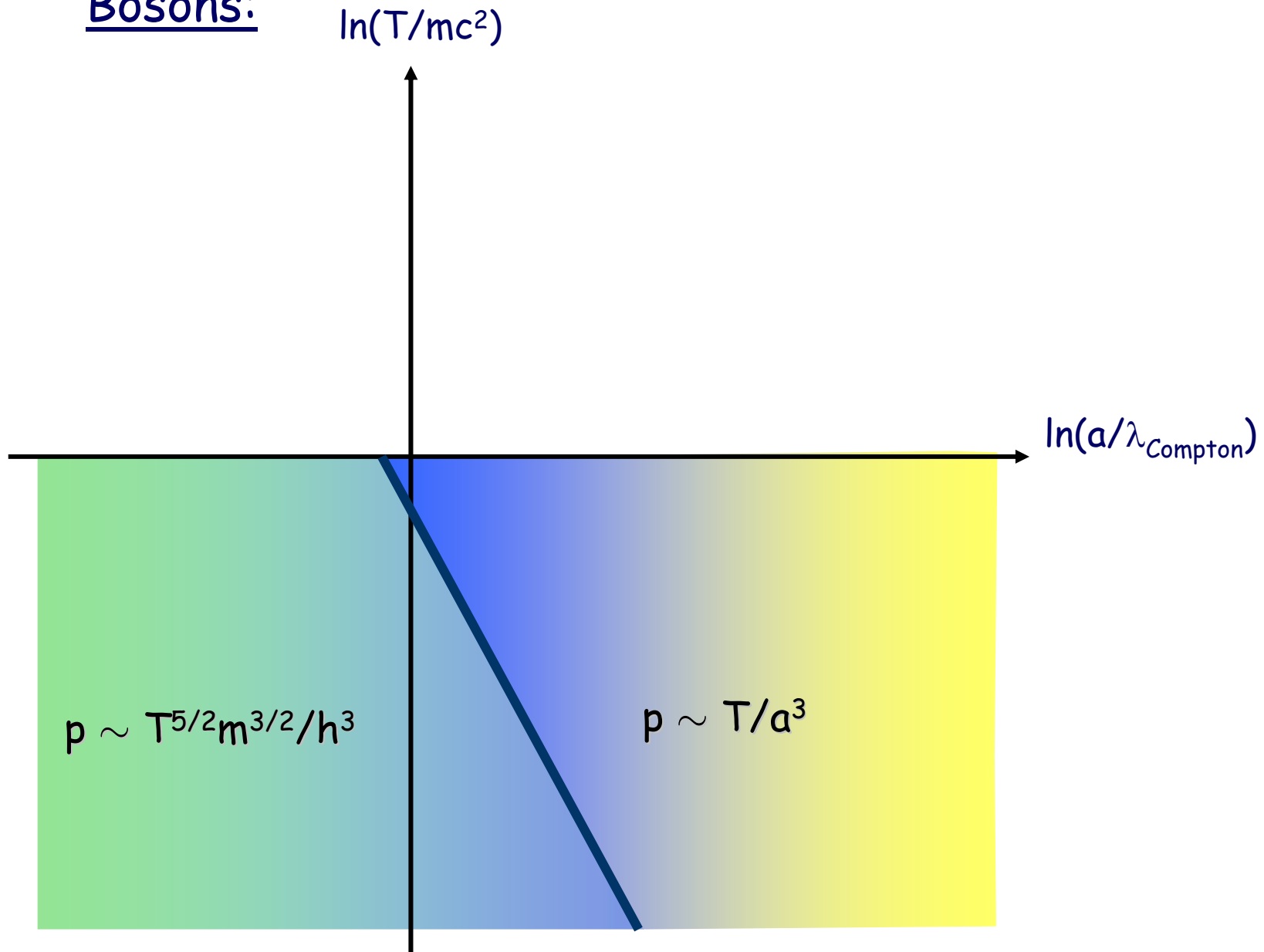
does not contribute to the pressure

$$T_c \sim h^2 / ma^2$$

$$\rightarrow N_0 = N(1 - (T/T_c)^{3/2})$$

particles in the ground state

Bosons:



# Ideal quantum gases - high temperatures :

$T \gg mc^2$  (fermions  $T > \max(hc/a = \varepsilon_F, mc^2)$ )

