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Advanced Quantum Mechanics

Exercise sheet 5

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **November 10th**, 2014 (10 am, i.e. before the lecture starts)

13. Jordan-Wigner transformation (10 points)

The Jordan-Wigner transformation transforms spin operators into fermionic ones. In the following, we consider spin-$\frac{1}{2}$ particles on a one-dimensional lattice with the Hamiltonian given by

$$\hat{H} = -\sum_{i=1}^{N} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y),$$

where i is the site index and we use periodic boundary conditions $\mathbf{S}_{N+1} = \mathbf{S}_1$. We represent the spin operators by fermionic creation and annihilation operators, $c_i^\dagger$ and c_i respectively, in the following form:

$$\begin{aligned} S_i^z &= c_i^\dagger c_i - \frac{1}{2} \\ S_i^+ &= \left(\prod_{j<i} (1 - 2c_j^\dagger c_j) \right) c_i^\dagger \\ S_i^- &= \left(\prod_{j<i} (1 - 2c_j^\dagger c_j) \right) c_i \, . \end{aligned}$$

a) Show that the spin operators defined above indeed satisfy the correct commutation relations, by using the fermionic commutation relations of the $c_i^\dagger$'s and c_i's.

b) Show that the Hamiltonian takes the form

$$\hat{H} = -\frac{1}{2} \sum_{i=1}^{N} c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + \frac{1}{2} \left(c_N^\dagger c_1 + c_1^\dagger c_N \right) \left((-1)^{\hat{N}} + 1 \right) ,$$

where $\hat{N} = \sum_{i=1}^{N} c_i^\dagger c_i$ counts the number of fermions. The original Hamiltonian commutes with $S^z = \sum_{i=1}^{N} S_i^z$, i.e. $[\hat{H}, S^z] = 0$. How does this conservation law read in terms of the fermionic operators?

c) In order to diagonalize the Hamiltonian, perform a Fourier transform

$$\hat{H} = \sum_k \epsilon(k) c_k^\dagger c_k$$

and determine the eigenenergies $\epsilon(k)$. As the Hamiltonian conserves the fermion number, you can do the Fourier transform separately for even and odd number of fermions in the chain. For even number of fermions, it is a good idea to incorporate the relative minus sign of the terms $c_N^\dagger c_1 + c_1^\dagger c_N$ by defining $c_{N+1} \equiv -c_1$, thus making the Hamiltonian translationally invariant. The resulting anti-periodic boundary conditions can then be taken care of by shifting the allowed momentum values.

14. Kitaev chain (10 points)

a) Fermionic creation and annihilation operators $c^\dagger$ and c can be expressed in terms of two *Majorana fermion* operators η and ξ. The Majorana operators are defined as

$$\eta = c + c^\dagger, \qquad \zeta = \frac{1}{i}(c - c^\dagger).$$

Show that $\eta^\dagger = \eta$ and $\zeta^\dagger = \zeta$. Compute the anticommutators $\{\eta,\eta\}$,$\{\zeta,\zeta\}$, and $\{\eta,\zeta\}$ and show that

$$c = \frac{1}{2}(\eta + i\zeta), \qquad c^\dagger = \frac{1}{2}(\eta - i\zeta).$$

Argue why a Majorana fermion cannot carry electric charge.

b) Consider the following one-dimensional tight-binding Hamiltonian with open boundary conditions:

$$\mathcal{H} = \tau \sum_{j=1}^{N-1} \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}^\dagger + \text{h.c.} \right),$$

where $\tau > 0$ and N is the total number of sites. Introduce two Majorana fermions η_j and ζ_j for each site j as shown above and simplify the Hamiltonian.

c) Now introduce new fermionic annihilation and creation operators by combining Majorana fermions of neighboring sites as:

$$f_j = \frac{1}{2}(\eta_{j+1} + i\zeta_j), \qquad f_j^\dagger = \frac{1}{2}(\eta_{j+1} - i\zeta_j)$$

and show that the Hamiltonian becomes diagonal in these new fermion operators. Determine the ground state and the ground state energy.

d) The Hamiltonian above does not depend on the Majorana operators η_1 and ζ_N. What does that imply for the ground state degeneracy? What would have been the ground state degeneracy if we had chosen periodic boundary conditions instead?