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Advanced Quantum Mechanics

Exercise sheet 10

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **December 15th**, 2014 (10 am, i.e. before the lecture starts)

25. γ-matrix identities (4 points)

In this exercise, we will derive a number of useful identities for products of γ matrices. These identities hold up independent of the specific representation for the γ matrices (Dirac-Pauli, Weyl, Majorana) that one might want to consider. Using only the anti-commutation relation of the γ matrices $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ show that

a) $$\gamma_\lambda \gamma^\lambda = 4$$

b) $$\gamma_\lambda \gamma^\alpha \gamma^\lambda = -2\gamma^\alpha$$

c) $$\gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\lambda = 4\eta^{\alpha\beta}$$

d) $$\gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\lambda = -2\gamma^\gamma \gamma^\beta \gamma^\alpha$$

e) $$[\gamma^\mu, \sigma_{\alpha\beta}] = -2i\gamma_\alpha {\delta_\beta}^\mu + 2i\gamma_\beta {\delta_\alpha}^\mu$$

where $\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$.

26. Lorentz covariance of the Dirac equation (16 points)

The Dirac equation should have the same form in all inertial frames in order to be a valid relativistic equation. This we want to show in this exercise. In the following, we therefor consider Lorentz transformations

$$x^\mu \to \tilde{x}^\mu = {\Lambda^\mu}_\nu x^\nu.$$

In the new reference frame, the Dirac equation has the form

$$(-i\hbar\gamma^\mu \tilde{\partial}_\mu + m_0 c)\tilde{\psi}(\tilde{x}) = 0.$$

In contrast to x^μ and ∂_μ, spinors are *not* four-vector, i.e. they do not transform as e.g. x^μ shown above. Instead, let us assume that the transformation of the spinor $\psi(x)$ can be described by

$$\tilde{\psi}(\tilde{x}) = S_\Lambda \psi(x) = S_\Lambda \psi(\Lambda^{-1}\tilde{x})$$

with the (yet unknown) 4×4 matrix S_Λ acting on the spinor index only. The matrix S_Λ depends of course on the Lorentz transformation Λ at hand, but is generically not identical to Λ.

a) Show that requiring the Dirac equation to be of the same form in the new reference frame is equivalent to

$$S_\Lambda^{-1} \gamma^\mu S_\Lambda = {\Lambda^\mu}_\nu \gamma^\nu. \tag{1}$$

In the remaining parts of this exercise, we will demonstrate that such a matrix indeed exists by explicitly constructing it for various Lorentz transformations.

b) *Lorentz covariance of the Dirac equation under boosts*
We consider a boost in the z-direction

$$\Lambda_{\mathrm{boost}} = \begin{pmatrix} \cosh(\theta) & 0 & 0 & -\sinh(\theta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\theta) & 0 & 0 & \cosh(\theta) \end{pmatrix}$$

with the rapidity θ satisfying $\tanh\theta = \frac{v}{c}$. For simplicity, we start by considering an infinitesimal Lorentz transformation, i.e. $\frac{v}{c} \ll 1$, for which Λ_{boost} can be expanded to first order as

$$\Lambda_{\mathrm{boost}} \approx \mathbb{1} + \Delta\omega, \qquad \Lambda^{-1}_{\mathrm{boost}} \approx \mathbb{1} - \Delta\omega$$

where $\mathbb{1}$ denotes the 4×4 identity matrix and $\Delta\omega$ is small in the sense that we can neglect terms $\Delta\omega^2$ and higher. A similar expansion is possible for the corresponding S_{boost} matrix with $S_{\mathrm{boost}} \approx 1 + \Delta\tau$ with $\Delta\tau$ small. Determine $\Delta\omega$ and show that the condition (1) is solved by

$$\Delta\tau = -\frac{v}{2c}\gamma^0\gamma^3.$$

Derive the expression for finite transformations by making the ansatz

$$S_{\mathrm{boost}} = \exp[\alpha\gamma^0\gamma^3]$$

and determine α by using (1). Determine also the expression for boosts in the x and y-direction (you don't need to do an explicit calculation, but motivate your result clearly).
Comment: The ansatz for finite boosts is motivated by regarding the finite boost with rapidity θ as N successive infinitesimal boosts with θ/N, where $N \to \infty$, and using

$$\lim_{N\to\infty}\left(1 + \frac{a}{N}\right)^N = e^a.$$

c) *Lorentz covariance of the Dirac equation under rotations*
Let us now consider a rotation around the z-axis

$$\Lambda_{\text{rot}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again, we start by considering infinitesimal rotations, $\Lambda_{\text{rot}} \approx \mathbb{1} + \Delta\omega$ (see part **b)**). Show that the corresponding infinitesimal transformation $S_{\text{rot}} \approx \mathbb{1} + \Delta\tau$ is obtained by choosing

$$\Delta\tau = \frac{\phi}{2}\gamma^1\gamma^2.$$

Derive the expression for finite transformations by making the ansatz

$$S_{\text{rot}} = \exp[\alpha\gamma^1\gamma^2]$$

and determine α by using (1). Determine also the expression for rotations around the x and y-direction (you don't need to do an explicit calculation, but motivate your result clearly).

d) *Lorentz covariance of the Dirac equation under space inversion (parity)*
While part **b)** and **c)** were concerned with continuous transformation, here we consider the discrete parity transformation $\mathcal{P}$:

$$\mathcal{P} : \mathbf{r} \to -\mathbf{r}.$$

Using condition (1) with the specific choice for Λ_P:

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

show that the corresponding matrix S_P is given by $S_P = \gamma^0$.
Comment: What is still missing is the transformation under time reversal. However, time reversal is special because it is not implemented by a linear operator, but also involves a complex conjugation. To be specific, it is described by the following transformations:

$$\begin{aligned} t &\to -t \\ \mathbf{r} &\to \mathbf{r} \\ i &\to -i \\ \psi(t, \mathbf{r}) &\to i\gamma^1\gamma^3\psi^\star(-t, \mathbf{r}). \end{aligned}$$