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Advanced Quantum Mechanics

Exercise sheet 0

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml
Due date: This sheet is discussed in the tutorials on Wednesday, **October 8th**, 2014.

0. General remarks

- The weekly exercise sheets are made accessible on the course webpage each Monday and must be handed in the following Monday **before 10 am**. No paper copies are provided in the lectures.
- You are allowed to hand in in groups if all of the group members are in the same tutorial. Make sure to mark your solutions clearly with your name and the number of your tutorial. Please do not hand in separate sheets but staple them!
- The solutions to the exercise sheets are discussed in the tutorials. There will be no solutions available online.
- Each exercise sheet gives 20 points. To be admitted to the final exam, you must have at least 50 % of the total number of points (most likely 240) and must have presented the solution to at least one exercise in the tutorials.

1. Harmonic oscillator

The Hamiltonian of the one-dimensional harmonic oscillator is given by

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{m\omega^2}{2}\mathbf{x}^2 = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right) \tag{1}$$

with

$$\mathbf{x} = \frac{\alpha}{\sqrt{2}}\left(\mathbf{a} + \mathbf{a}^\dagger\right), \quad \mathbf{p} = \frac{\hbar}{i\alpha\sqrt{2}}\left(\mathbf{a} - \mathbf{a}^\dagger\right) \tag{2}$$

and $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. The oscillator length is $\alpha = \sqrt{\hbar/(m\omega)}$. The stationary states are given by $|n\rangle = \frac{1}{\sqrt{n!}}(\mathbf{a}^\dagger)^n|0\rangle$ with $n = 0, 1, 2, 3...$, and the eigenenergies read $\varepsilon_n = \hbar\omega(n + \frac{1}{2})$.

a) Compute the expectation values of the operators $\mathbf{p}, \mathbf{x}, \mathbf{p}^2$ und $\mathbf{x}^2$ in the state $|n\rangle$. Compute the uncertainty product $\Delta x \Delta p$ with respect to the same state. For which n is the uncertainty product minimal?

b) Consider the state $|\Psi(t)\rangle$. At $t = 0$, it is given by $|\Psi(0)\rangle = C\left(2|0\rangle + |1\rangle\right)$. First, determine the normalization constant C so that $\langle\Psi(0)|\Psi(0)\rangle = 1$. What are the expectation values $\langle\mathbf{x}\rangle(t)$ and $\langle\mathbf{p}\rangle(t)$ as a function of t? Show that they obey the Ehrenfest theorem.

c) Determine the representation of the wavefunction in real space, $\Psi(x,t) = \langle x|\Psi(t)\rangle$? Sketch the probability density $|\Psi(x,t)|^2$ qualitatively at time $t = 0$ and $t = T/2$ where T is the oscillation period.

2. Two-particle wavefunction

Two identical free particles possess finite momenta p_1 and p_2 along the x-axis. Determine the two-particle wavefunction $\phi(x_1, x_2)$ for

(i) bosonic particles

(ii) fermionic particles.

What is the probability density $|\phi(x_1, x_2)|^2$ for the two cases? Discuss, in particular, the limit $x_2 - x_1 \to 0$ and $p_2 - p_1 \to 0$. Compare with the case of two distinguishable particles where $\phi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)}$.