Search for Simplicity*

Qualitative Methoden in der (theoretischen) Physik

T. Nattermann
Universität zu Köln

*Von den Keplerschen Gesetzen bis zur asymptotischen Freiheit

NATURE [MARCH 18, 1915

THE PRINCIPLE OF SIMILITUDE.

Mich hat oft die Geringschätzung verblüfft mit der originelle Forscher das Prinzip der Dimensionsanalyse behandeln. Es passiert nicht selten, dass Resultate in der Form neuer "Gesetze" vorgetragen werden, die man nach kurzer Überlegung vorausgesagt haben könnte. Wie nützlich auch immer deren Überprüfung sein mag, entweder um Zweifel zu beseitigen oder um Studenten zu beschäftigen, es scheint eine Umkehrung der natürlichen Ordnung zu sein...

One reason for the neglect of the principle may be that, at any rate in its applications to particular cases, it does not much interest mathematicians. On the other hand, engineers, who might make much more use of it than they have done, employ a notation which tends to obscure it.

RAYLEIGH.

John William Strutt, 3. Baron Rayleigh

DIMENSIONAL ANALYSIS

BY

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SIMILARITY AND DIMENSIONAL METHODS IN MECHANICS

Translated from the Russian by V. I. Kisin

Victor F. Weisskopf

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AMERICAN JOURNAL of PHYSICS

Search for Simplicity: Atoms with several electrons

Search for Simplicity: The molecular bond

Search for Simplicity: Thermal expansion

Search for Simplicity: Quantum mechanics and the Pauli principle

Search for Simplicity: Chemical energy

ODERN PHYSICS FROM AN ELEMENTARY POINT OF VIEW

V.F. Weisskopf

CONTEMP. PHYS., 1981, VOL. 22, NO. 4, 375–395

The Formation of Cooper Pairs and the Superconducting Currents†

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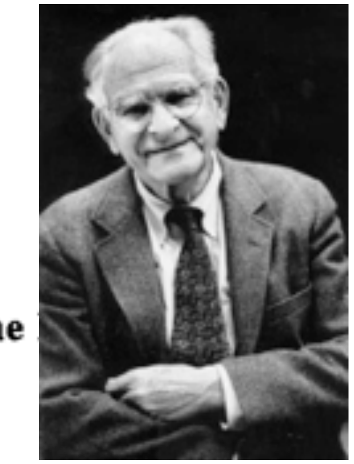


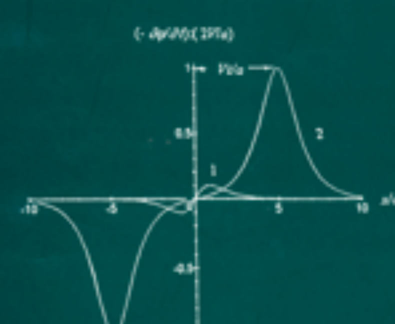
Qualitative Methods in Physical Kinetics and Hydrodynamics

Vladimir P. Krainov

American Institute of Physics New York

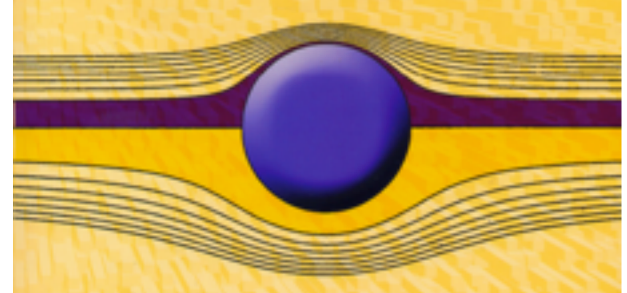
Selected Mathematical Methods in Theoretical Physics

Vladimir P. Krainov



Dimensional Analysis

EXAMPLES OF THE USE OF SYMMETRY



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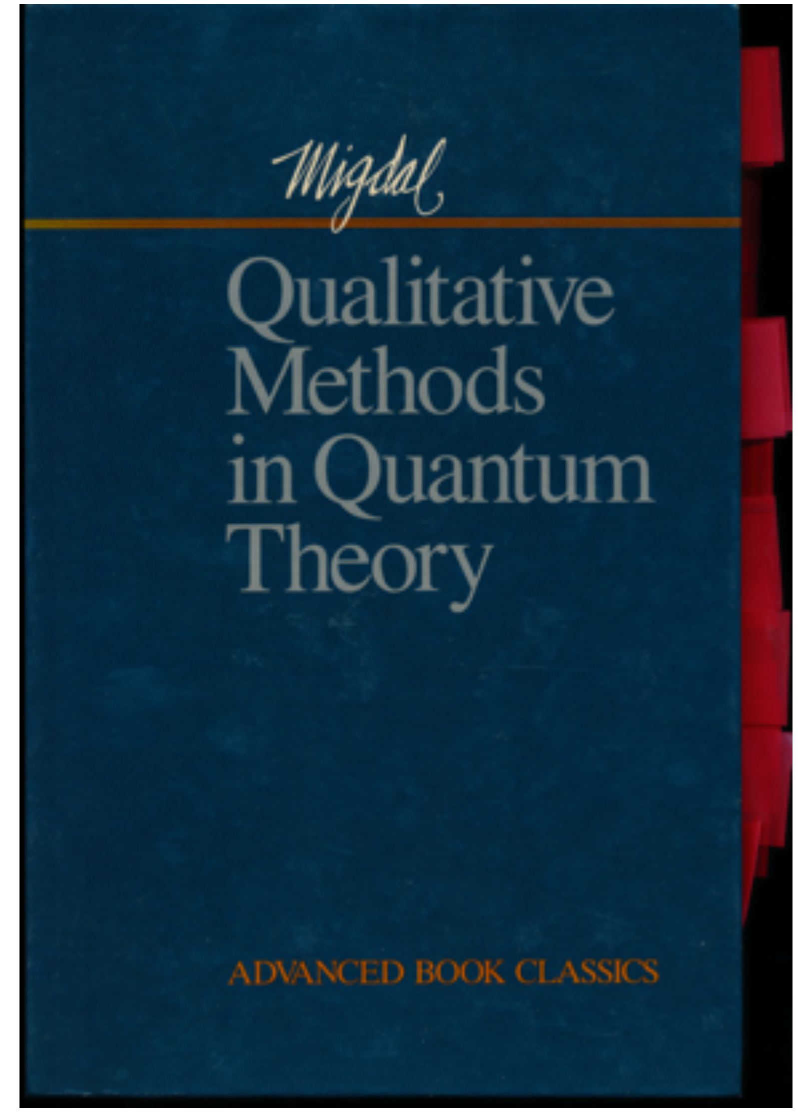
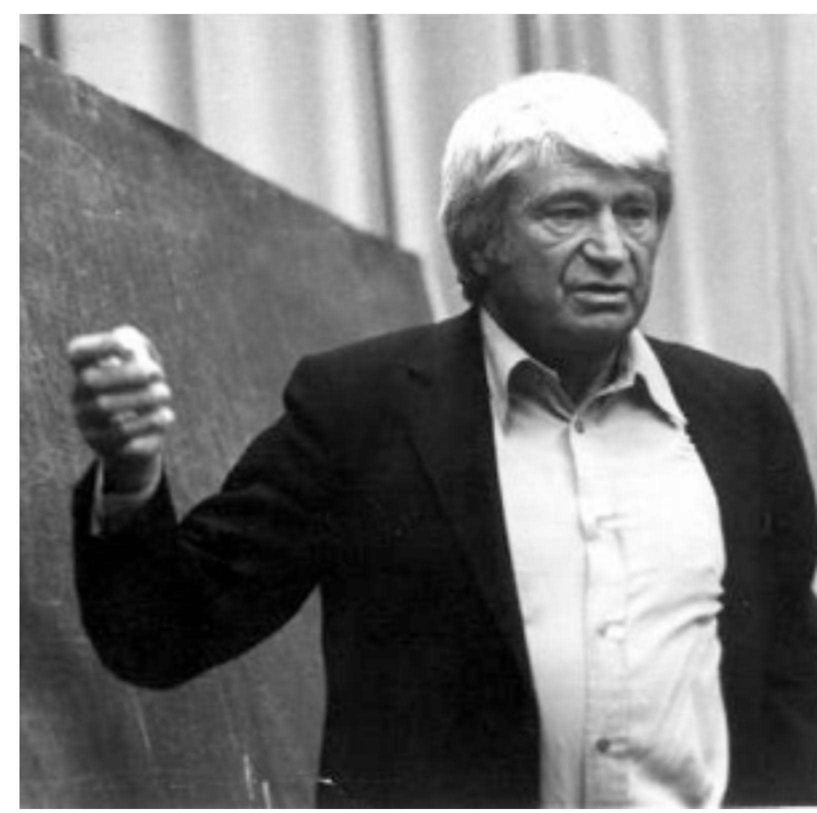
A.B. Migdal, Qualitative methods in quantum theory

Die Lösung der meisten Probleme der theoretischen Physik beginnt mit der Anwendung qualitativer Methoden, die die schönste und attraktivste Charakteristik dieser Disziplin ist. Unter qualitativen Methoden verstehe ich einfache Dimensionsüberlegungen, ..die Ausnutzung eines kleinen Parameters, analytischer Eigenschaften oder Symmetrieüberlegungen.