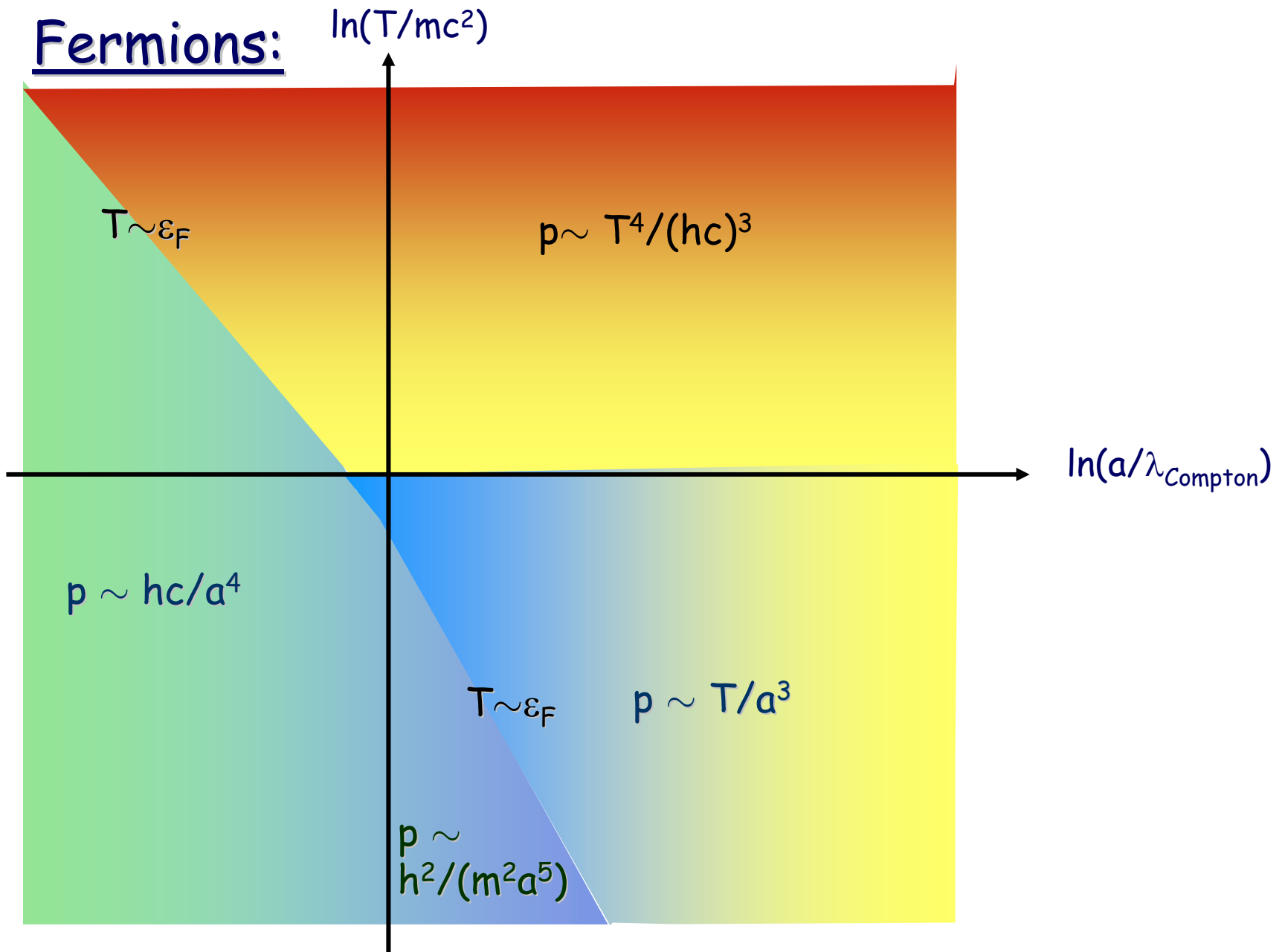
→ Particle-antiparticle generation

$a$  only good parameter if conservation law exist  
(charge, baryon number etc.)

