

Search for Simplicity*

Qualitative Methoden in der (theoretischen) Physik

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*Von den Keplerschen Gesetzen bis zur asymptotischen Freiheit

THE PRINCIPLE OF SIMILITUDE.

Mich hat oft die Geringschätzung verblüfft mit der originelle Forscher das Prinzip der Dimensionsanalyse behandeln. Es passiert nicht selten, dass Resultate in der Form neuer "Gesetze" vorgetragen werden, die man nach kurzer Überlegung vorausgesagt haben könnte. Wie nützlich auch immer deren Überprüfung sein mag, entweder um Zweifel zu beseitigen oder um Studenten zu beschäftigen, es scheint eine Umkehrung der natürlichen Ordnung zu sein...

One reason for the neglect of the principle may be that, at any rate in its applications to particular cases, it does not much interest mathematicians. On the other hand, engineers, who might make much more use of it than they have done, employ a notation which tends to obscure it.

RAYLEIGH.



**John William Strutt,
3. Baron Rayleigh**

DIMENSIONAL ANALYSIS

BY
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ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

L. I. Sedov
SIMILARITY AND DIMENSIONAL METHODS IN MECHANICS

Translated from the Russian
 by
 V. I. Kisin

Victor F. Weisskopf

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AMERICAN JOURNAL of PHYSICS

MODERN PHYSICS FROM AN ELEMENTARY POINT OF VIEW

V.F. Weisskopf

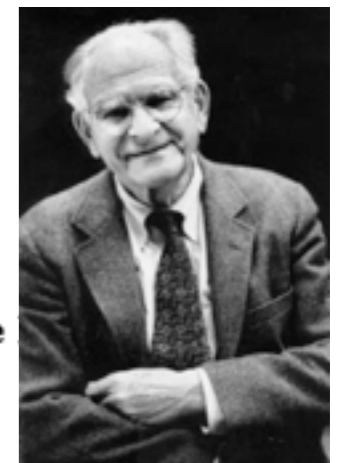
CONTEMP. PHYS., 1981, VOL. 22, NO. 4, 375-395

- Search for Simplicity: Atoms with several electrons**
- Search for Simplicity: The molecular bond**
- Search for Simplicity: Thermal expansion**
- Search for Simplicity: Quantum mechanics and the Pauli principle**
- Search for Simplicity: Chemical energy**

The Formation of Cooper Pairs and the Superconducting Currents†

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CERN, Geneva, Switzerland, and Massachusetts Institute of Technology, Cambridge, Mass., U.S.A.



Qualitative Methods in Physical Kinetics and Hydrodynamics

Vladimir P. Krainov

American Institute of Physics New York

Selected Mathematical Methods in Theoretical Physics

Vladimir P. Krainov

Dimensional Analysis

EXAMPLES OF THE USE OF SYMMETRY

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'SCALING PHENOMENA IN FLUID MECHANICS'

AN INAUGURAL LECTURE
 AT THE UNIVERSITY OF CAMBRIDGE



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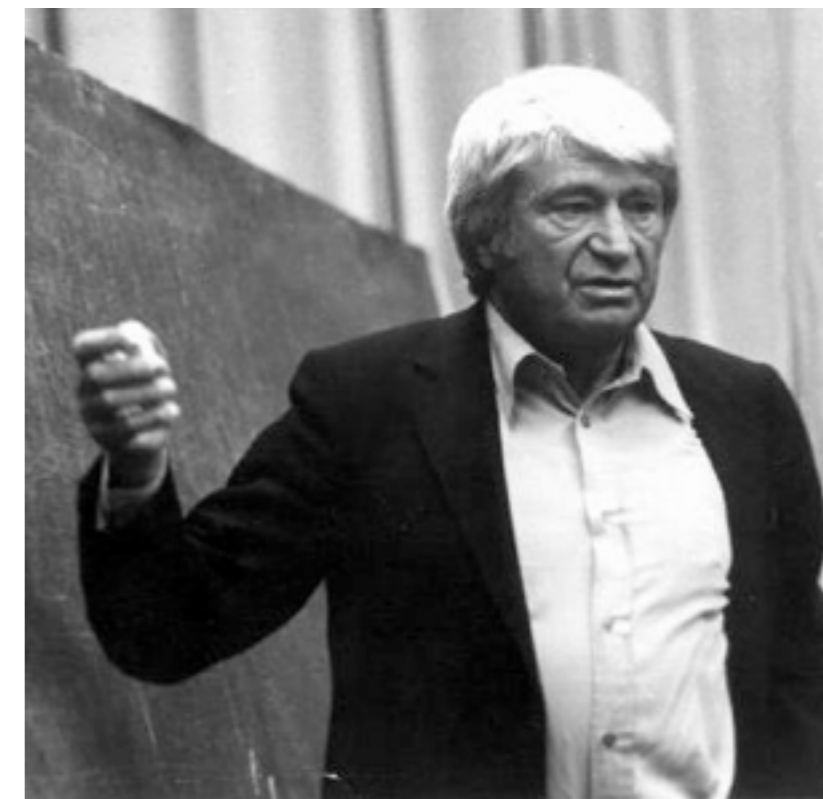
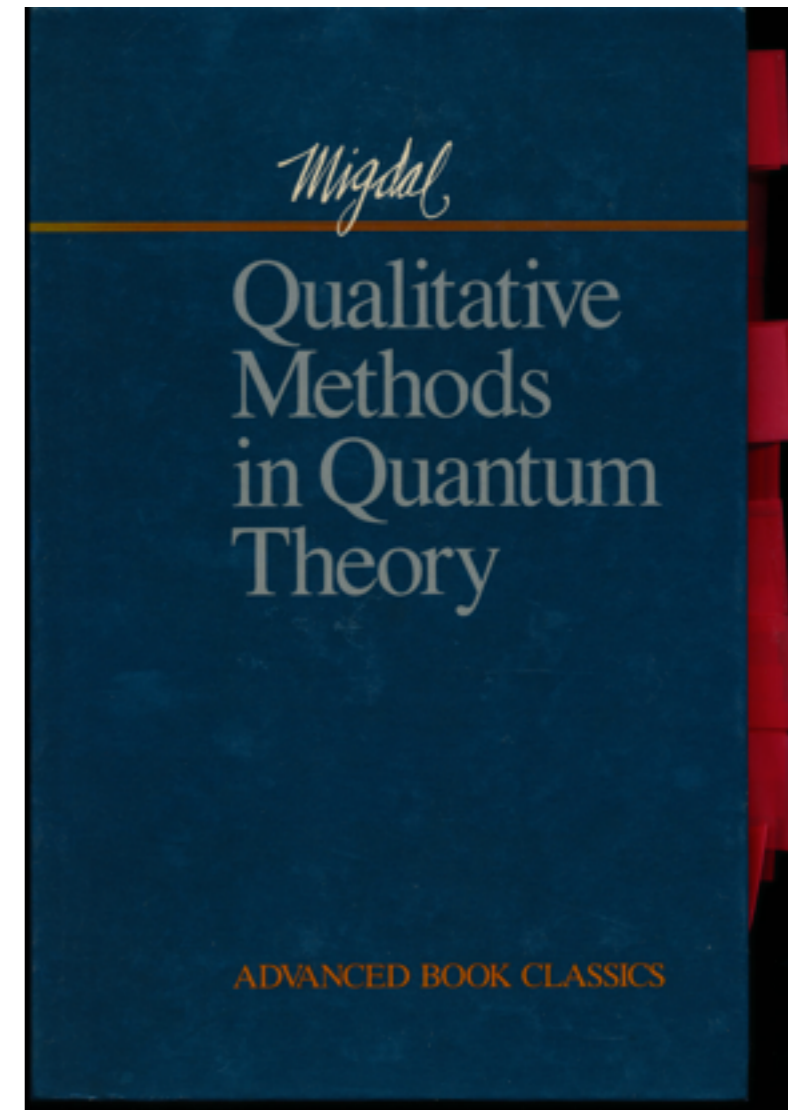
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A.B. Migdal, Qualitative methods in quantum theory

