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Advanced Quantum Mechanics

Exercise sheet 4

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

Due date: Monday, **November 16th**, 2015 (10 am, i.e. before the lecture starts)

10. Schwinger boson representation (5 points)

The Schwinger boson provides a representation of quantum mechanical spins in terms of bosons. The spin is written in terms of two bosonic operators a and b in the form

$$\hat{S}^+ = a^\dagger b, \;\hat{S}^- = \left(\hat{S}^+\right)^\dagger$$
$$\hat{S}_z = \frac{1}{2}(a^\dagger a - b^\dagger b)\,.$$

a) Show that this definition is consistent with the commutation relations for the spin operator.

b) Derive the constraint on the bosonic Hilbert space that comes from requiring a fixed spin quantum number S.

c) Show that

$$|S,m\rangle = \frac{\left(a^\dagger\right)^{S+m}}{\sqrt{(S+m)!}}\frac{\left(b^\dagger\right)^{S-m}}{\sqrt{(S-m)!}}|\Omega\rangle,$$

with Ω being the vacuum state of the Schwinger bosons, is an eigenstate of $\mathbf{S}^2$ and S_z.

11. Bose condensate wavefunction (5 points)

The ground state of a Bose condensate $|\psi_0\rangle$ is defined by the property $\tilde{\mathbf{c}}_\mathbf{k}|\psi_0\rangle = 0$, where $\tilde{\mathbf{c}}_\mathbf{k} = \mathbf{c}_\mathbf{k} - \frac{\alpha}{\mu}\delta_{\mathbf{k},0}$ and $\tilde{\mathbf{c}}^\dagger_\mathbf{k} = \mathbf{c}^\dagger_\mathbf{k} - \frac{\alpha^*}{\mu}\delta_{\mathbf{k},0}$ are shifted bosonic operators. Determine the normalized ground state wave function.

Hint: Use the ansatz $|\psi_0\rangle = \sum_{n=0}^\infty a_n(\mathbf{c}^\dagger_0)^n|0\rangle$ and determine the coefficients a_n.

12. Jordan-Wigner transformation (10 points)

The Jordan-Wigner transformation transforms spin operators into fermionic ones. In the following, we consider spin-$\frac{1}{2}$ particles on a one-dimensional lattice with the Hamiltonian given by

$$\hat{H} = -\sum_{i=1}^{N} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y),$$

where i is the site index and we use periodic boundary conditions $\mathbf{S}_{N+1} = \mathbf{S}_1$. We represent the spin operators by fermionic creation and annihilation operators, $c_i^\dagger$ and c_i respectively, in the following form:

$$\begin{aligned} S_i^z &= c_i^\dagger c_i - \frac{1}{2} \\ S_i^+ &= \left(\prod_{j<i} (1 - 2c_j^\dagger c_j) \right) c_i^\dagger \\ S_i^- &= \left(\prod_{j<i} (1 - 2c_j^\dagger c_j) \right) c_i \,. \end{aligned}$$

a) Show that the spin operators defined above indeed satisfy the correct commutation relations, by using the fermionic commutation relations of the $c_i^\dagger$'s and c_i's.

b) Show that the Hamiltonian takes the form

$$\hat{H} = -\frac{1}{2} \sum_{i=1}^{N} c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + \frac{1}{2} \left(c_N^\dagger c_1 + c_1^\dagger c_N \right) \left((-1)^{\hat{N}} + 1 \right) ,$$

where $\hat{N} = \sum_{i=1}^{N} c_i^\dagger c_i$ counts the number of fermions. The original Hamiltonian commutes with $S^z = \sum_{i=1}^{N} S_i^z$, i.e. $[\hat{H}, S^z] = 0$. How does this conservation law read in terms of the fermionic operators?

c) In order to diagonalize the Hamiltonian, perform a Fourier transform

$$\hat{H} = \sum_k \epsilon(k) c_k^\dagger c_k$$

and determine the eigenenergies $\epsilon(k)$. As the Hamiltonian conserves the fermion number, you can do the Fourier transform separately for even and odd number of fermions in the chain. For even number of fermions, it is a good idea to incorporate the relative minus sign of the terms $c_N^\dagger c_1 + c_1^\dagger c_N$ by defining $c_{N+1} \equiv -c_1$, thus making the Hamiltonian translationally invariant. The resulting anti-periodic boundary conditions can then be taken care of by shifting the allowed momentum values.