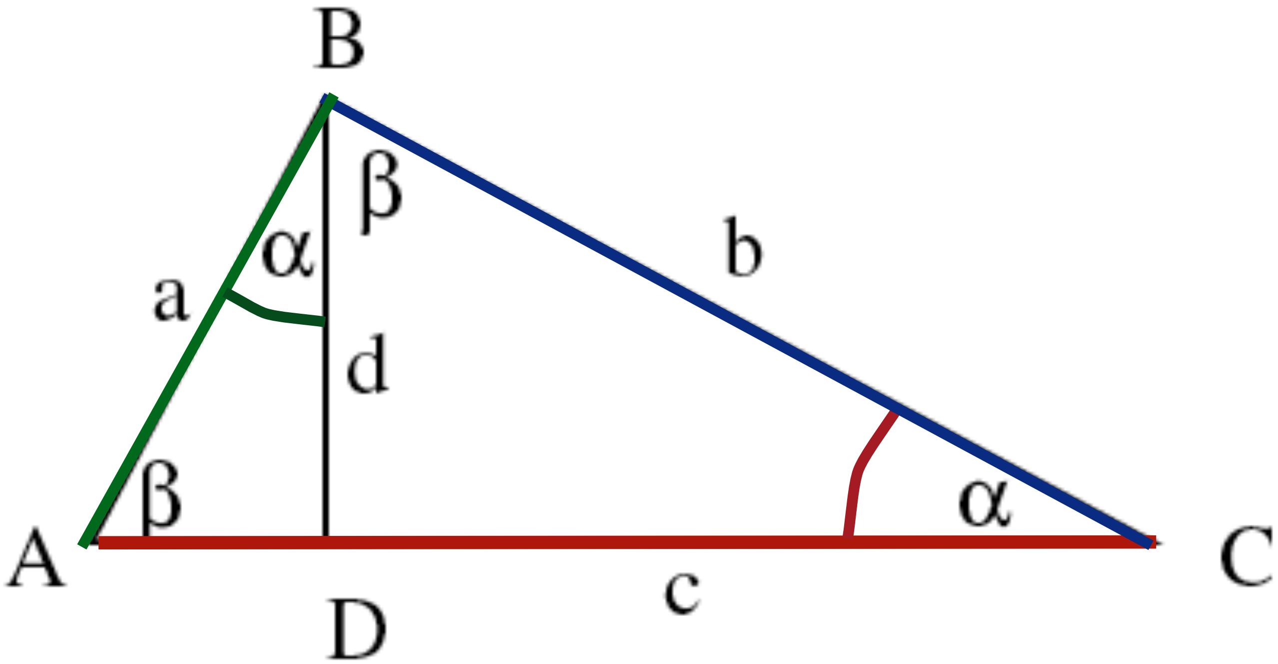
Wie jedoch die Erfahrung des Hörsaals zeigt, ist es gerade dieser Aspekt der theoretischen Physik, der der schwierigste für den Anfänger ist.... Unglücklicherweise werden die Methoden der theoretischen Physik in einer formalen, mathematischen Weise präsentiert, und nicht in konstruktiver Form, in der sie in der wissenschaftlichen Arbeit benutzt werden.

Ein Fehler des Anfängers ist, alles sofort verstehen zu wollen. Im realen Leben kommt das Verständnis allmählich, so wie man sich an neue Ideen langsam gewöhnt.





Pythagoras



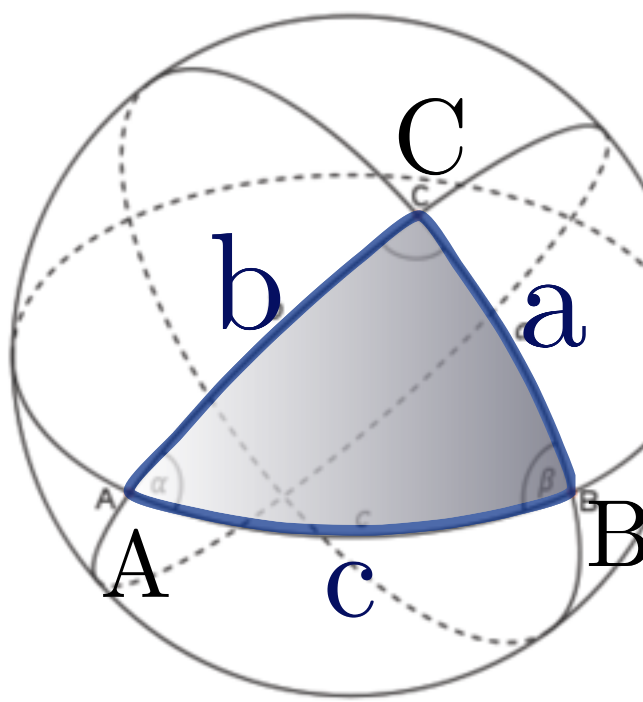
$$\alpha + \beta = \pi/2$$

$$F_{ABC} = c^2 f(\alpha)$$

$$F_{BDC} = b^2 f(\alpha)$$

$$F_{ABD} = a^2 f(\alpha)$$

$$c^2 \cancel{f(\alpha)} = (a^2 + b^2) \cancel{f(\alpha)}$$



$$\cos(c/R) = \cos(a/R)\cos(b/R).$$

Dimensionsanalyse

$$a = A^{(1)}\{a\}^{(1)} = A^{(2)}\{a\}^{(2)}$$

numerischer Wert Einheit

$$\{a\}^{(2)} = \frac{\{a\}^{(1)}}{A^{(2)}/A^{(1)}} \equiv \frac{\{a\}^{(1)}}{[a]_{1\to 2}}$$

$$[a]_{Fuss\to Elle} = 0,47061$$

Umrechnungsfaktor identisch fur alle gemessenen Grössen gleicher Natur -> eigener Name: <u>Dimension</u> $[a] = \mathsf{A}$

$$\mathrm{cm} = \frac{\mathrm{m}}{[\ell]_{SI\to G}}, \quad \mathrm{s} = \frac{\mathrm{s}}{[t]_{SI\to G}}, \quad \mathrm{g} = \frac{\mathrm{kg}}{[m]_{SI\to G}}.$$

dimensionslose Grösse A=1

abgeleitete Grösse

$$[b]_{1\to 2} = f([a_1]_{1\to 2}, ..., [a_k]_{1\to 2}) = [a_1]^{\alpha_1}_{1\to 2}[a_2]^{\alpha_2}_{1\to 2}...[a_k]^{\alpha_k}_{1\to 2}$$

$$[v] = \frac{[x]}{[t]} = \frac{\mathsf{L}}{\mathsf{T}}$$

$$\boxed{\Pi = \frac{b}{a_1^{\alpha_1}\cdot ... \cdot a_k^{\alpha_k}}, \qquad [\Pi] = 1}$$

$$\Pi_i = \frac{b_i}{a_1^{\alpha_{i1}}\cdot ... \cdot a_k^{\alpha_{ik}}}, \quad \boxed{i = 1, ..., m,} \quad [\Pi_i] = 1$$