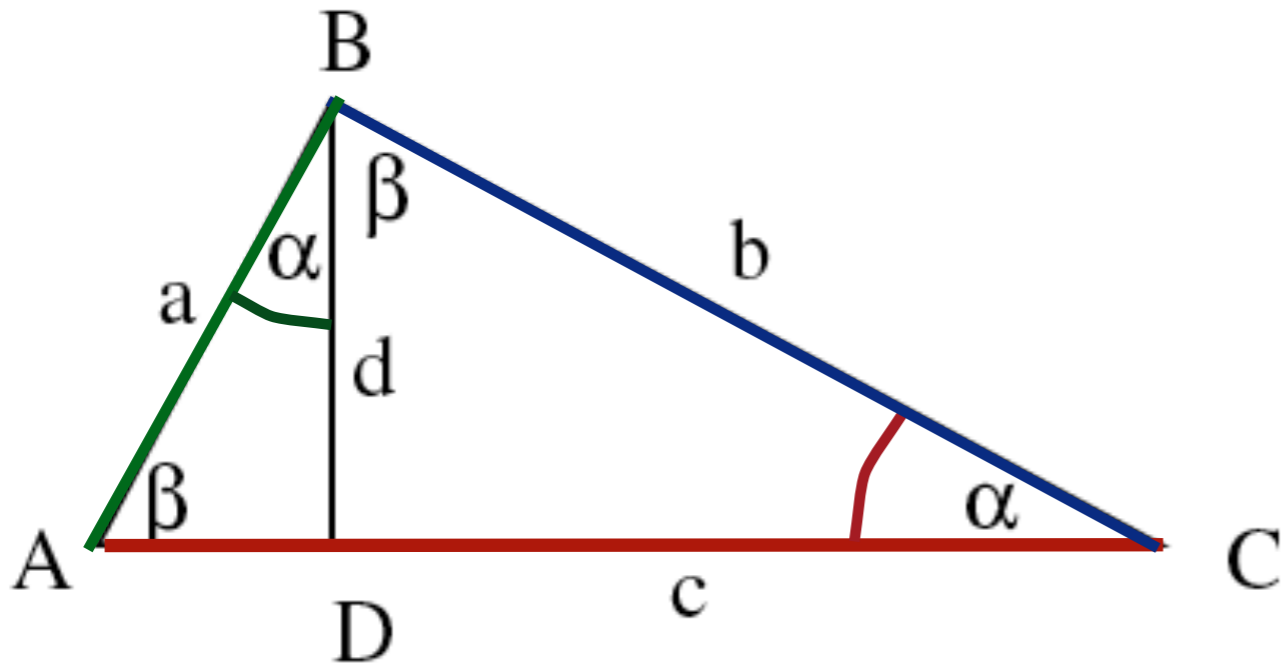
Die Lösung der meisten Probleme der theoretischen Physik beginnt mit der Anwendung qualitativer Methoden, die die schönste und attraktivste Charakteristik dieser Disziplin ist. Unter qualitativen Methoden verstehe ich einfache Dimensionsüberlegungen, ..die Ausnutzung eines kleinen Parameters, analytischer Eigenschaften oder Symmetrieüberlegungen.

Wie jedoch die Erfahrung des Hörsaals zeigt, ist es gerade dieser Aspekt der theoretischen Physik, der der schwierigste für den Anfänger ist.... Unglücklicherweise werden die Methoden der theoretischen Physik in einer formalen, mathematischen Weise präsentiert, und nicht in konstruktiver Form, in der sie in der wissenschaftlichen Arbeit benutzt werden.

Ein Fehler des Anfängers ist, alles sofort verstehen zu wollen. Im realen Leben kommt das Verständnis allmählich, so wie man sich an neue Ideen langsam gewöhnt.



Pythagoras



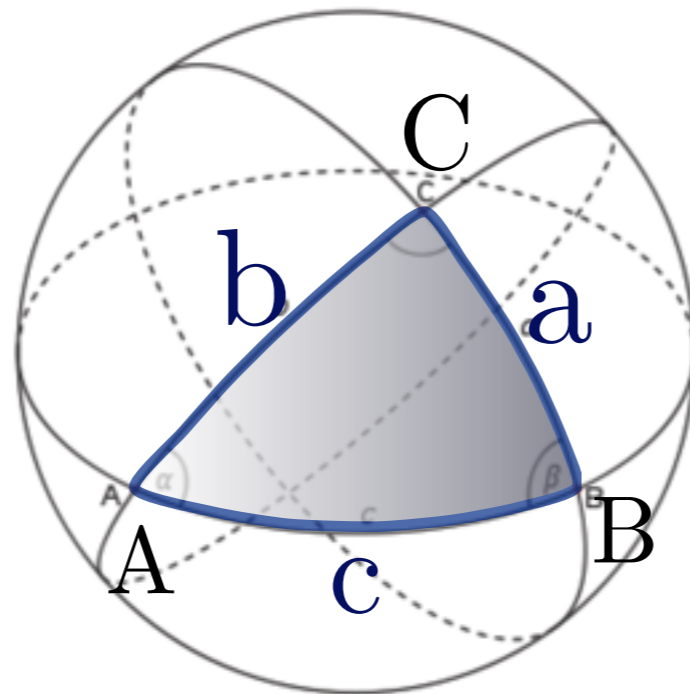
$$\alpha + \beta = \pi/2$$

$$F_{ABC} = c^2 f(\alpha)$$

$$F_{BDC} = b^2 f(\alpha)$$

$$F_{ABD} = a^2 f(\alpha)$$

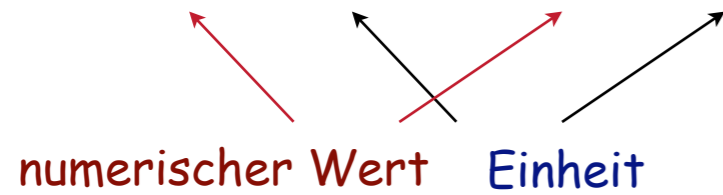
$$c^2 \cancel{f(\alpha)} = (a^2 + b^2) \cancel{f(\alpha)}$$



$$\cos(c/R) = \cos(a/R) \cos(b/R).$$

Dimensionsanalyse

$$a = A^{(1)} \{a\}^{(1)} = A^{(2)} \{a\}^{(2)}$$



$$\{a\}^{(2)} = \frac{\{a\}^{(1)}}{A^{(2)}/A^{(1)}} \equiv \frac{\{a\}^{(1)}}{[a]_{1 \rightarrow 2}}$$

$$[a]_{Fuss \rightarrow Elle} = 0,47061$$



Umrechnungsfaktor identisch für alle gemessenen Größen gleicher Natur → eigener Name: Dimension $[a] = A$

$$\text{cm} = \frac{\text{m}}{[\ell]_{SI \rightarrow G}}, \quad \text{s} = \frac{\text{s}}{[t]_{SI \rightarrow G}}, \quad \text{g} = \frac{\text{kg}}{[m]_{SI \rightarrow G}}$$

dimensionslose Grösse $A=1$

abgeleitete Grösse

$$[b]_{1 \rightarrow 2} = f([a_1]_{1 \rightarrow 2}, \dots, [a_k]_{1 \rightarrow 2}) = [a_1]_{1 \rightarrow 2}^{\alpha_1} [a_2]_{1 \rightarrow 2}^{\alpha_2} \dots [a_k]_{1 \rightarrow 2}^{\alpha_k}$$

$$[v] = \frac{[x]}{[t]} = \frac{\text{L}}{\text{T}}$$

$$\Pi = \frac{b}{a_1^{\alpha_1} \cdot \dots \cdot a_k^{\alpha_k}}, \quad [\Pi] = 1$$

$$\Pi_i = \frac{b_i}{a_1^{\alpha_{i1}} \cdot \dots \cdot a_k^{\alpha_{ik}}}, \quad i = 1, \dots, m, \quad [\Pi_i] = 1$$

