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Advanced Quantum Mechanics

Exercise sheet 6

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

Due date: Monday, **November 30th**, 2015 (10 am, i.e. before the lecture starts)

15. Fermionic Bogoliubov transformation (10 points)

a) Consider fermionic creation and annihilation operators, $\mathbf{c}^\dagger_{\mathbf{k}\sigma}$ and $\mathbf{c}_{\mathbf{k}\sigma}$, respectively, where $\mathbf{k}$ labels the momentum and $\sigma = \uparrow, \downarrow$. New operators are introduced with the help of the transformation

$$\mathbf{d}_{\mathbf{k}\uparrow} = u_{\mathbf{k}} \mathbf{c}_{\mathbf{k}\uparrow} + v_{\mathbf{k}} \mathbf{c}^\dagger_{-\mathbf{k}\downarrow} \tag{1}$$

$$\mathbf{d}_{\mathbf{k}\downarrow} = u_{\mathbf{k}} \mathbf{c}_{\mathbf{k}\downarrow} - v_{\mathbf{k}} \mathbf{c}^\dagger_{-\mathbf{k}\uparrow}, \tag{2}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are real and even functions of $\mathbf{k}$, i.e., $v_{-\mathbf{k}} = v_{\mathbf{k}}$ and $u_{-\mathbf{k}} = u_{\mathbf{k}}$. What are the requirements on $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ so that the new operators $\mathbf{d}^\dagger_{\mathbf{k}\sigma}$ and $\mathbf{d}_{\mathbf{k}\sigma}$ can be identified with fermionic creation and annihilation operators?

b) Consider the Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} \mathbf{c}^\dagger_{\mathbf{k}\sigma} \mathbf{c}_{\mathbf{k}\sigma} - \Delta \sum_{\mathbf{k}} \left(\mathbf{c}^\dagger_{\mathbf{k}\uparrow} \mathbf{c}^\dagger_{-\mathbf{k}\downarrow} + \mathbf{c}_{-\mathbf{k}\downarrow} \mathbf{c}_{\mathbf{k}\uparrow} \right) \tag{3}$$

Use the transformation of part (a) to diagonalize the Hamiltonian, i.e., to write it in the form $\mathcal{H} = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \mathbf{d}^\dagger_{\mathbf{k}\sigma} \mathbf{d}_{\mathbf{k}\sigma} + \text{const.}$. What is the eigenenergy $E_{\mathbf{k}}$?

16. Dynamics of field operators (10 points)

a) In the Heisenberg picture, we can introduce a time-dependent operator $A^H(t)$ by

$$A^H(t) = U_t^\dagger A U_t,$$

where U_t denotes the unitary time-evolution operator. Show that for a time-independent Hamiltonian the time evolution of $A^H(t)$ is given by

$$i\hbar\partial_t A^H(t) = \left[A^H(t), H\right].$$

b) Let us now consider the Hamiltonian

$$\begin{aligned} H = &\int d^3r \, -\frac{\hbar^2}{2m}\hat{\psi}^\dagger(\mathbf{r},t)\nabla^2\hat{\psi}(\mathbf{r},t) + U(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}(\mathbf{r},t) \\ &+ \frac{1}{2}\int d^3r\, d^3r'\, V(\mathbf{r}-\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r},t)\hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)\hat{\psi}(\mathbf{r},t), \end{aligned}$$

where $\hat{\psi}^\dagger(\mathbf{r},t)$ and $\hat{\psi}(\mathbf{r},t)$ are fermionic creation and annihilation operators. Show that the full time-evolution of the fermionic annihilation operator is given by

$$\begin{aligned} i\hbar\partial_t\hat{\psi}(\mathbf{r},t) &= \left[\hat{\psi}(\mathbf{r},t), H\right] \\ &= -\frac{\hbar^2\nabla^2}{2m}\hat{\psi}(\mathbf{r},t) + U(\mathbf{r})\hat{\psi}(\mathbf{r},t) + \int d^3r'\, \hat{\psi}^\dagger(\mathbf{r}',t)\hat{\psi}(\mathbf{r}',t)V(\mathbf{r}'-\mathbf{r})\hat{\psi}(\mathbf{r},t). \end{aligned}$$

Show that you obtain the same end result also for bosonic operators.