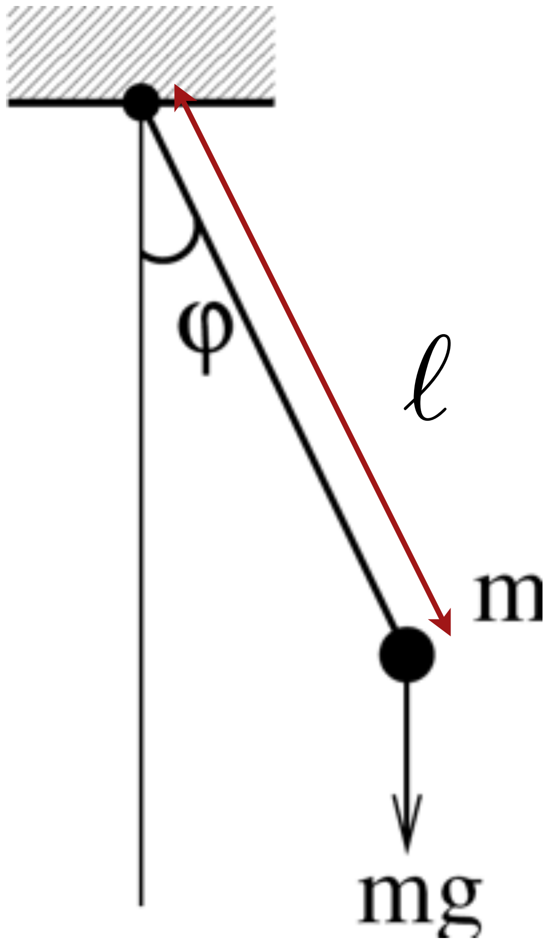
Buckingham Theorem : jede physikalische Relation hat die Form $\boxed{F(\Pi_1, \Pi_2, ..., \Pi_m) = 0}$

$$\Pi_1 = f(\Pi_2, ..., \Pi_m)$$

Die Tatsache, dass wenn eine bestimmte physikalische Grösse in zwei verschiedenen Einheitensystemen gemessen wird, diese nicht nur unterschiedliche numerische Werte sondern auch unterschiedliche Dimension hat, ist oft als Inkonsistenz interpretiert worden, die einer Erklärung bedarf, und Anlass gegeben hat, nach der "wahren" Dimension der physikalischen Grössen zu fragen. Nach dieser Diskussion ist klar, dass diese Frage nicht mehr Sinn hat als nach dem "wahren" Namen eines Gegenstands zu fragen.

Max Planck

Das mathematische Pendel



Relevante Grössen $\varphi,\ \varphi_0, t,\ m,\ \ell,\ g$

$$[t] = \mathsf{T},\ [\varphi_0] = 1,\ [\ell] = \mathsf{L},\ [g] = \mathsf{LT}^{-2},\ \cancel{[m] = \mathsf{M}}$$

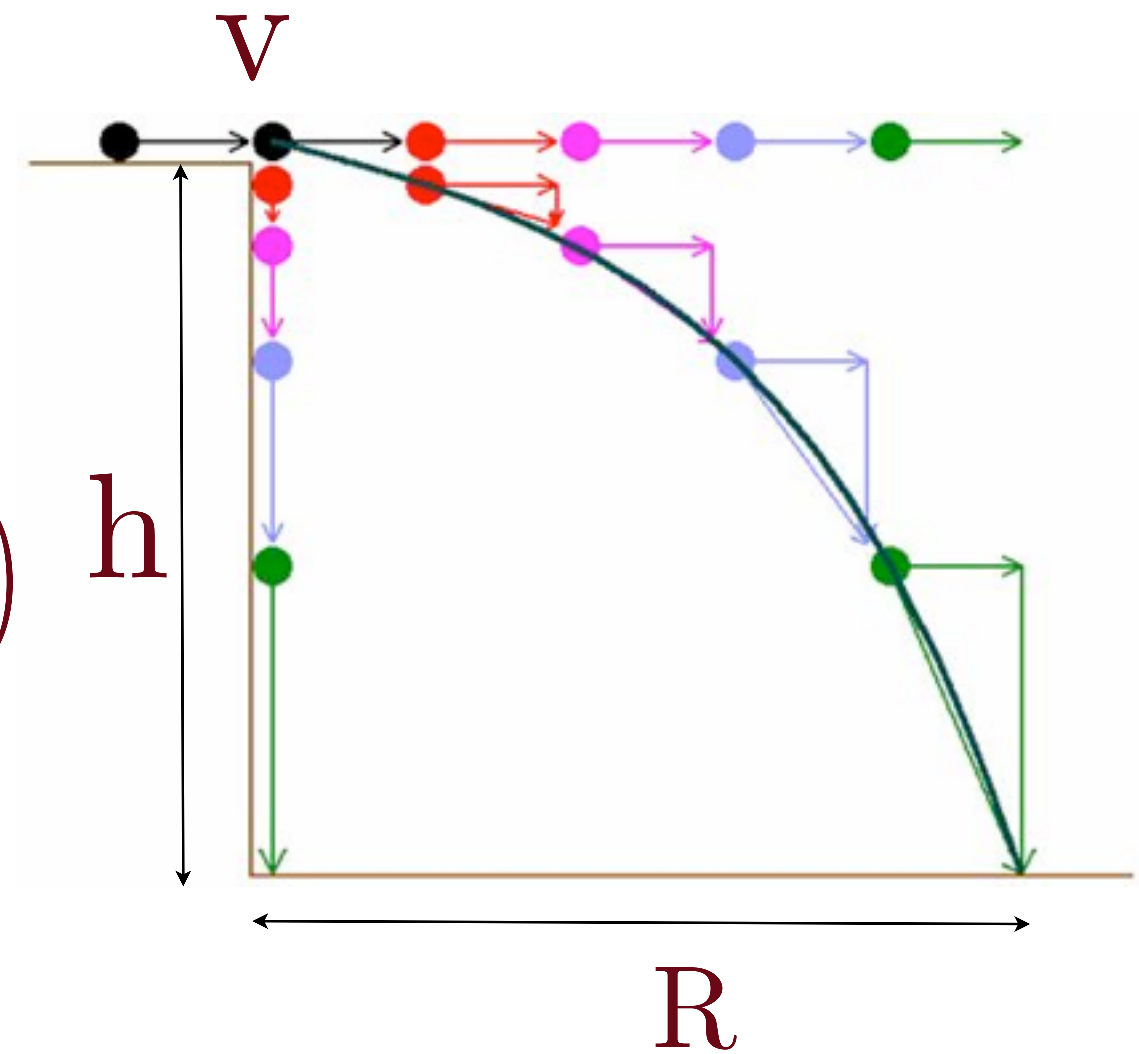
Es existiert keine dimensionslose Grösse, die m enthält!

$$\rightarrow \qquad \Pi_1 = \frac{gt^2}{\ell}, \quad \Pi_2 = \varphi_0, \quad \Pi_3 = \varphi$$

$$\boxed{\begin{gathered}\varphi(t) = f\left(\varphi_0, t\sqrt{g/\ell}\right) \\ \\ \tau = \sqrt{\ell/g}\,\theta(\varphi_0)\,.\end{gathered}}$$

$$\theta(\varphi_0) = 4K\left(\sin\frac{\varphi_0}{2}\right),\ \ K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}$$

Flugweite eines Projektils



$$[R] = [h] = \mathsf{L},\ [v] = \mathsf{LT}^{-1},\ [g] = \mathsf{LT}^{-2}$$

$$\Pi_1 = \frac{h}{R}, \quad \Pi_2 = \frac{gh}{v^2} \quad \rightarrow \quad R = h\,f\left(\frac{gh}{v^2}\right)$$