Buckingham Theorem : jede physikalische Relation hat die Form

$$F(\Pi_1, \Pi_2, \dots, \Pi_m) = 0$$

$$\Pi_1 = f(\Pi_2, \dots, \Pi_m)$$

Die Tatsache, dass wenn eine bestimmte physikalische Grösse in zwei verschiedenen Einheitensystemen gemessen wird, diese nicht nur unterschiedliche numerische Werte sondern auch unterschiedliche Dimension hat, ist oft als Inkonsistenz interpretiert worden, die einer Erklärung bedarf, und Anlass gegeben hat, nach der "wahren" Dimension der physikalischen Grössen zu fragen. Nach dieser Diskussion ist klar, dass diese Frage nicht mehr Sinn hat als nach dem "wahren" Namen eines Gegenstands zu fragen.

Max Planck

Das mathematische Pendel

Relevante Grössen $\varphi, \varphi_0, t, m, \ell, g$

$$[t] = T, [\varphi_0] = 1, [\ell] = L, [g] = LT^{-2}, [m] \neq M$$

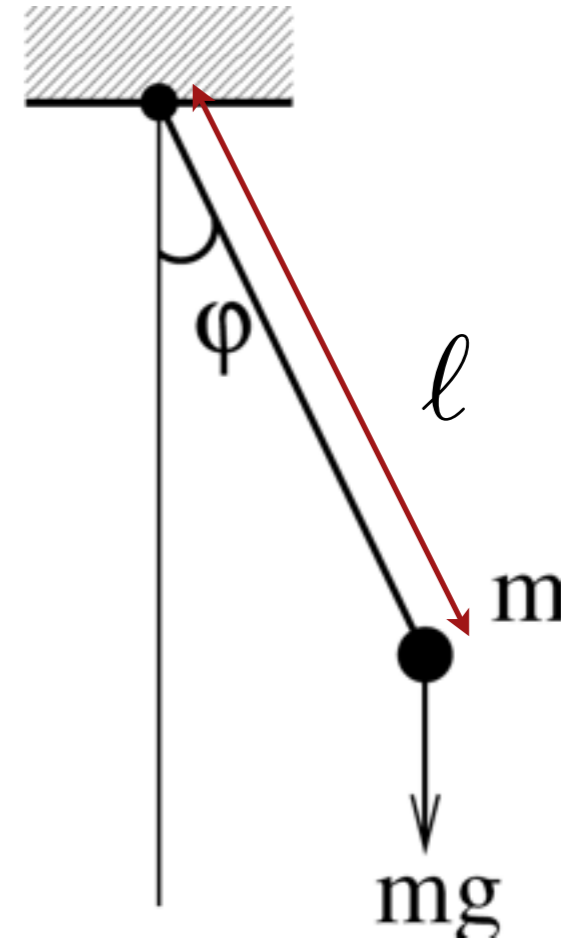
Es existiert keine dimensionslose Grösse, die m enthält!

$$\rightarrow \Pi_1 = \frac{gt^2}{\ell}, \quad \Pi_2 = \varphi_0, \quad \Pi_3 = \varphi$$

$$\varphi(t) = f\left(\varphi_0, t\sqrt{g/\ell}\right)$$

$$\tau = \sqrt{\ell/g} \theta(\varphi_0).$$

$$\theta(\varphi_0) = 4K\left(\sin \frac{\varphi_0}{2}\right), \quad K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

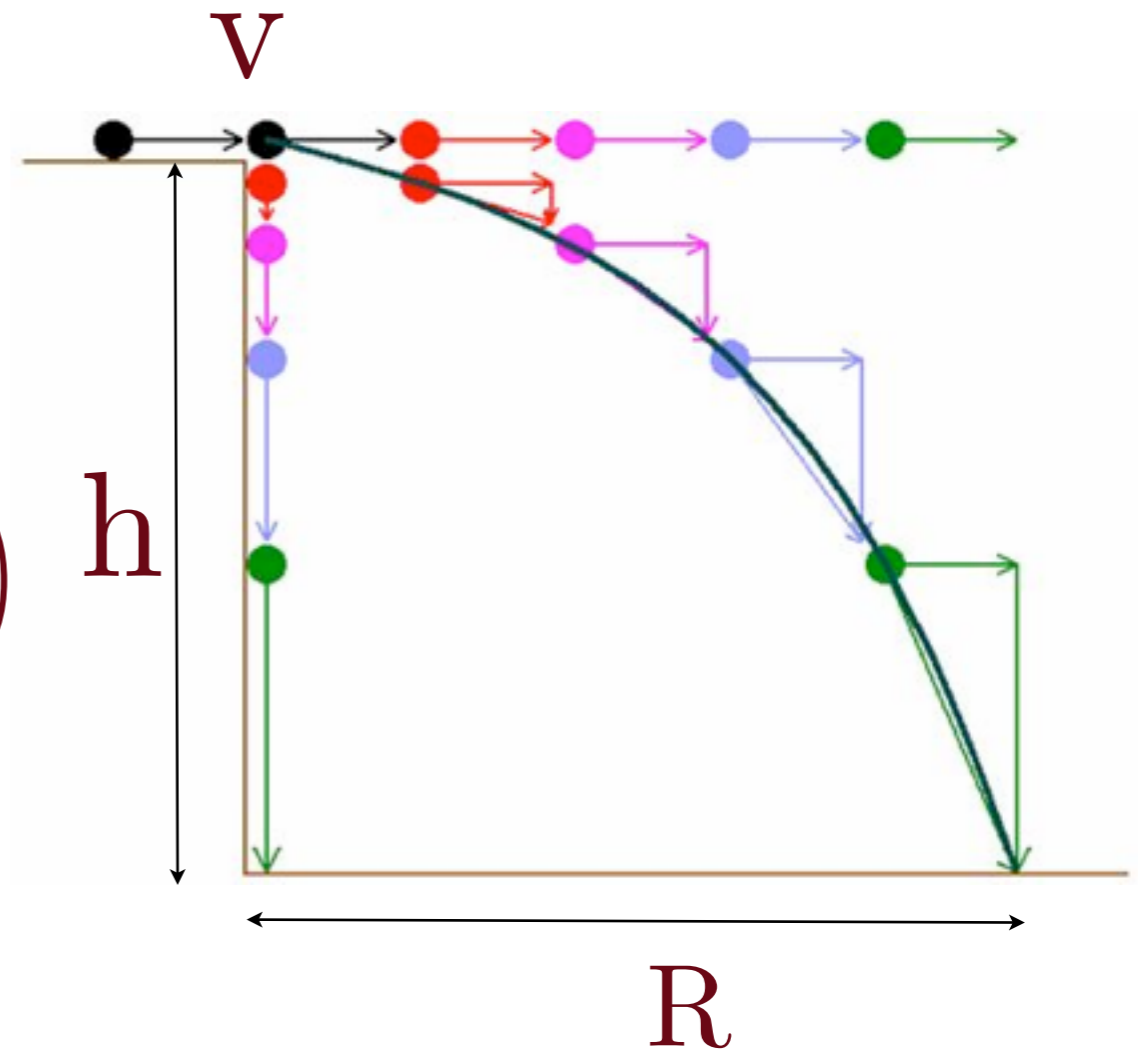


Flugweite eines Projektils

$$[R] = [h] = L, \quad [v] = LT^{-1}, \quad [g] = LT^{-2}$$

$$\Pi_1 = \frac{h}{R}, \quad \Pi_2 = \frac{gh}{v^2} \quad \rightarrow \quad R = h f\left(\frac{gh}{v^2}\right)$$

