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# Advanced Quantum Mechanics

Exercise sheet 6

*Winter term 2014/15*

**Homepage**: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml
**Due date**: Monday, **November 17th**, 2014 (10 am, i.e. before the lecture starts)

## 15. Dynamics of field operators (8 points)

**a)** In the Heisenberg picture, we can introduce a time-dependent operator $A^H(t)$ by

$$A^H(t) = U_t^\dagger A U_t,$$

where $U_t$ denotes the unitary time-evolution operator. Show that for a time-independent Hamiltonian the time evolution of $A^H(t)$ is given by

$$i\hbar\partial_t A^H(t) = \left[A^H(t), H\right].$$

**b)** Let us now consider the Hamiltonian

$$\begin{aligned} H = &\int d^3r \; -\frac{\hbar^2}{2m}\hat{\psi}^\dagger(\mathbf{r},t)\nabla^2\hat{\psi}(\mathbf{r},t) + U(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) \\ &+ \frac{1}{2}\int d^3r\, d^3r'\, V(\mathbf{r}-\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t), \end{aligned}$$

where $\hat{\psi}^\dagger(\mathbf{r},t)$ and $\hat{\psi}(\mathbf{r},t)$ are fermionic creation and annihilation operators. Show that the full time-evolution of the fermionic annihilation operator is given by

$$\begin{aligned} i\hbar\partial_t\hat{\psi}(\mathbf{r},t) &= \left[\hat{\psi}(\mathbf{r},t), H\right] \\ &= -\frac{\hbar^2\nabla^2}{2m}\hat{\psi}(\mathbf{r},t) + U(\mathbf{r})\hat{\psi}(\mathbf{r},t) + \int d^3r'\, \hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)V(\mathbf{r}'-\mathbf{r})\hat{\psi}(\mathbf{r},t). \end{aligned}$$

Argue why you obtain the same end result also for bosonic operators.

## 16. Quantization of the electrodynamic fields (12 points)

In the lecture, the electrodynamic fields were quantized using bosonic creation and annihilation operators, $\mathbf{a}_{\mathbf{k}\lambda}$ and $\mathbf{a}^{\dagger}_{\mathbf{k}\lambda}$:

$$\vec{A}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_{\mathbf{k}\lambda} \frac{\hat{e}_{\mathbf{k}\lambda}}{\sqrt{c|\mathbf{k}|}} \left( e^{i\mathbf{kr}} \mathbf{a}^{\dagger}_{\mathbf{k}\lambda} + e^{-i\mathbf{kr}} \mathbf{a}_{\mathbf{k}\lambda} \right),$$

$$\vec{E}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar}{V}} \sum_{\mathbf{k}\lambda} i\sqrt{c|\mathbf{k}|}\hat{e}_{\mathbf{k}\lambda} \left( e^{i\mathbf{kr}} \mathbf{a}^{\dagger}_{\mathbf{k}\lambda} - e^{-i\mathbf{kr}} \mathbf{a}_{\mathbf{k}\lambda} \right),$$

$$\vec{B}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_{\mathbf{k}\lambda} \frac{i\mathbf{k} \times \hat{e}_{\mathbf{k}\lambda}}{\sqrt{c|\mathbf{k}|}} \left( e^{i\mathbf{kr}} \mathbf{a}^{\dagger}_{\mathbf{k}\lambda} - e^{-i\mathbf{kr}} \mathbf{a}_{\mathbf{k}\lambda} \right),$$

where $\lambda = 1, 2$ labels the two polarizations, $\hat{e}_{\mathbf{k},\lambda}$ is the polarization vector, and $V$ is the volume of the system

**a)** The momentum $\vec{P}$ of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, respectively, is given by

$$\vec{P} = \frac{1}{4\pi c} \int d^3r \, \vec{E}(\mathbf{r}) \times \vec{B}(\mathbf{r}).$$

Express it in terms of creation and annihilation operators.

**b)** Compute the commutators $[B_i(\mathbf{r}), B_j(\mathbf{r}')]$, $[E_i(\mathbf{r}), E_j(\mathbf{r}')]$, and $[B_i(\mathbf{r}), E_j(\mathbf{r}')]$, where $B_i(\mathbf{r})$ and $E_i(\mathbf{r})$ are the $i = x, y, z$ component of the magnetic and electric field operator, respectively.

Hint: For the last commutator, you can use the following identity

$$\sum_{\lambda} \left(\hat{e}_{\mathbf{k},\lambda}\right)_i \left(\hat{e}_{\mathbf{k},\lambda}\right)_j = \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2},$$

where $\left(\hat{e}_{\mathbf{k},\lambda}\right)_i$ denotes the $i = x, y, z$ component of the polarization vector $\hat{e}_{\mathbf{k},\lambda}$. You are not required to prove this identity.