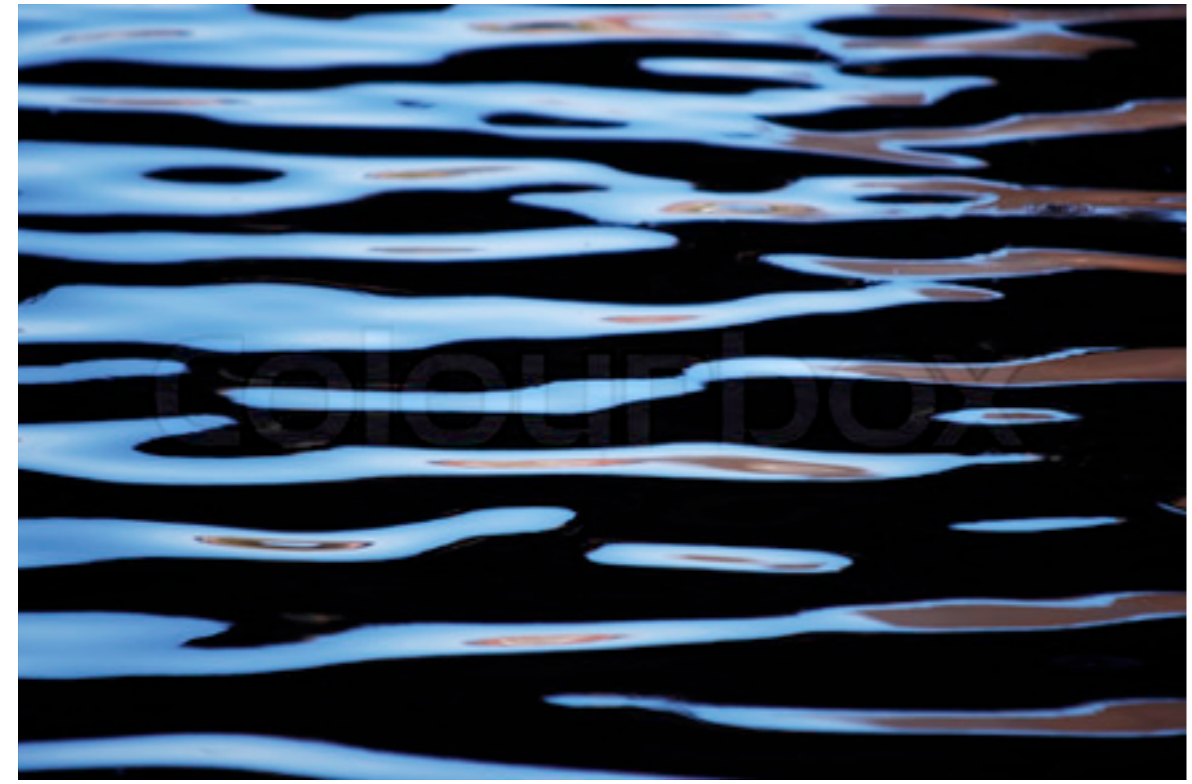
$$k = 2, \quad m = 2$$

$$[R] = \mathsf{L}_x,\ [h] = \mathsf{L}_z,\ [v] = \mathsf{L_x T}^{-1},\ [g] = \mathsf{L_z T}^{-2}.$$

$$a_1 = R, \quad a_2 = v, \quad a_3 = h, \quad b_1 = g \quad \rightarrow \quad \Pi = \frac{gR^2}{v^2 h} \quad \boxed{\rightarrow R \sim v\sqrt{h/g}}$$

$$k = 3, \quad m = 1$$

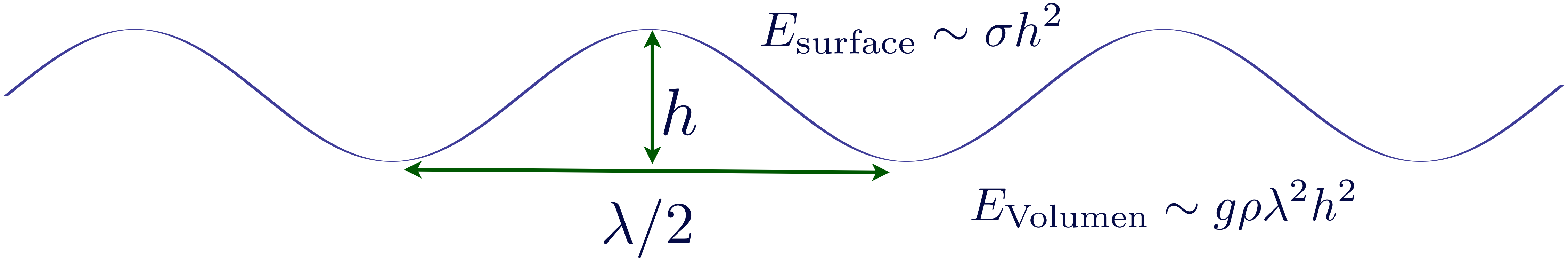
Oberflächenwellen



$[\rho] = \mathsf{ML}^{-3}, \quad [g] = \mathsf{LT}^{-2}, \quad [k] = \mathsf{L}^{-1}, \quad [\omega] = \mathsf{T}^{-1}, \quad [\sigma] = \mathsf{MT}^{-2}$

$$\rightarrow \quad \Pi_1 = \frac{gk}{\omega^2}, \qquad \Pi_2 = \frac{\sigma k^2}{\rho g} \quad \rightarrow$$

$$\boxed{\omega = \sqrt{gk}\, f\left(\frac{\sigma k^2}{\rho g}\right)}$$



$$E_{\text{surface}} \sim \sigma h^2$$

$$E_{\text{Volumen}} \sim g\rho\lambda^2 h^2$$

short wave length: surface tension dominant

long wave length: gravity dominant

exact result Lord Kelvin 1871:

$$f(x \gg 1) \sim x^{1/2} \quad \rightarrow \quad \omega \sim \left(\frac{\sigma k^3}{\rho}\right)^{1/2}$$

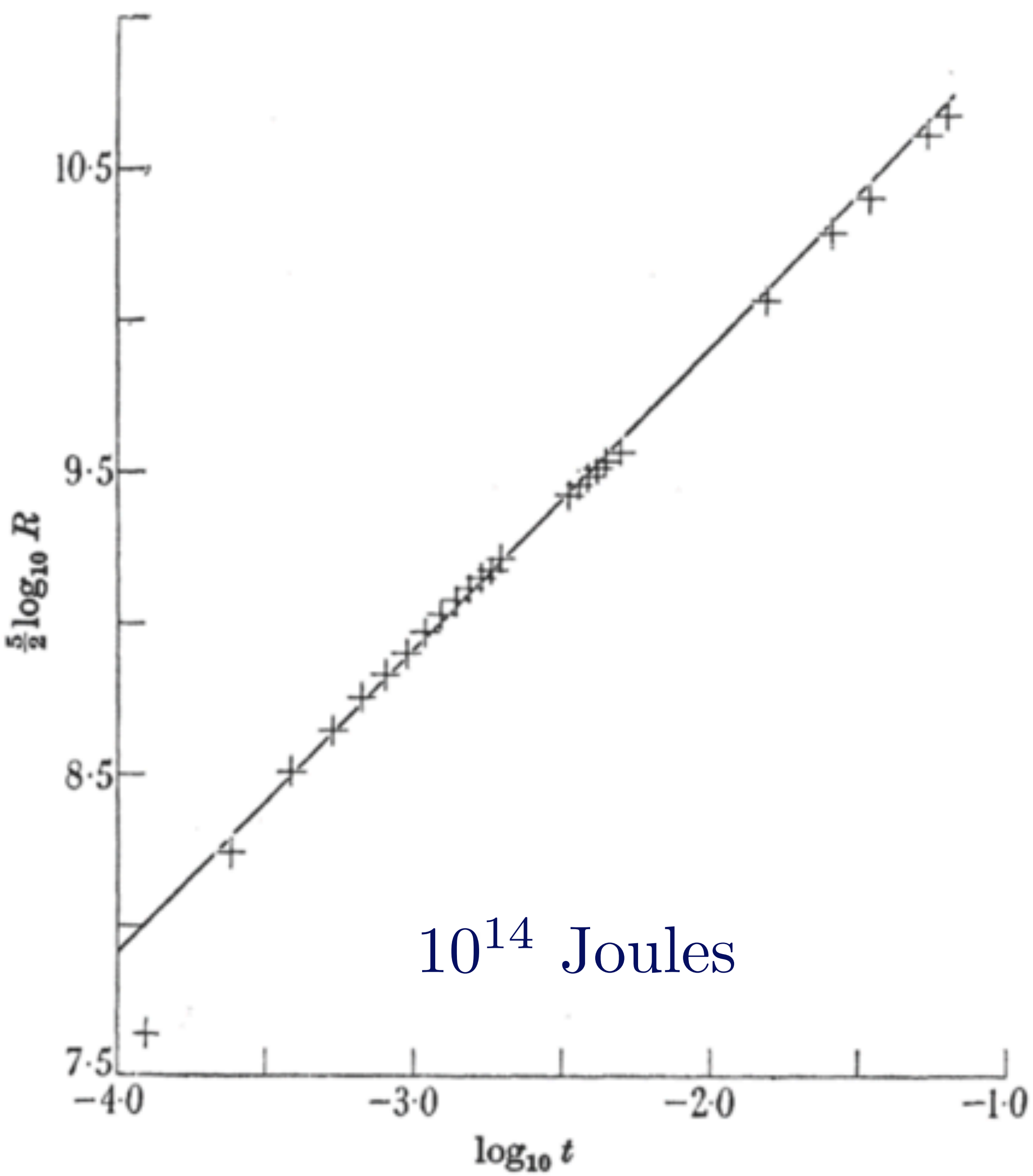
$$f(x \ll 1) \quad \rightarrow \quad \text{constant}$$