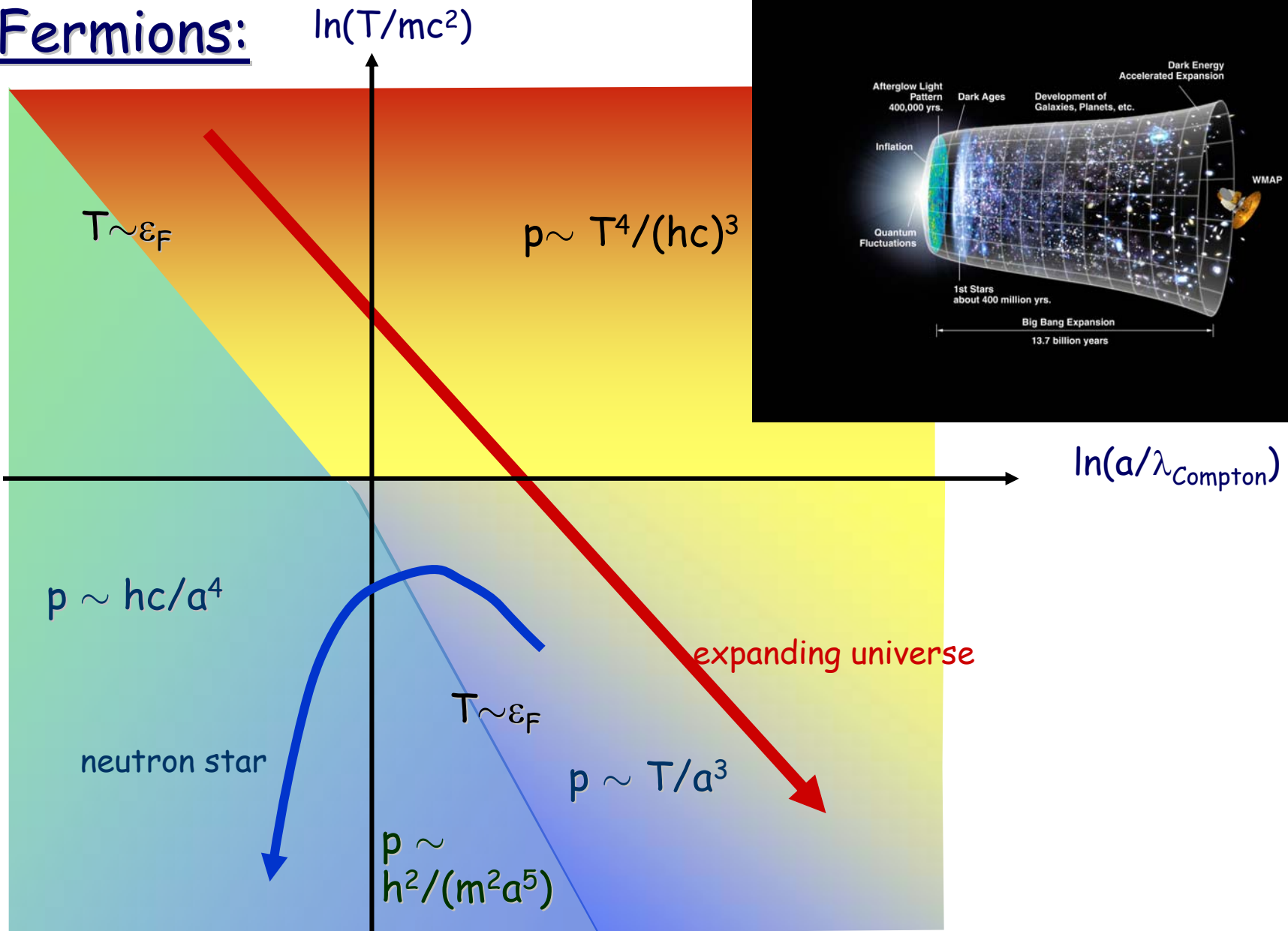
→ Pressure depends only on  $T, h, c$

$$\rightarrow p \sim