$$k = 2, \quad m = 2$$



$$[R] = L_x, \quad [h] = L_z, \quad [v] = L_x T^{-1}, \quad [g] = L_z T^{-2}.$$

$$a_1 = R, \quad a_2 = v, \quad a_3 = h, \quad b_1 = g \quad \rightarrow \quad \Pi = \frac{gR^2}{v^2 h} \quad \rightarrow R \sim v \sqrt{h/g}$$

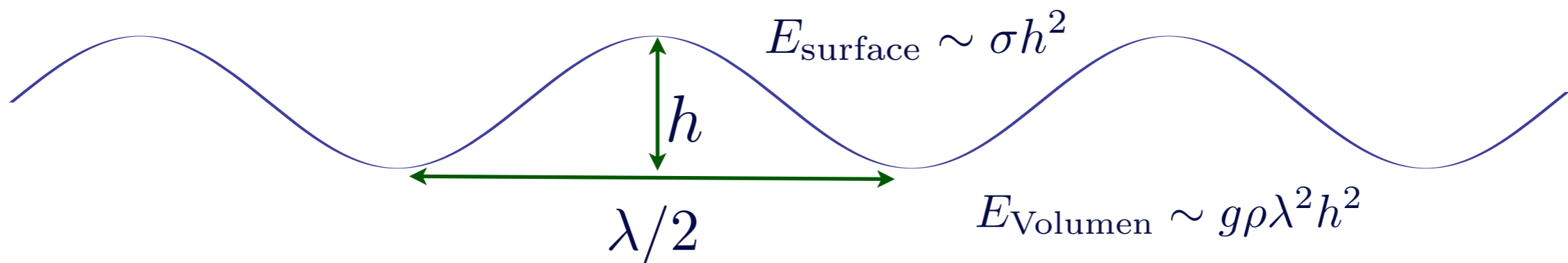
$$k = 3, \quad m = 1$$

Oberflächenwellen

$$[\rho] = \text{ML}^{-3}, [g] = \text{LT}^{-2}, [k] = \text{L}^{-1}, [\omega] = \text{T}^{-1}, [\sigma] = \text{MT}^{-2}$$

$$\rightarrow \Pi_1 = \frac{gk}{\omega^2}, \quad \Pi_2 = \frac{\sigma k^2}{\rho g} \rightarrow$$

$$\omega = \sqrt{gk} f\left(\frac{\sigma k^2}{\rho g}\right)$$



short wave length: surface tension dominant

long wave length: gravity dominant

exact result Lord Kelvin 1871:

$$f(x \gg 1) \sim x^{1/2} \rightarrow \omega \sim \left(\frac{\sigma k^3}{\rho}\right)^{1/2}$$

$$f(x \ll 1) \rightarrow \text{constant}$$

$$f(x) = \sqrt{1 + x}$$

Taylor's blast, 1947

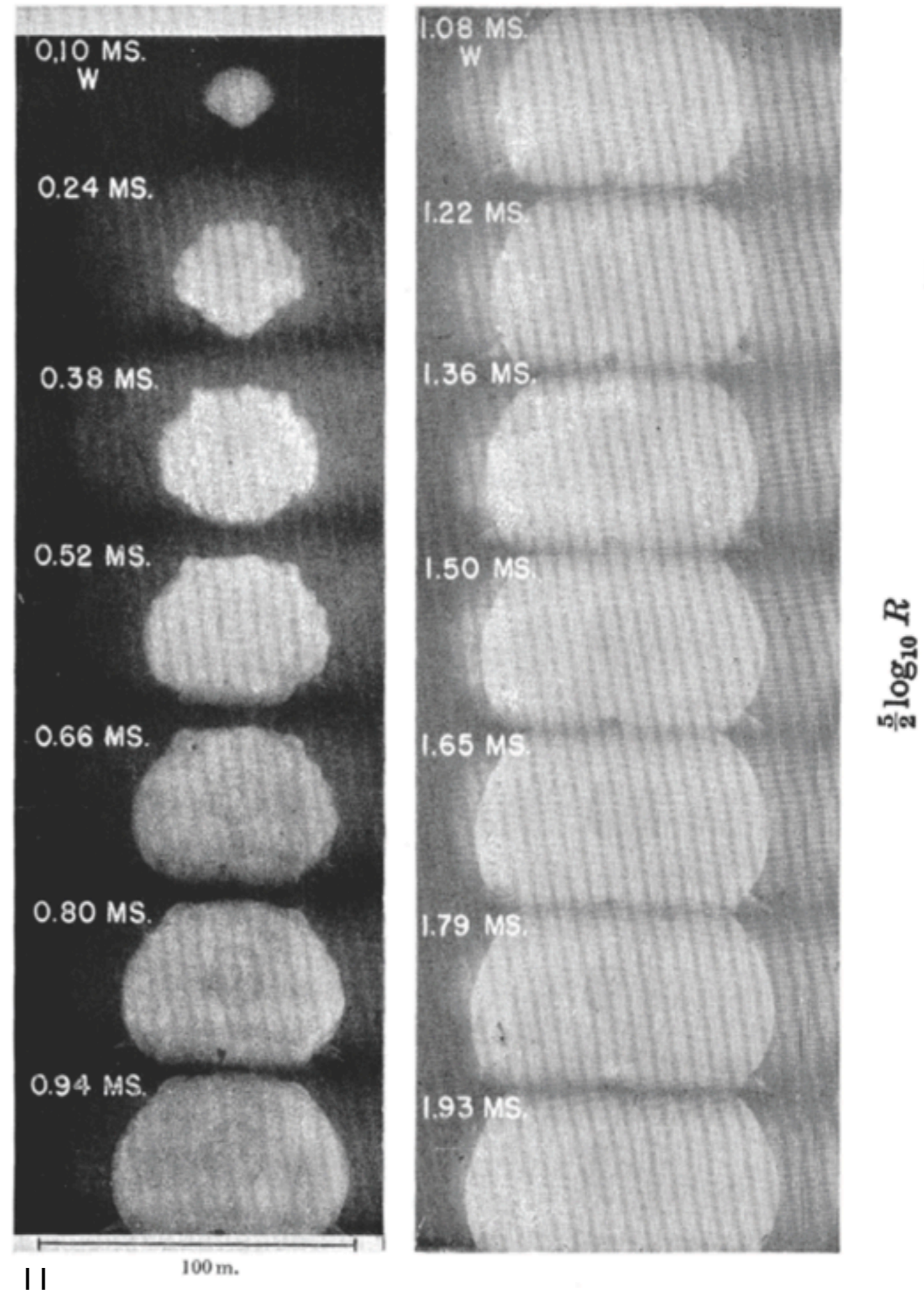
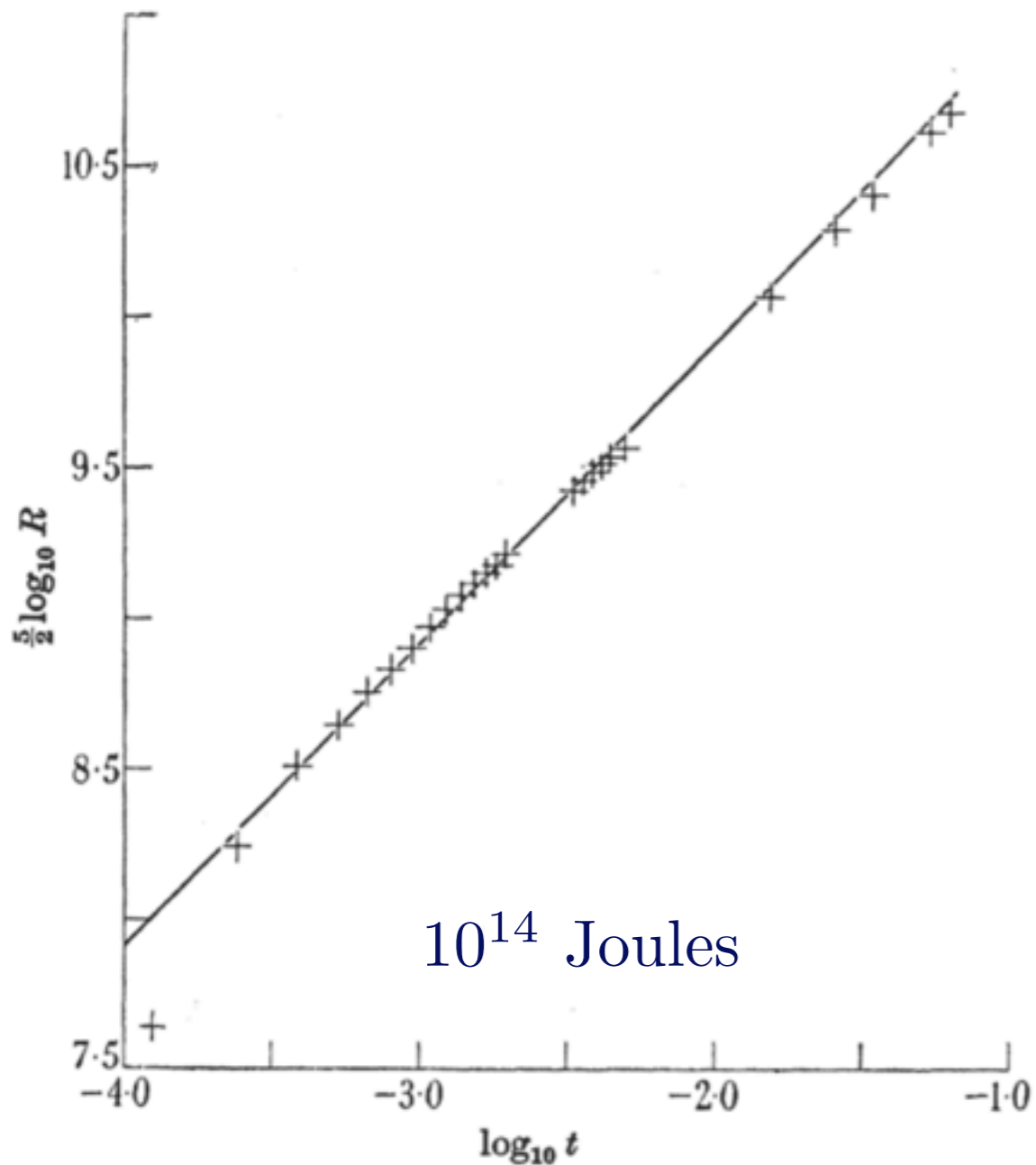
$$[E] = ML^2T^{-1}$$

