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Advanced Quantum Mechanics

Exercise sheet 9

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **December 8th**, 2014 (10 am, i.e. before the lecture starts)

22. Klein-Gordon equation (10 points)

The complex Klein-Gordon field is expressed in terms of creation operators as

$$\phi^\dagger(\mathbf{r},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}}} \left(e^{i\omega_{\mathbf{k}}t - i\mathbf{k}\mathbf{r}} c^\dagger_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}t + i\mathbf{k}\mathbf{r}} b_{\mathbf{k}} \right) \tag{1}$$

where $c^\dagger_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ are bosonic creation operators and $\hbar\omega_{\mathbf{k}} = \sqrt{(m_0c^2)^2 + (\hbar\mathbf{k}c)^2}$. The Hamiltonian reads

$$\mathcal{H} = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(c^\dagger_{\mathbf{k}} c_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + 1 \right). \tag{2}$$

The conjugate momentum density to the Klein-Gordon field is given by

$$\pi(\mathbf{r},t) = \partial_t \phi^\dagger(\mathbf{r},t). \tag{3}$$

a) Use the Heisenberg equations of motion to express $\pi(\mathbf{r},t)$ in terms of $c_{\mathbf{k}}$ and $b_{\mathbf{k}}$. Show that the equal-time commutators are given by $[\phi^\dagger(\mathbf{r},t), \phi(\mathbf{r}',t)] = [\pi^\dagger(\mathbf{r},t), \pi(\mathbf{r}',t)] = 0$ and $[\pi(\mathbf{r},t), \phi(\mathbf{r}',t)] = -i\hbar\delta(\mathbf{r}-\mathbf{r}')$.

b) Express the creation operators $c^\dagger_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ in terms of $\phi^\dagger(\mathbf{r},t)$ and $\pi(\mathbf{r},t)$ using the inverse Fourier transform.

c) Using the result of **b)** show that the Hamiltonian can be written as

$$\mathcal{H} = \int d\mathbf{r} \left(\pi^\dagger \pi + c^2 (\boldsymbol{\nabla}\phi^\dagger)(\boldsymbol{\nabla}\phi) + \frac{m_0^2 c^4}{\hbar^2} \phi^\dagger \phi \right). \tag{4}$$

23. Real Klein-Gordon field (4 points)

In the lectures, you considered the complex Klein-Gordon field and derived its second quantized expression. Here, we consider the real Klein-Gordon field, which upon quantization becomes a Hermitian operator $\phi^\dagger(\mathbf{r},t) = \phi(\mathbf{r},t)$.

a) Show that $\phi^\dagger(\mathbf{r},t) = \phi(\mathbf{r},t)$ implies that $\phi(\mathbf{r},t)$ is of the form

$$\phi(\mathbf{r},t) = \sqrt{\frac{\hbar}{V}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_K}} \left(e^{i\omega_{\mathbf{k}}t - i\mathbf{k}\cdot\mathbf{r}} c^\dagger_{\mathbf{k}} + e^{-i\omega_{\mathbf{k}}t + i\mathbf{k}\cdot\mathbf{r}} c_{\mathbf{k}} \right)$$

What consequences has $\phi^\dagger(\mathbf{r},t) = \phi(\mathbf{r},t)$ for the charge of the field operator?

b) Derive the Hamiltonian H and the momentum P in terms of the creation and annihilation operators.

24. Parity operator of the real Klein-Gordon field (6 points)

The parity transformation (i.e. space inversion) is defined by

$$\phi(\mathbf{r},t) \to \mathcal{P}\phi(\mathbf{r},t)\mathcal{P}^{-1} \equiv \eta_P \phi(-\mathbf{r},t) \qquad (*)$$

where the parity operator $\mathcal{P}$ is a unitary operator which leaves the vacuum invariant $\mathcal{P}|0\rangle = |0\rangle$, and $\eta_P = \pm 1$ is called the *intrinsic* parity of the field.

a) Show that the parity transformation leaves the Hamiltonian density $\mathcal{H}$

$$\mathcal{H}(\mathbf{r},t) = \frac{1}{2}\left[(\partial_t\phi)^2 + c^2(\vec{\nabla}\phi)^2 + c^2\mu^2\phi^2\right]$$

invariant and, thus, describes a symmetry of our system.

b) Show that

$$\mathcal{P}|\mathbf{k}_1,\ldots,\mathbf{k}_n\rangle = \eta_P^n|-\mathbf{k}_1,\ldots,-\mathbf{k}_n\rangle,$$

where $|\mathbf{k}_1,\ldots,\mathbf{k}_n\rangle = \prod_{j=1}^n c^\dagger_{\mathbf{k}_j}|0\rangle$.

c) Prove that the parity operator is given by $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2$, where

$$\mathcal{P}_1 = \exp\left[-i\frac{\pi}{2}\sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}}\right], \qquad \mathcal{P}_2 = \exp\left[i\frac{\pi}{2}\eta_P\sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{-\mathbf{k}}\right].$$

To this end, show that $\mathcal{P}_1\mathcal{P}_2$ is a unitary operator that leaves the vacuum invariant and satisfies $(*)$.
Hint: In order to prove $(*)$ it is useful to first derive

$$\mathcal{P}_1 c_{\mathbf{k}} \mathcal{P}_1^{-1} = i c_{\mathbf{k}}, \qquad \mathcal{P}_2 c_{\mathbf{k}} \mathcal{P}_2^{-1} = -i\eta_P c_{-\mathbf{k}}.$$