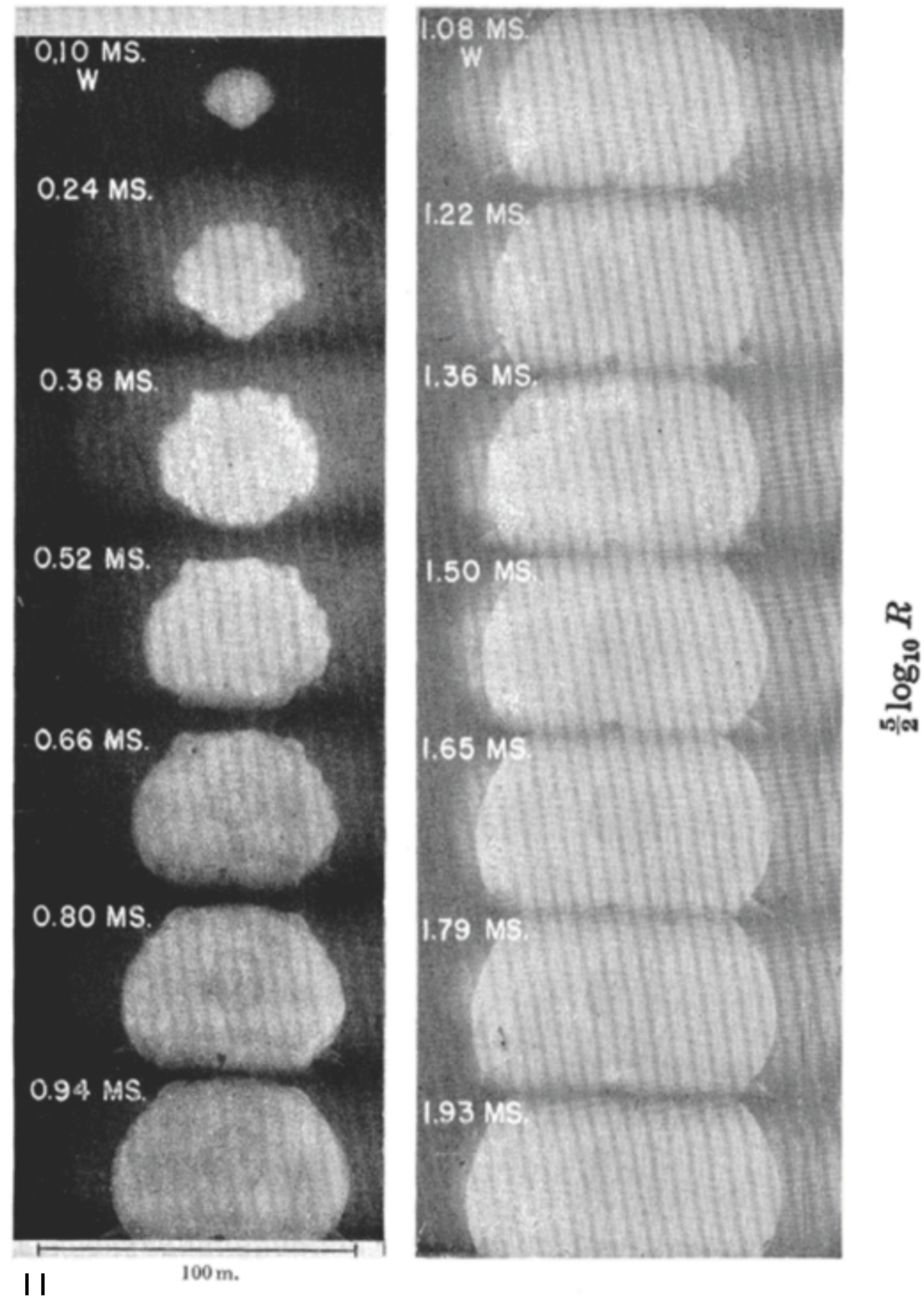
$$f(x) = \sqrt{1 + x}$$

Taylor's blast, 1947

$$[E] = \mathrm{ML}^2\mathrm{T}^{-1}$$
$$[\rho] = \mathrm{ML}^{-3}$$
$$[t] = \mathrm{T}$$
$$[R] = \mathrm{L}$$

$$\boxed{R \sim \left(\frac{Et^2}{\rho}\right)^{1/5}}$$



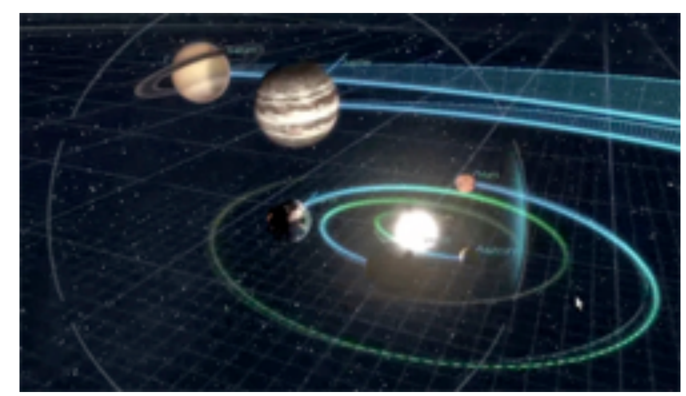


Elektrodynamik

"Einheiten sind eine kulturelle Angelegenheit, keine wissenschaftliche. Jedes System das E und B verschiedene Einheiten gibt, obwohl sie durch eine relativistische Transformation verbunden sind, ist jenseits aller Vernunft."

Val. F. Fitch

quantity	symbol	dimension Gaussian class	dimension SI class
vacuum permittivity	ϵ	1	$\mathsf{I}^2\mathsf{T}^4\mathsf{M}^{-1}\mathsf{L}^{-4}$
vacuum permeability	μ	1	$\mathsf{MLI}^{-2}\mathsf{T}^{-2}$
charge	q	$\mathsf{M}^{1/2}\mathsf{L}^{3/2}\mathsf{T}^{-1}$	IT
electric field	$\mathbf{E}$	$\mathsf{M}^{1/2}\mathsf{L}^{-1/2}\mathsf{T}^{-1}$	$\mathsf{MLT}^{-3}\mathsf{I}^{-1}$
magnetic induction	$\mathbf{B}$	$\mathsf{M}^{1/2}\mathsf{L}^{-1/2}\mathsf{T}^{-1}$	$\mathsf{MT}^{-2}\mathsf{I}^{-1}$
current	$\mathcal{J}$	$\mathsf{M}^{1/2}\mathsf{L}^{3/2}\mathsf{T}^{-2}$	I
voltage	U	$\mathsf{M}^{1/2}\mathsf{L}^{1/2}\mathsf{T}^{-1}$	$\mathsf{ML}^2\mathsf{T}^{-3}\mathsf{I}^{-1}$
conductance	G	LT^{-1}	$\mathsf{T}^3\mathsf{I}^{2\mathsf{M}^{-1}\mathsf{L}^{-2}}$
inductance	L	$\mathsf{T}^2\mathsf{L}^{-1}$	$\mathsf{ML}^2\mathsf{T}^{-2}\mathsf{I}^{-2}$
capacitance	C	L	$\mathsf{T}^4\mathsf{I}^2\mathsf{M}^{-1}\mathsf{L}^{-2}$



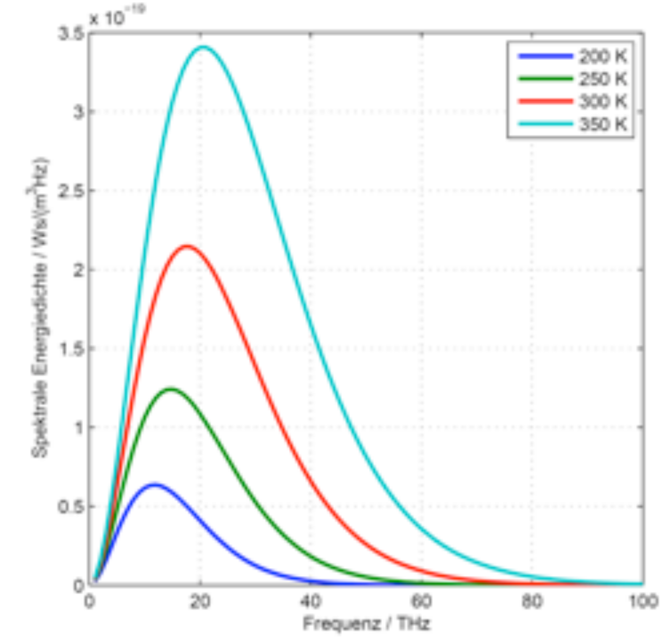
Kapazitat einer Kugel vom Radius r ist r, 1 Farad = $9 \times 10^9 m$