$$[\rho] = ML^{-3}$$

$$[t] = T$$

$$[R] = L$$

$$R \sim \left(\frac{Et^2}{\rho} \right)^{1/5}$$



Elektrodynamik

"Einheiten sind eine kulturelle Angelegenheit, keine wissenschaftliche. Jedes System das E und B verschiedene Einheiten gibt, obwohl sie durch eine relativistische Transformation verbunden sind, ist jenseits aller Vernunft."

Val. F. Fitch

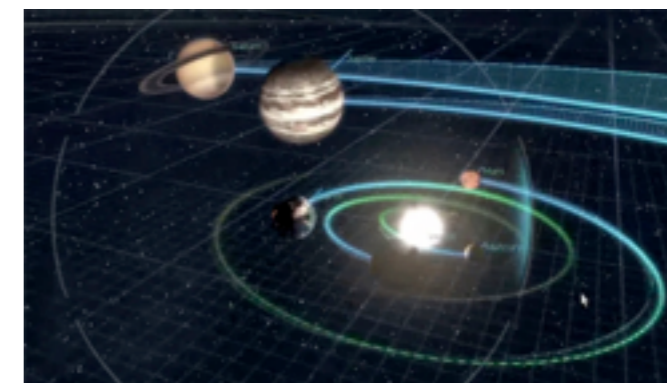
quantity	symbol	dimension Gaussian class	dimension SI class
vacuum permittivity	ϵ	1	$I^2 T^4 M^{-1} L^{-4}$
vacuum permeability	μ	1	$M L I^{-2} T^{-2}$
charge	q	$M^{1/2} L^{3/2} T^{-1}$	$I T$
electric field	\mathbf{E}	$M^{1/2} L^{-1/2} T^{-1}$	$M L T^{-3} I^{-1}$
magnetic induction	\mathbf{B}	$M^{1/2} L^{-1/2} T^{-1}$	$M T^{-2} I^{-1}$
current	\mathcal{J}	$M^{1/2} L^{3/2} T^{-2}$	I
voltage	U	$M^{1/2} L^{1/2} T^{-1}$	$M L^2 T^{-3} I^{-1}$
conductance	G	$L T^{-1}$	$T^3 I^2 M^{-1} L^{-2}$
inductance	L	$T^2 L^{-1}$	$M L^2 T^{-2} I^{-2}$
capacitance	C	L	$T^4 I^2 M^{-1} L^{-2}$

Kapazität einer Kugel vom Radius r ist r , 1 Farad = $9 \times 10^9 m$

Drude Leitwert=1/Widerstand $G = \frac{e^2 \ell}{\pi \hbar L} N$

abgestrahlte Energie pro Zeiteinheit einer bewegten Ladung: $q, c, |\ddot{\mathbf{r}}| \quad \dot{E}$

$$\dot{E} = c \frac{(q\ddot{\mathbf{r}})^2}{c^3}$$



Die Plancksche Konstante \hbar

spektrale Energiedichte (Wien 1893)

$$u(\omega, k_B T, c) \sim \frac{k_B T \omega^2}{c^3} \quad U = \int_0^\infty u(\omega) d\omega \rightarrow \infty$$

$$[U] = \text{ML}^{-1}\text{T}^{-2}, \quad [c] = \text{LT}^{-1}, \quad [k_B T] = \text{ML}^2\text{T}^{-2}$$

→ Es ist unmöglich U nur durch $k_B T$ und c auszudrücken!