T^4/(hc)^3 \sim T/\lambda_{\text{Broglie}}^3$$

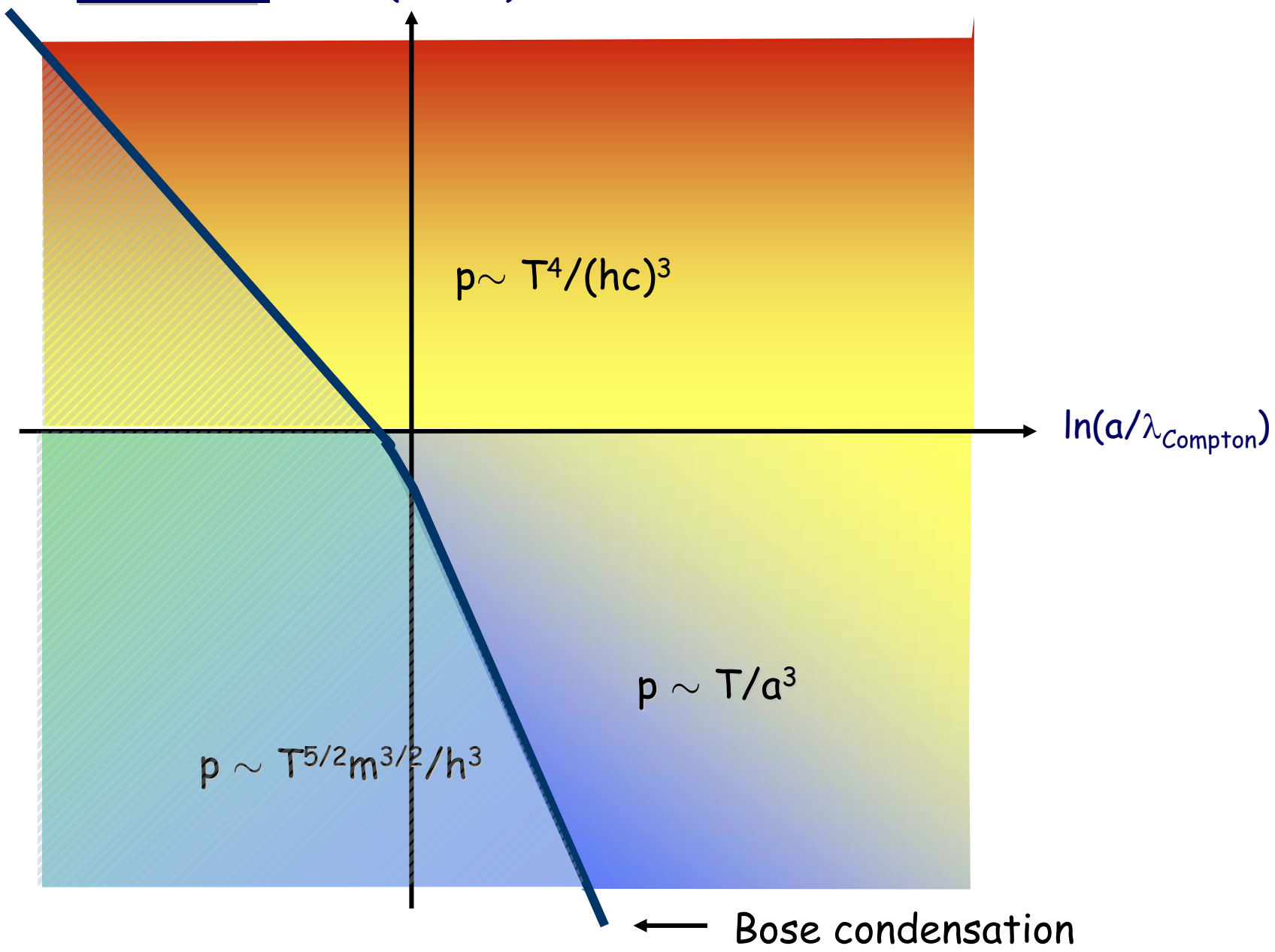
# Fermions:

