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Advanced Quantum Mechanics

Exercise sheet 13

Winter term 2014/15

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2014-QM2.shtml

Due date: Monday, **January 19th**, 2015 (10 am, i.e. before the lecture starts)

34. Scattering on a spherical potential well (10 points)

In this exercise, we consider a spherical well with $U(r) = -U_0\theta(R-r)$, where $\theta(x)$ is a step function with $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$ and $U_0 > 0$. The wave function is ϕ-independent and can be expanded as

$$\psi(r,\theta) = \sum_{l=0}^{\infty} \frac{u_l(r)}{r} P_l(\cos(\theta)),$$

where the $u_l(r)$ obey $u_l(r=0) = 0$ and

$$\left[\partial_r^2 - \frac{l(l+1)}{r^2} - V(r) + k^2\right] u_l(r) = 0,$$

where $V(r) = \frac{2mU(r)}{\hbar^2}$ is the effective potential.

a) At low energies, the $l = 0$ (s-wave) channel dominates the scattering amplitude. Compute the $l = 0$ phase shift δ_0 by requiring that u_0 and its derivative are continuous at $r = R$. Show that the $l = 0$ partial cross section becomes

$$\begin{aligned}\sigma_0 &= \frac{4\pi}{k^2}\sin^2(\delta_0) \\ &\to 4\pi R^2 \left(\frac{\tan(K_0 R)}{K_0 R} - 1\right)^2 \text{ in the limit } k \to 0,\end{aligned}$$

where $K_0 = \frac{\sqrt{2mU_0}}{\hbar}$, and compute the scattering length a_0 defined by

$$a_0 = -\lim_{k\to 0} \frac{\delta_0(k)}{k}.$$

b) For $K_0 R = \pi(n + \frac{1}{2})$ the partial cross section and the scattering length diverge and the scattering length also changes its sign. This phenomenon is called **resonant scattering** and originates from the presence of **bound states** in the system. In order to see this, consider bound states with $E < 0$, where the function $u_0(r)$ decays exponentially for $r > R$. Show that bound states can only exist for $K_0 R > \frac{\pi}{2}$ and that the condition $K_0 R = \pi(n + \frac{1}{2})$ marks the emergence of an additional bound state.

35. $1/r^2$ scattering potential (10 points)

Consider the scattering of a particle in three spatial dimensions from the potential

$$V(\mathbf{r}) = \frac{\lambda}{|\mathbf{r}|^2}. \tag{1}$$

The aim of this exercise is the computation of the differential scattering cross-section $\frac{d\sigma}{d\Omega}$.

a) You can solve the scattering problem exactly. Write the Schrödinger equation for the radial part of the wavefunction $u_{k\ell}(r)$. Show that it assumes the form for a free particle

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{\hbar^2 \ell'(\ell'+1)}{2mr^2}\right) u_{k\ell}(r) = \frac{\hbar^2 k^2}{2m} u_{k\ell}(r). \tag{2}$$

Determine ℓ' that is a function of the angular momentum ℓ and the potential strength λ. We know that for large distances $r \to \infty$ the asymptotic behavior of $u_{k\ell}(r)$ is given by $u_{k\ell}(r) \propto \sin(kr - \frac{\pi\ell'}{2})$. This also holds for a non-integer ℓ'. Using this asymptotic behavior of the radial wavefunction, determine the phase shift $\delta_\ell(k)$! The differential scattering cross section in terms of the phase shift is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell(k)} \sin\delta_\ell(k)\mathcal{P}_\ell(\cos\theta)\right|^2. \tag{3}$$

b) In the lectures, you derived the Born approximation for scattering from weak potentials. In the Born approximation the differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 |V_{\mathbf{k_f}-\mathbf{k_0}}|^2, \tag{4}$$

where $\mathbf{k_f}$ and $\mathbf{k}_0$ is the out- and in-going momentum, respectively, with $|\mathbf{k}_0| = |\mathbf{k_f}|$ and $\mathbf{k}_0\mathbf{k_f} = |\mathbf{k}|^2\cos\theta$, and $V_\mathbf{k}$ is the Fourier transform of the scattering potential.

Evaluate the differential scattering cross-section perturbatively using the Born approximation, Eq (4).
Hint #1: $\int_0^\infty dx \frac{\sin x}{x} = \pi/2$.

Now expand the result from part (a) in lowest order in λ and recover the result of the Born approximation.
Hint #2: Use that $\sum_{\ell=0}^{\infty}\mathcal{P}_\ell(x) = \frac{1}{\sqrt{2(1-x)}}$.