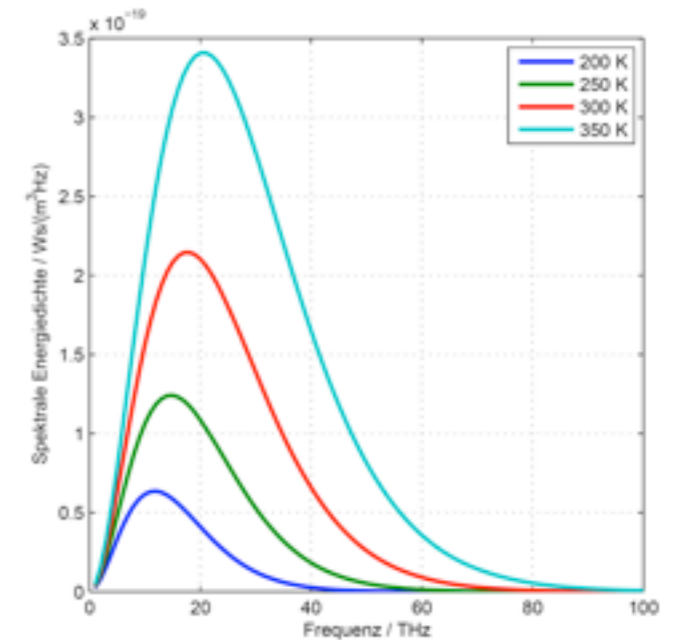
$$\rightarrow \mathcal{U} \sim k_B T c^{-3} \omega_0^3 \quad [\omega_0] = \text{T}^{-1}$$

follows with $\omega_0 = k_B T / h$, $[h] = \text{ML}^{-2}\text{T}^{-1}$

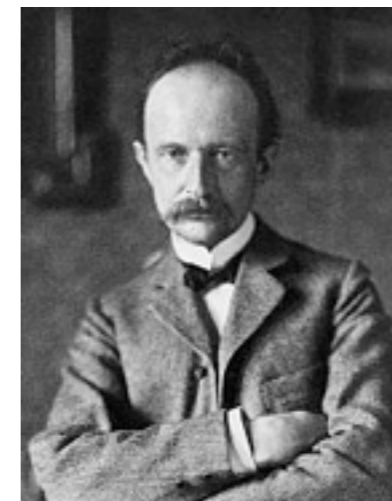
Stefan-Boltzmann Gesetz (1879)

$$\mathcal{U} = \frac{4\sigma}{c} T^4, \quad \sigma = 5.67 \times 10^{-8} \text{kg s}^{-3} \text{K}^{-4}$$

$$h = \left(\frac{c^2 k_B^4}{4\sigma} \right)^{1/3}$$



$$\rightarrow u(\omega) = \frac{k_B T \omega^2}{\pi^2 c^3} \Phi \left(\frac{h\omega}{k_B T} \right) \quad \begin{aligned} \Phi(x \rightarrow 0) &= 1 \\ \Phi(x \rightarrow \infty) &\rightarrow 0 \end{aligned}$$



$$\Phi(x) = \frac{x}{e^x - 1}$$

Teilchen im r^k Potenzial

$$\mathcal{H} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{u}{k} \left| \frac{\mathbf{r}}{a} \right|^k$$

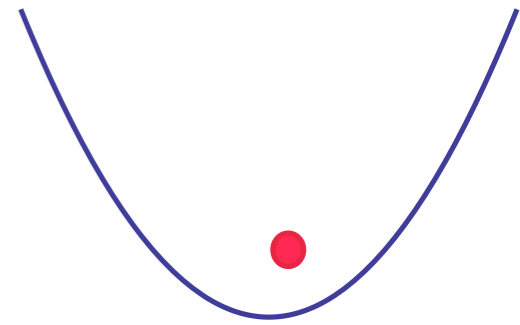
$$[E] = \text{ML}^2\text{T}^{-2}, \quad \left[\frac{\hbar^2}{m} \right] = \text{ML}^4\text{T}^{-2}, \quad \left[\frac{u}{ka^k} \right] = \text{ML}^{2-k}\text{T}^{-2}$$

Ansatz

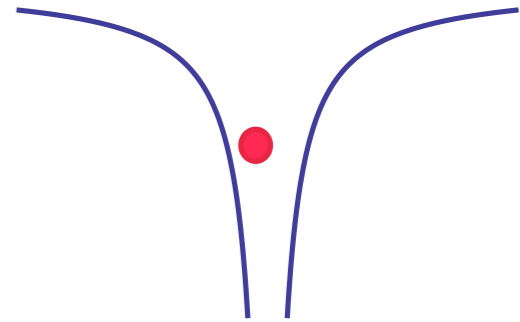
$$E \sim \left(\frac{\hbar^2}{m} \right)^\alpha \left(\frac{u}{ka^k} \right)^\beta \rightarrow \alpha = \frac{k}{2+k}, \quad \beta = \frac{2}{2+k}$$

$$E_0 \sim \frac{u}{k} \left(\frac{\hbar^2 k}{2ma^2 u} \right)^{\frac{k}{2+k}} \rightarrow E_n \sim \frac{u}{k} \left(\frac{n^2 \hbar^2 k}{2ma^2 u} \right)^{\frac{k}{2+k}}$$

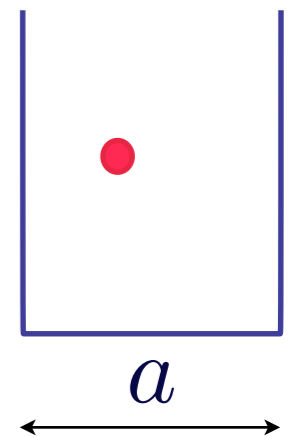
$k=2$ harmonischer Oszillator



$k=-1$ Coulomb Problem



$k \rightarrow \infty$ unendlich hoher Potenzialtopf



$$k = 2 : \quad \omega = \sqrt{\frac{u}{ma^2}}$$

$$k = -1 : \quad ua = Ze^2,$$

$$k \rightarrow \infty :$$

$$\rightarrow E_n \sim n\hbar\omega$$

$$\rightarrow E_n \sim -\frac{2mZ^2e^4}{(n+1)^2\hbar^2}$$

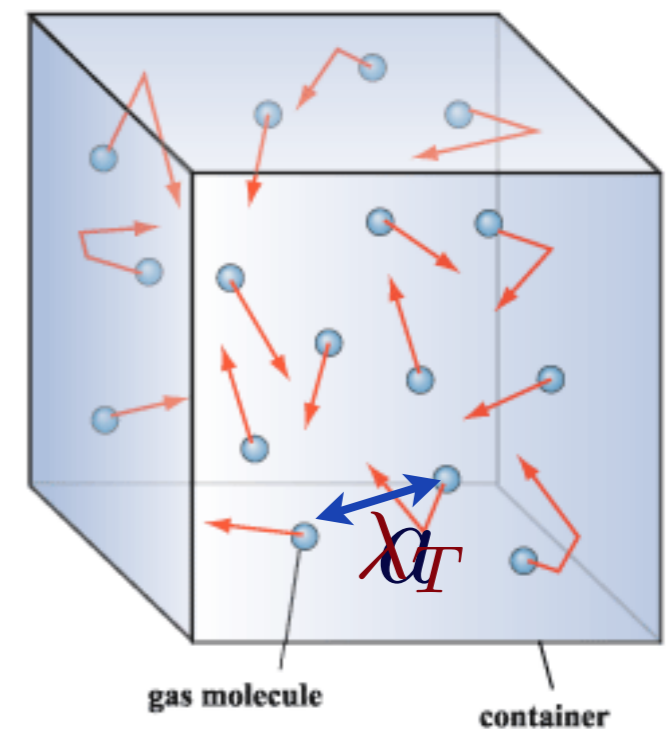
$$\rightarrow E_n \sim \frac{n^2\hbar^2}{2ma^2}$$