Drude Leitwert=1/Widerstand $\boxed{G = \frac{e^2}{\pi\hbar}\frac{\ell}{L}N}$

abgestrahlte Energie pro Zeiteineheit einer bewegten Ladung: q, c, $|\ddot{\mathbf{r}}|$ $\dot{E}$

$$\boxed{\dot{E} = \mathcal{C}\frac{(q\ddot{\mathbf{r}})^2}{c^3}}$$

Die Plancksche Konstante $\hbar$



spektrale Energiedichte (Wien 1893}

$$u(\omega, k_B T, c) \sim \frac{k_B T \omega^2}{c^3} \qquad U = \int_0^\infty u(\omega) d\omega \quad \to \infty$$

$$[\mathcal{U}] = \mathsf{ML}^{-1}\mathsf{T}^{-2}, \quad [c] = \mathsf{LT}^{-1}, \quad [k_B T] = \mathsf{ML}^2\mathsf{T}^{-2}$$

$\to$ Es its unmöglich $\mathcal{U}$ nur durch $k_B T$ und c auszudrücken!

$$\to \quad \mathcal{U} \sim k_B T c^{-3} \omega_0^3 \quad [\omega_0] = \mathsf{T}^{-1}$$

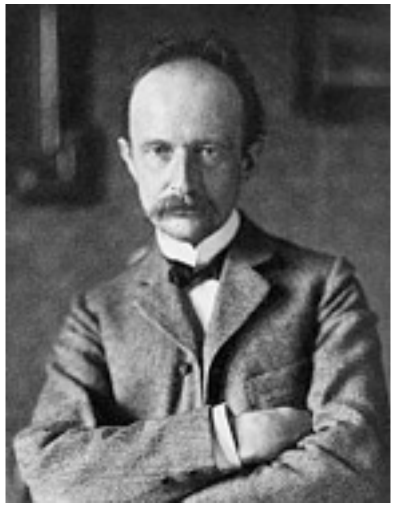
follows with $\omega_0 = k_\mathrm{B} T / h, \quad [h] = \mathsf{ML}^{-2}\mathsf{T}^{-1}$

$$h = \left(\frac{c^2 k_B^4}{4\sigma} \right)^{1/3}$$

Stefan-Boltzmann Gesetz (1879)

$$\mathcal{U} = \frac{4\sigma}{c} T^4, \qquad \sigma = 5.67 \times 10^{-8} \mathrm{kg\ s}^{-3}\mathrm{K}^{-4}$$

$$\boxed{\to \quad u(\omega) = \frac{k_B T \omega^2}{\pi^2 c^3} \Phi\left(\frac{h\omega}{k_B T}\right) \qquad \begin{array}{r} \Phi(x \to 0) = 1 \\ \Phi(x \to \infty) \to 0 \end{array}}$$

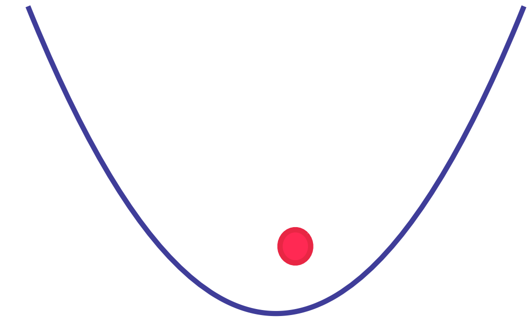


$$\Phi(x) = \frac{x}{e^x - 1}$$

Teilchen im r^k Potenzial

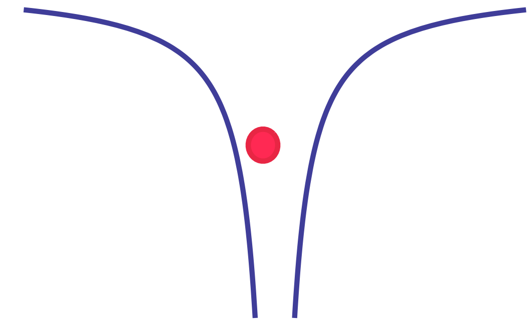
$$\mathcal{H} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \frac{u}{k}\left|\frac{\mathbf{r}}{a}\right|^k$$

k=2 harmonischer Oszillator



$$[E] = \mathsf{ML^2T^{-2}}, \quad \left[\frac{\hbar^2}{m}\right] = \mathsf{ML^4T^{-2}}, \quad \left[\frac{u}{ka^k}\right] = \mathsf{ML^{2-k}T^{-2}}$$

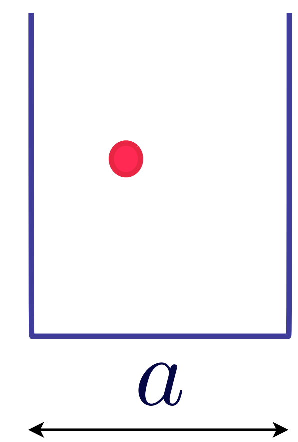
k=-1 Coulomb Problem



Ansatz

$$E \sim \left(\frac{\hbar^2}{m}\right)^\alpha \left(\frac{u}{ka^k}\right)^\beta \quad \to \quad \alpha = \frac{k}{2+k}, \quad \beta = \frac{2}{2+k}$$

k-> ∞ unendlich hoher Potenzialtopf



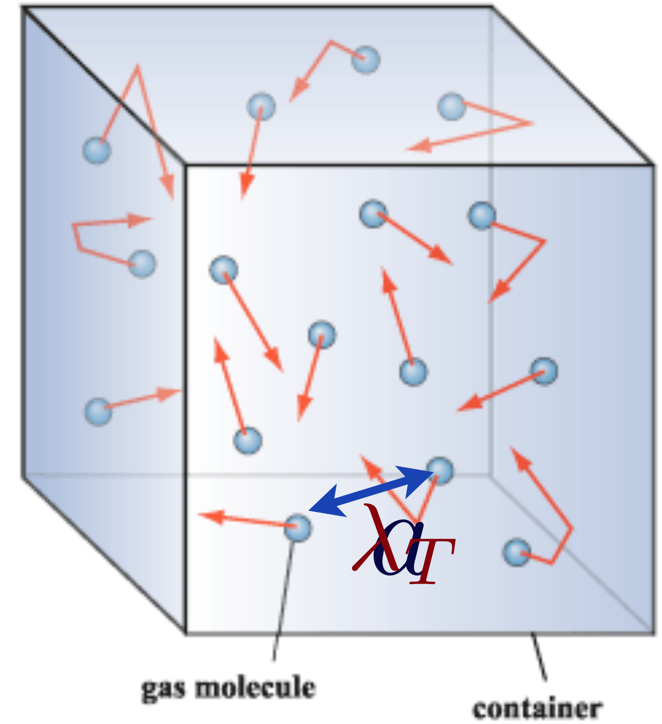
$$E_0 \sim \frac{u}{k}\left(\frac{\hbar^2 k}{2ma^2u}\right)^{\frac{k}{2+k}} \quad \to \quad \boxed{E_n \sim \frac{u}{k}\left(\frac{n^2\hbar^2 k}{2ma^2u}\right)^{\frac{k}{2+k}}}$$

$$k = 2: \quad \omega = \sqrt{\frac{u}{ma^2}} \quad \to \quad E_n \sim n\hbar\omega$$

$$k = -1: \quad ua = Ze^2, \quad \to \quad E_n \sim -\frac{2mZ^2e^4}{(n+1)^2\hbar^2}$$

$$k \to \infty: \quad \to \quad E_n \sim \frac{n^2\hbar^2}{2ma^2}$$

Ideale Gase



1. Ideales klassisches Gas : $m, \quad k_B T, \quad \frac{N}{V} = n = \frac{1}{a^3}$

$$[m] = \mathsf{M}, \quad [k_B T] = \mathsf{ML^2T^{-2}}, \quad [n] = \mathsf{L^{-3}}, \quad [p] = \mathsf{ML^{-1}T^{-2}}$$

$$\Pi = \frac{k_B T N}{pV} \quad \rightarrow \quad \boxed{p = k_B T \frac{N}{V}}$$

2. Ideales Quantengas $\quad [\hbar] = \mathsf{ML^2T^{-1}} \quad \Pi_2 = \frac{m k_B T V^{2/3}}{N^{2/3}\hbar^2} \equiv \frac{a^2}{\lambda_T^2} \quad \rightarrow \quad \boxed{p = k_B T \frac{N}{V} f(\Pi_2)}$