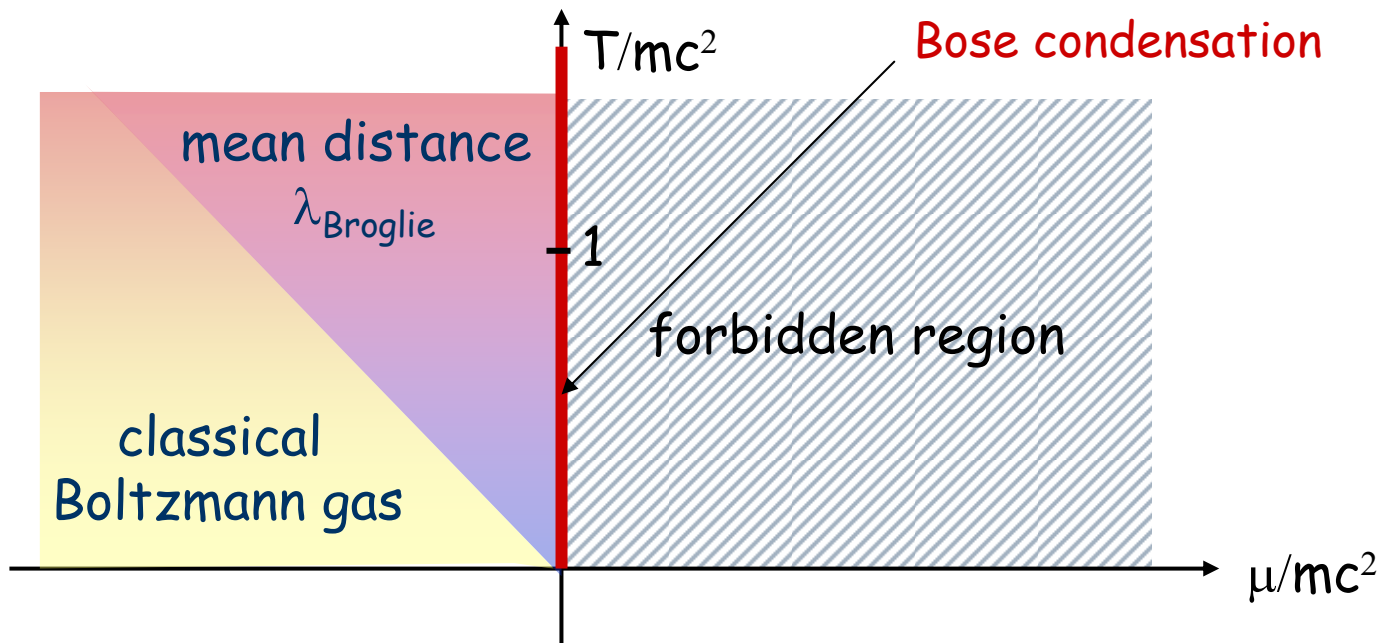
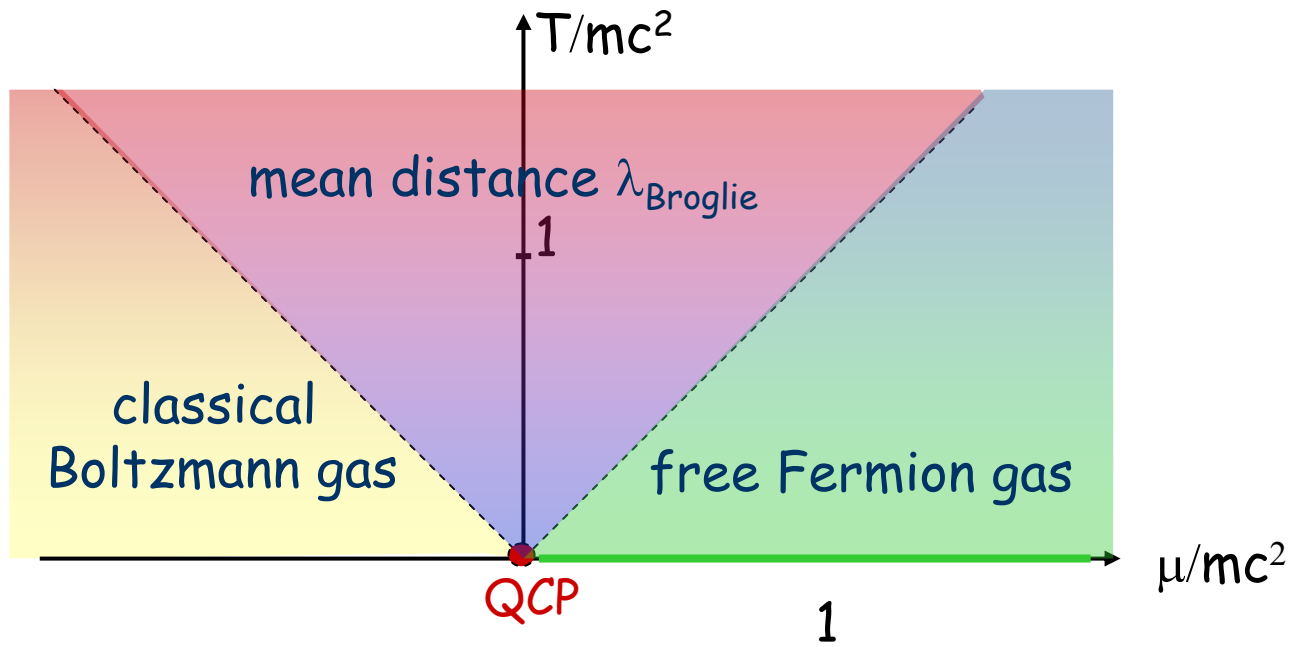


# Fermions:



Bosons:





# Conclusions:

Landau & Lifshitz Vol. 5 Statistical Physics

(45.5)

(56.8)

(56.9)

(56.15)

(57.7)

(58.6)

(61.4)

(61.7)

(62.4)

(62.6)

(62.9)

(63.16)

(63.15)

(63.16)

(63.17)

(101.2)

(106.5)

+superfluids