Ideale Gase

1. Ideales klassisches Gas : $m, k_B T, \frac{N}{V} = n = \frac{1}{a^3}$

$[m] = M, [k_B T] = ML^2 T^{-2}, [n] = L^{-3}, [p] = ML^{-1} T^{-2}$

$$\Pi = \frac{k_B T N}{pV} \rightarrow p = k_B T \frac{N}{V}$$



2. Ideales Quantengas

$$[\hbar] = ML^2 T^{-1} \quad \Pi_2 = \frac{mk_B T V^{2/3}}{N^{2/3} \hbar^2} \equiv \frac{a^2}{\lambda_T^2} \rightarrow p = k_B T \frac{N}{V} f(\Pi_2)$$

Fermionen: Druck durch Pauliverbot selbst bei $T=0 \rightarrow f(\Pi_2) \sim \Pi_2^{-1} \rightarrow p \sim \frac{\hbar^2 n^{5/3}}{m} \sim E_F n$

Bosonen: bei $T=0$ sind alle Teilchen im Grundzustand \rightarrow Druck hängt nicht mehr vom Volumen ab

$$\rightarrow f(\Pi_2) \sim \Pi_2^{3/2} \rightarrow p \sim \frac{(k_B T)^{5/3} m^{3/2}}{\hbar^3} \sim \frac{k_B T}{\lambda_T^3}$$

Zahl der Teilchen im Kondensat:

$$N_0 = N - \frac{V}{\lambda_T^3} = N \left(1 - \frac{a^3}{\lambda_T^3} \right) = N \left(1 - \frac{T^{3/2}}{T_c^{3/2}} \right), \quad T_c \sim \frac{\hbar^2}{ma^2}$$

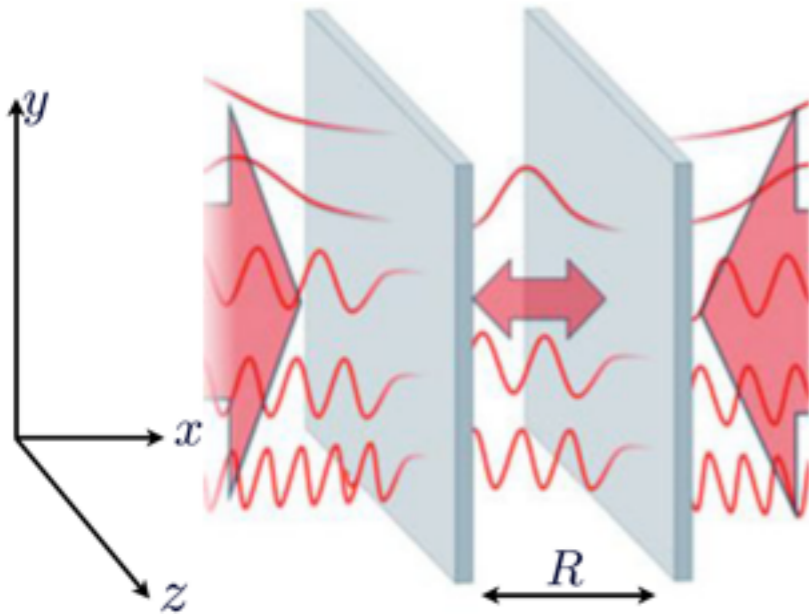
Quantenelektrodynamik

$$\left[\hat{E}_{\alpha\mathbf{k}}, \hat{B}_{\beta\mathbf{k}'} \right]_- = \hbar c \epsilon_{\alpha\beta\gamma} k_{\gamma} \delta_{\mathbf{k},\mathbf{k}'}$$

Grundzustandsenergie des elektromagnetischen Feldes

$$\Delta \mathbf{E}_{\mathbf{k}}^2 + \Delta \mathbf{B}_{\mathbf{k}}^2 > \Delta \mathbf{E}_{\mathbf{k}}^2 + \frac{(\hbar c k)^2}{\Delta \mathbf{E}_{\mathbf{k}}^2} > \hbar c k \quad \rightarrow \quad E_{GZ} = \sum_{\mathbf{k}} \hbar c k$$

Casimir-Kraft



'I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that is something new. I told him that I was puzzled by the extremely simple form of the expressions for the interaction at very large distances and he mumbled something about zero-point energy. That was all, but it put me on a new track''. H.G.Casimir

$$[\hbar c] = \text{ML}^3\text{T}^{-2}, \quad [R] = \text{L}, \quad [p] = \text{ML}^{-1}\text{T}^{-2} \quad \rightarrow \quad p = C \frac{\hbar c}{R^4}, \quad C = \frac{\pi^2}{240} \approx 0.04$$

Search for Simplicity

From Kepler's Laws to Asymptotic Freedom

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- 1.2 Examples
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 - Casimir effect

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 - Decoherence

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Turbulenz

Navier-Stokes Gleichung

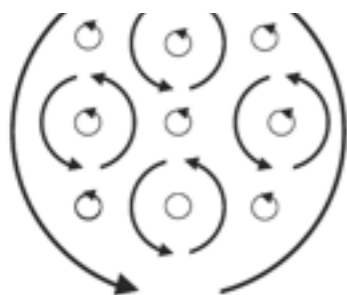
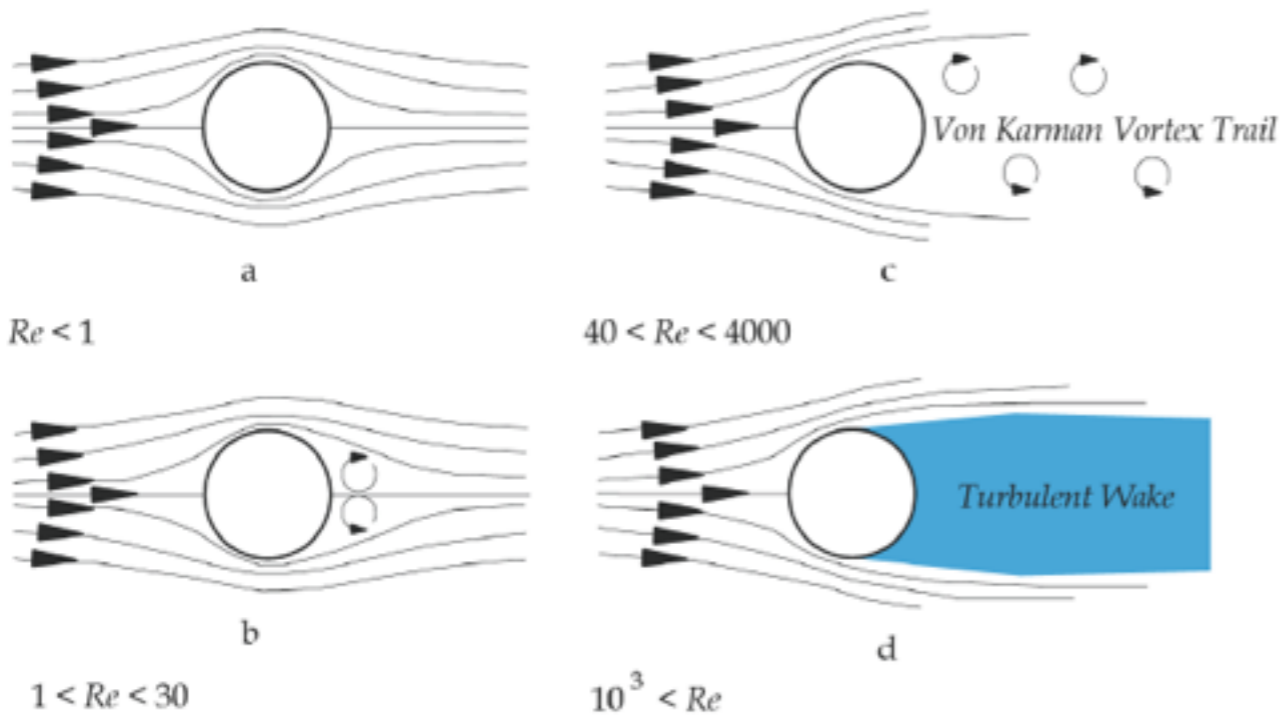
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \Delta \mathbf{v}.$$

$$\dot{\mathcal{E}}_v = \frac{\partial}{\partial t} \left(\frac{v^2}{2} - \mathbf{g} \cdot \mathbf{r} \right) = -\frac{1}{2} \nu u_{\alpha\beta}^2$$

