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Advanced Quantum Mechanics

Exercise sheet 5

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

Due date: Monday, **November 23rd**, 2015 (10 am, i.e. before the lecture starts)

13. Fermionic holes (10 points)

a) Fermionic creation and annihilation operators $\mathbf{c}^\dagger_{\mathbf{k}\sigma}$ and $\mathbf{c}_{\mathbf{k}\sigma}$, respectively, are given where $\mathbf{k}$ labels the momentum and $\sigma =\uparrow,\downarrow$. Consider the following operators:
i) energy $\mathcal{H}_0 = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} \mathbf{c}^\dagger_{\mathbf{k}\sigma} \mathbf{c}_{\mathbf{k}\sigma}$ with the dispersion $\varepsilon_{-\mathbf{k}} = \varepsilon_{\mathbf{k}}$,
ii) momentum $\mathbf{K} = \sum_{\mathbf{k}\sigma} \mathbf{k}\, \mathbf{c}^\dagger_{\mathbf{k}\sigma} \mathbf{c}_{\mathbf{k}\sigma}$, spin $\vec{\mathbf{S}} = \sum_{\mathbf{k}\mu,\nu} \frac{1}{2} \mathbf{c}^\dagger_{\mathbf{k}\mu} \vec{\sigma}_{\mu\nu} \mathbf{c}_{\mathbf{k}\nu}$,
iii) charge density $\rho = q\frac{1}{V} \sum_{\mathbf{k}\sigma} \mathbf{c}^\dagger_{\mathbf{k}\sigma} \mathbf{c}_{\mathbf{k}\sigma}$, where V is the volume and q is the charge, and
iv) charge current density $\mathbf{J} = q\frac{1}{V} \sum_{\mathbf{k}\sigma} \mathbf{v}_{\mathbf{k}} \mathbf{c}^\dagger_{\mathbf{k}\sigma} \mathbf{c}_{\mathbf{k}\sigma}$ with the velocity $\mathbf{v}_{\mathbf{k}} = \nabla_{\mathbf{k}} \varepsilon_{\mathbf{k}}$. Show that the momentum and the spin operator assume the same form in terms of creation and annihilation operators of fermionic holes,

$$
\begin{aligned}
\mathbf{h}^\dagger_{\mathbf{k}\uparrow} &= \mathbf{c}_{-\mathbf{k},\downarrow}, & \mathbf{h}_{\mathbf{k}\uparrow} &= \mathbf{c}^\dagger_{-\mathbf{k},\downarrow} \\
\mathbf{h}^\dagger_{\mathbf{k}\downarrow} &= -\mathbf{c}_{-\mathbf{k},\uparrow}, & \mathbf{h}_{\mathbf{k}\downarrow} &= -\mathbf{c}^\dagger_{-\mathbf{k},\uparrow}.
\end{aligned} \tag{1}
$$

b) Bring the energy operator to the form $\mathcal{H}_0 = \text{const.} + \sum_{\mathbf{k}\sigma} \varepsilon^{\text{hole}}_{\mathbf{k}} \mathbf{h}^\dagger_{\mathbf{k}\sigma} \mathbf{h}_{\mathbf{k}\sigma}$ and determine the energy dispersion $\varepsilon^{\text{hole}}_{\mathbf{k}}$.

c) Similarly, bring the charge density to the form $\rho = \text{const.} + q^{\text{hole}} \frac{1}{V} \sum_{\mathbf{k}\sigma} \mathbf{h}^\dagger_{\mathbf{k}\sigma} \mathbf{h}_{\mathbf{k}\sigma}$ and determine the charge q^{hole}.

d) Show that the charge current can be written as $\mathbf{J} = q^{\text{hole}} \frac{1}{V} \sum_{\mathbf{k}\sigma} \mathbf{v}^{\text{hole}}_{\mathbf{k}} \mathbf{h}^\dagger_{\mathbf{k}\sigma} \mathbf{h}_{\mathbf{k}\sigma}$ and express the velocity $\mathbf{v}^{\text{hole}}_{\mathbf{k}}$ as a derivative of the hole energy $\varepsilon^{\text{hole}}_{\mathbf{k}}$.

14. Kitaev chain (10 points)

a) Fermionic creation and annihilation operators $c^\dagger$ and c can be expressed in terms of two *Majorana fermion* operators η and ξ. The Majorana operators are defined as

$$\eta = c + c^\dagger, \qquad \zeta = \frac{1}{i}(c - c^\dagger).$$

Show that $\eta^\dagger = \eta$ and $\zeta^\dagger = \zeta$. Compute the anticommutators $\{\eta, \eta\}$,$\{\zeta, \zeta\}$, and $\{\eta, \zeta\}$ and show that

$$c = \frac{1}{2}(\eta + i\zeta), \qquad c^\dagger = \frac{1}{2}(\eta - i\zeta).$$

Argue why a Majorana fermion cannot carry electric charge.

b) Consider the following one-dimensional tight-binding Hamiltonian with open boundary conditions:

$$\mathcal{H} = \tau \sum_{j=1}^{N-1} \left(c^\dagger_{j+1} c_j + c^\dagger_j c^\dagger_{j+1} + \text{h.c.} \right),$$

where $\tau > 0$ and N is the total number of sites. Introduce two Majorana fermions η_j and ζ_j for each site j as shown above and simplify the Hamiltonian.

c) Now introduce new fermionic annihilation and creation operators by combining Majorana fermions of neighboring sites as:

$$f_j = \frac{1}{2}(\eta_{j+1} + i\zeta_j), \qquad f^\dagger_j = \frac{1}{2}(\eta_{j+1} - i\zeta_j)$$

and show that the Hamiltonian becomes diagonal in these new fermion operators. Determine the ground state and the ground state energy.

d) The Hamiltonian above does not depend on the Majorana operators η_1 and ζ_N. What does that imply for the ground state degeneracy? What would have been the ground state degeneracy if we had chosen periodic boundary conditions instead?