Fermionen: Druck durch Pauliverbot selbst bei T=0 $\rightarrow \quad f(\Pi_2) \sim \Pi_2^{-1} \quad \rightarrow \quad \boxed{p \sim \frac{\hbar^2 n^{5/3}}{m} \sim E_F n}$

Bosonen: bei T=0 sind alle Teilchen im Grundzustand -> Druck hängt nicht mehr vom Volumen ab

$$\rightarrow \quad f(\Pi_2) \sim \Pi_2^{3/2} \quad \rightarrow \quad \boxed{p \sim \frac{(k_B T)^{5/3} m^{3/2}}{\hbar^3} \sim \frac{k_B T}{\lambda_T^3}}$$

Zahl der Teilchen im Kondensat: $\quad N_0 = N - \frac{V}{\lambda_T^3} = N\left(1 - \frac{a^3}{\lambda_T^3}\right) = N\left(1 - \frac{T^{3/2}}{T_c^{3/2}}\right), \quad T_c \sim \frac{\hbar^2}{m a^2}$

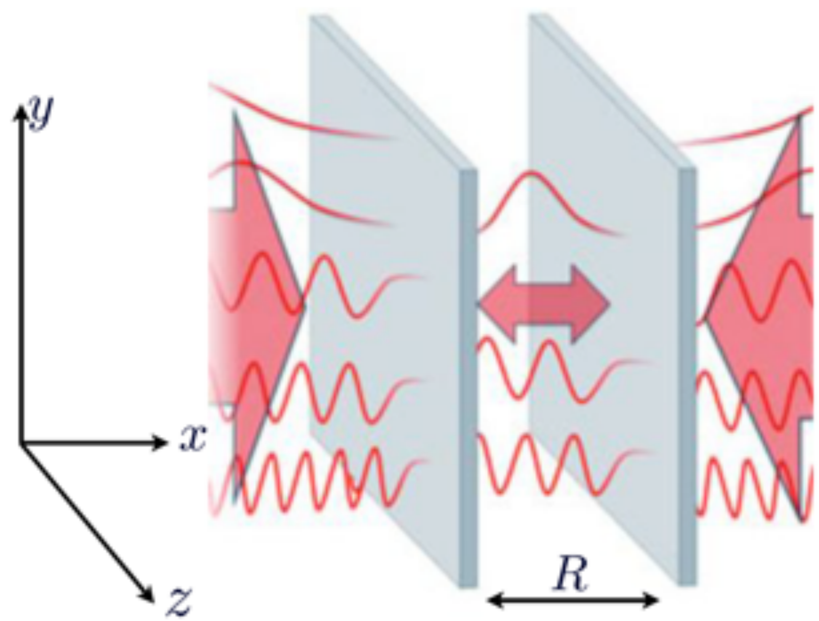
Quantenelektrodynamik

$$\left[\widehat{E}_{\alpha\mathbf{k}}, \widehat{B}_{\beta\mathbf{k}'}\right]_- = \hbar c \epsilon_{\alpha\beta\gamma} k_\gamma \delta_{\mathbf{k},\mathbf{k}'}$$

Grunzustandsenergie des elektromagnetischen Feldes

$$\Delta\mathbf{E}_\mathbf{k}^2 + \Delta\mathbf{B}_\mathbf{k}^2 > \Delta\mathbf{E}_\mathbf{k}^2 + \frac{(\hbar c k)^2}{\Delta\mathbf{E}_\mathbf{k}^2} > \hbar c k \quad \rightarrow \quad E_{GZ} = \sum_\mathbf{k} \hbar c k$$

Casimir-Kraft



'I mentioned my results to Niels Bohr, during a walk. That is nice, he said, that is something new. I told him that I was puzzled by the extremely simple form of the expressions for the interaction at very large distances and he mumbled something about zero-point energy. That was all, but it put me on a new track''. H.G.Casimir

$$[\hbar c] = \mathsf{ML}^3\mathsf{T}^{-2}, \quad [R] = \mathsf{L}, \quad [p] = \mathsf{ML}^{-1}\mathsf{T}^{-2} \quad \rightarrow \quad p = \mathcal{C}\frac{\hbar c}{R^4}, \quad \mathcal{C} = \frac{\pi^2}{240} \approx 0.04$$

Search for Simplicity

From Kepler's Laws to Asymptotic Freedom

Turbulenz

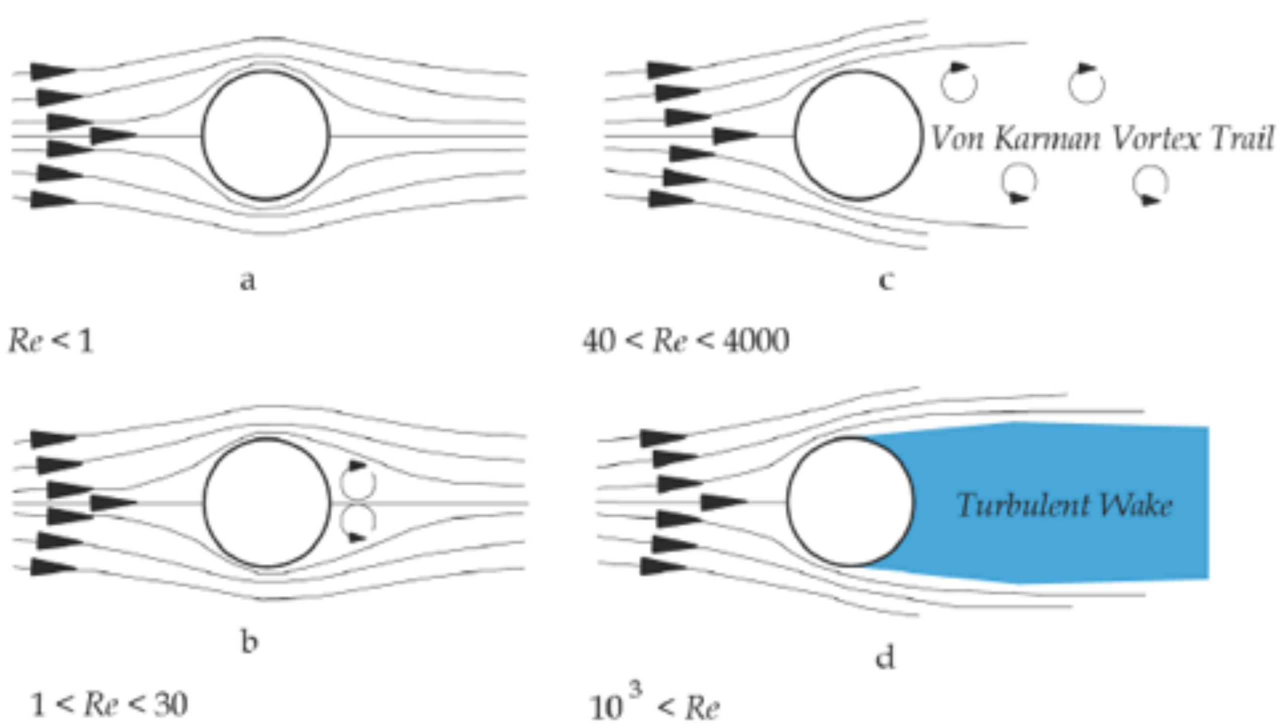
$$\mathcal{R}e = \frac{u_{l_0} l_0}{\nu}$$

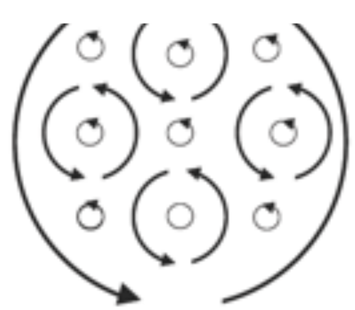
Navier-Stokes Gleichung

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \cancel{-\frac{1}{\rho}\boldsymbol{\nabla} p + \mathbf{g}} + \nu \Delta \boldsymbol{v}.$$