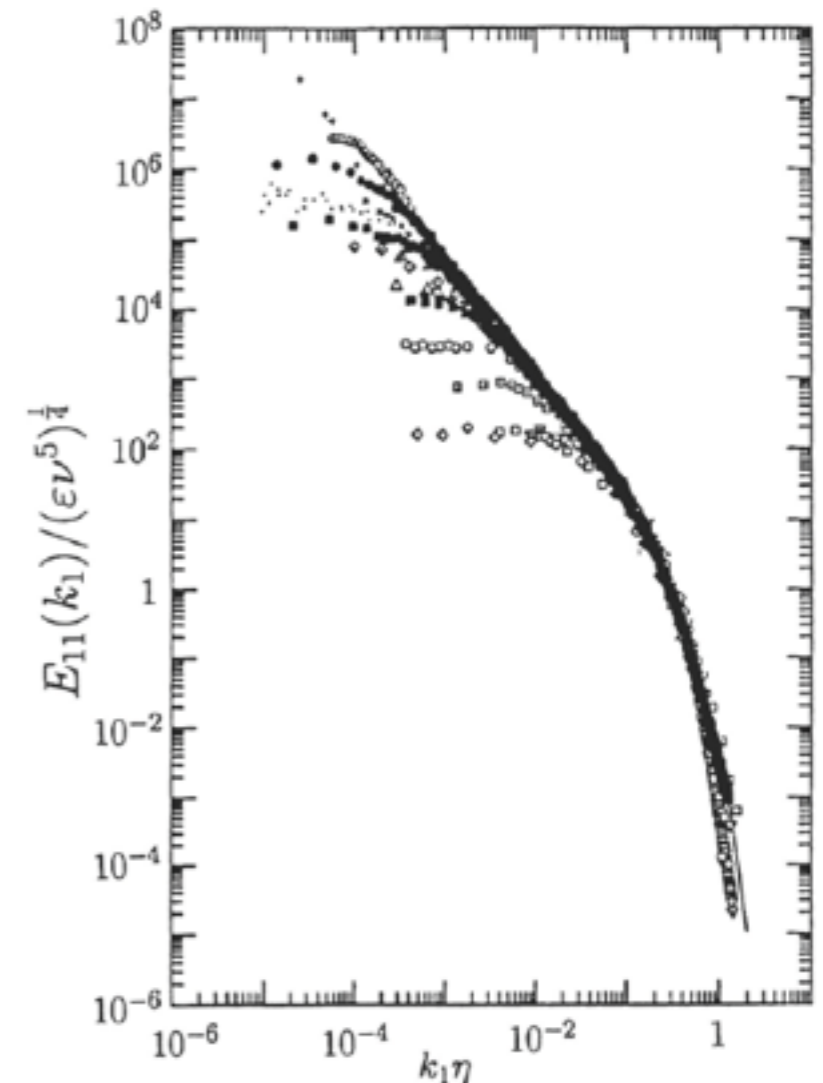
$$Re = \frac{u_{l_0} l_0}{\nu}$$

object Reynolds number Re

ideal fluid	∞
airplane	$10^6 - 10^7$
swimming human	10^4
goldfish	10^2
bacteria	$10^{-4} \dots 10^{-5}$
virus	10^{-7}
earth mantle	10^{-9}



$$\dot{\mathcal{E}} \sim \frac{u_l^3}{l} \quad \text{or} \quad u_l \sim (\dot{\mathcal{E}} l)^{1/3}$$

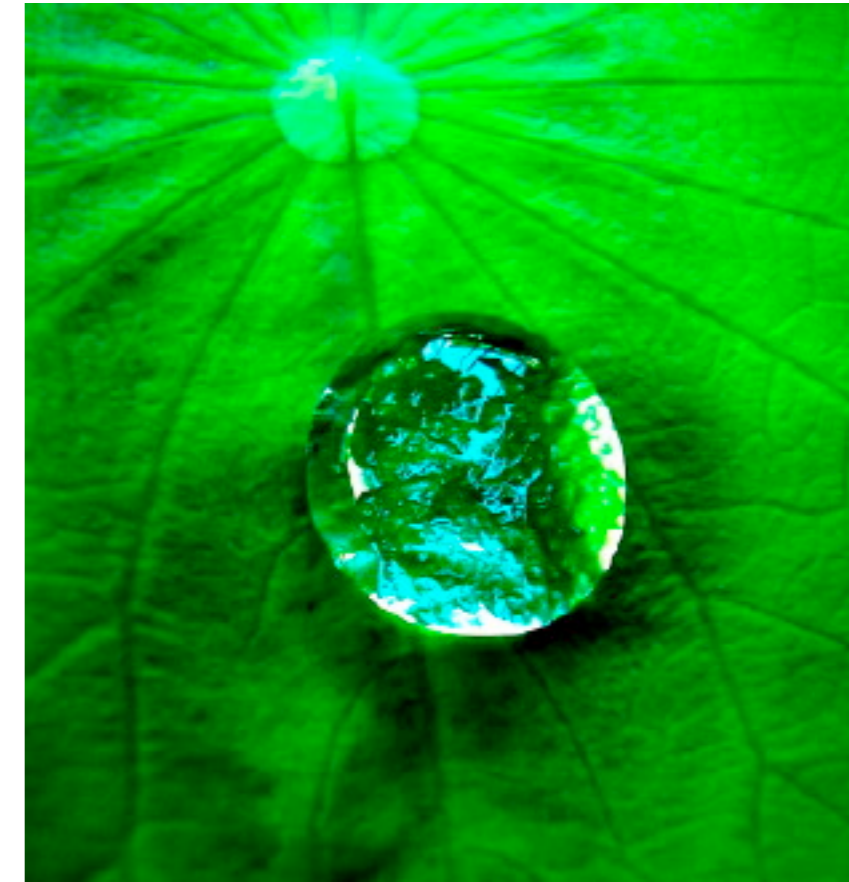


The oscillation frequency of a water droplet and a star

$$[\sigma] = \text{MT}^{-2}, \quad [\rho] = \text{ML}^{-3}, \quad [r] = \text{L}$$

$$\omega \sim \sqrt{\frac{\sigma}{\rho r^3}}$$

With $\sigma \approx 0.07 \text{ kg}/(\text{ms}^2)$ one gets a frequency of 3 Hz



$$\omega \sim \sqrt{G_N \rho} \quad \text{independent of its radius } r!$$

$$\rho_{\text{sun}} \approx 1.4 \times 10^3 \text{ kg m}^{-3} \quad \omega_{\text{sun}}^{-1} \approx 1 \text{ h}$$

$$\rho_{\text{n-star}} \approx 7 \cdot 10^{11} \text{ kg m}^{-3} \quad \omega_{\text{n-star}}^{-1} \approx 10^{-4} \text{ s}$$