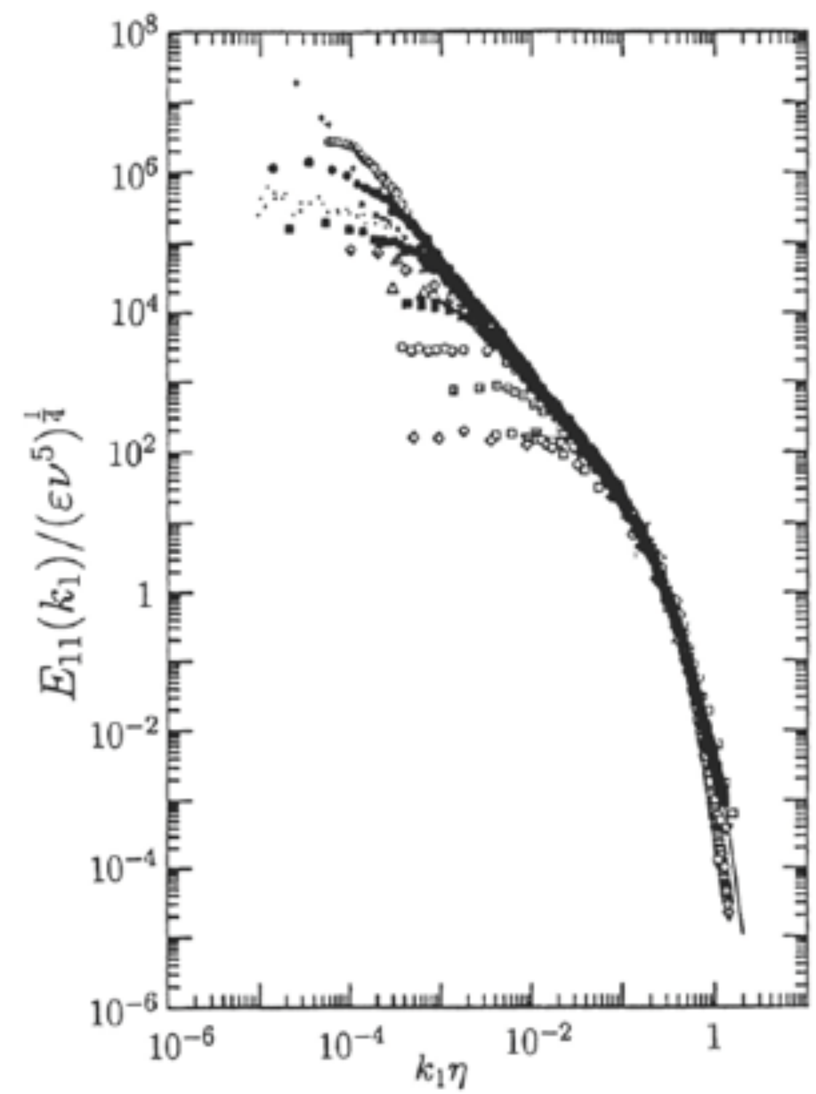
$$\dot{\mathcal{E}}_\nu = \frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}^2}{2} - \mathbf{g} \cdot \mathbf{r}\right) = -\frac{1}{2}\nu v_{\alpha\beta}^2$$

object	Reynolds number $\mathcal{R}e$
ideal fluid	∞
airplane	$10^6 - 10^7$
swimming human	10^4
goldfish	10^2
bacteria	$10^{-4} \ldots 10^{-5}$
virus	10^{-7}
earth mantle	10^{-9}

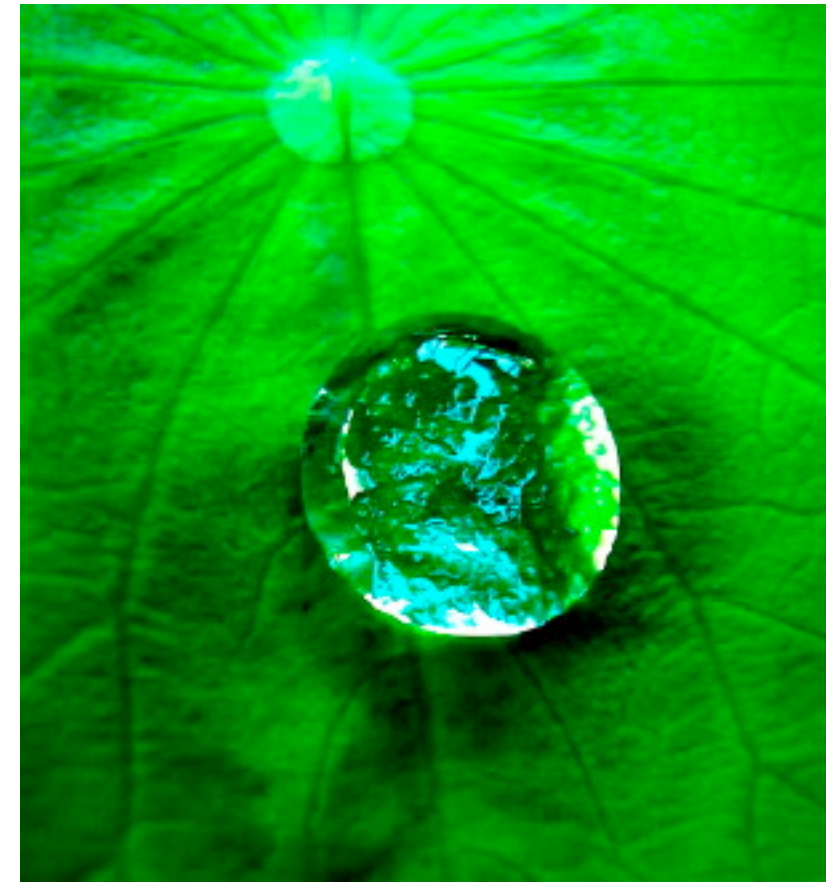




$$\dot{\mathcal{E}} \sim \frac{u_l^3}{l} \quad \text{or} \quad u_l \sim (\dot{\mathcal{E}}\, l)^{1/3}$$



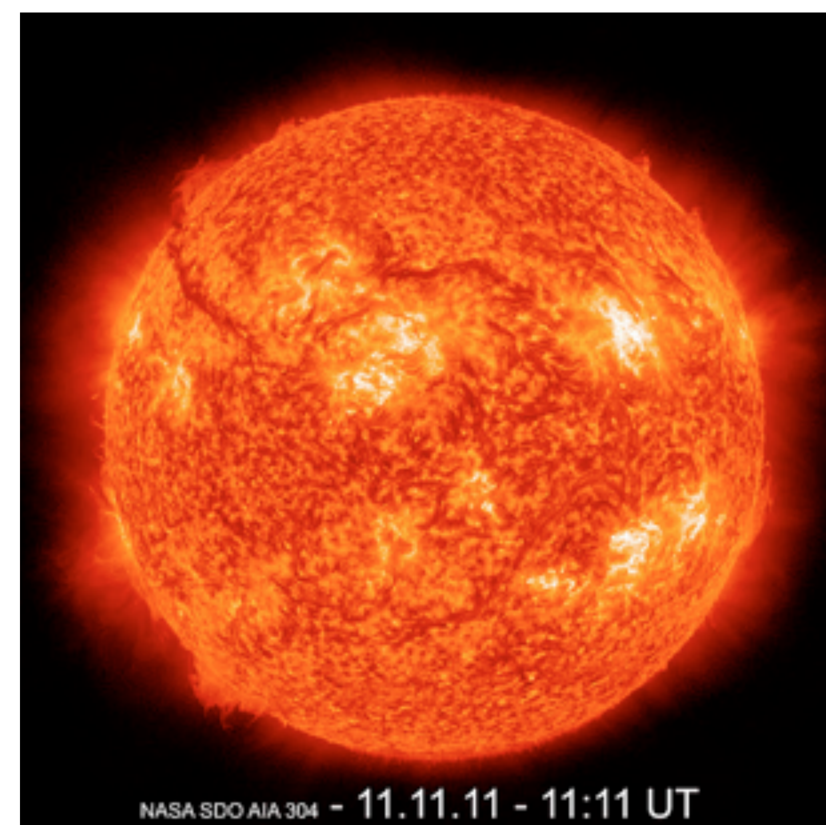
The oscillation frequency of a water droplet and a star



$$[\sigma] = \mathsf{MT}^{-2}, \quad [\rho] = \mathsf{ML}^{-3}, \quad [r] = \mathsf{L}$$

$$\omega \sim \sqrt{\frac{\sigma}{\rho r^3}}$$

With $\sigma \approx 0.07$ kg/(ms^2) one gets a frequency of 3 Hz



$\omega \sim \sqrt{G_N \rho}$ independent of its radius r!

$$\rho_{\text{sun}} \approx 1.4 \times 10^3 \text{ kg m}^{-3} \qquad \omega_{\text{sun}}^{-1} \approx 1\text{h}$$

$$\rho_{\text{n-star}} \approx 7 \cdot 10^{11} \text{kg m}^{-3} \quad \omega_{\text{n-star}}^{-1} \approx 10^{-4}\text{s}$$