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Advanced Quantum Mechanics

Exercise sheet 7

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

Due date: Monday, **December 7th**, 2015 (10 am, i.e. before the lecture starts)

17. Quantization of the electrodynamic fields (12 points)

In the lecture, the electrodynamic fields were quantized using bosonic creation and annihilation operators, $\mathbf{a}_{\mathbf{k}\lambda}$ and $\mathbf{a}^\dagger_{\mathbf{k}\lambda}$:

$$\vec{A}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_{\mathbf{k}\lambda} \frac{\hat{e}_{\mathbf{k}\lambda}}{\sqrt{c|\mathbf{k}|}} \left(e^{i\mathbf{kr}} \mathbf{a}^\dagger_{\mathbf{k}\lambda} + e^{-i\mathbf{kr}} \mathbf{a}_{\mathbf{k}\lambda} \right),$$

$$\vec{E}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar}{V}} \sum_{\mathbf{k}\lambda} i\sqrt{c|\mathbf{k}|}\,\hat{e}_{\mathbf{k}\lambda} \left(e^{i\mathbf{kr}} \mathbf{a}^\dagger_{\mathbf{k}\lambda} - e^{-i\mathbf{kr}} \mathbf{a}_{\mathbf{k}\lambda} \right),$$

$$\vec{B}(\mathbf{r}) = \sqrt{\frac{2\pi\hbar c^2}{V}} \sum_{\mathbf{k}\lambda} \frac{i\mathbf{k} \times \hat{e}_{\mathbf{k}\lambda}}{\sqrt{c|\mathbf{k}|}} \left(e^{i\mathbf{kr}} \mathbf{a}^\dagger_{\mathbf{k}\lambda} - e^{-i\mathbf{kr}} \mathbf{a}_{\mathbf{k}\lambda} \right),$$

where $\lambda = 1, 2$ labels the two polarizations, $\hat{e}_{\mathbf{k},\lambda}$ is the polarization vector, and V is the volume of the system

a) The momentum $\vec{P}$ of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, respectively, is given by

$$\vec{P} = \frac{1}{4\pi c} \int d^3r\, \vec{E}(\mathbf{r}) \times \vec{B}(\mathbf{r}).$$

Express it in terms of creation and annihilation operators.

b) Compute the commutators $[B_i(\mathbf{r}), B_j(\mathbf{r}')]$, $[E_i(\mathbf{r}), E_j(\mathbf{r}')]$, and $[B_i(\mathbf{r}), E_j(\mathbf{r}')]$, where $B_i(\mathbf{r})$ and $E_i(\mathbf{r})$ are the $i = x, y, z$ component of the magnetic and electric field operator, respectively.

Hint: For the last commutator, you can use the following identity

$$\sum_\lambda \left(\hat{e}_{\mathbf{k},\lambda}\right)_i \left(\hat{e}_{\mathbf{k},\lambda}\right)_j = \delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2},$$

where $\left(\hat{e}_{\mathbf{k},\lambda}\right)_i$ denotes the $i = x, y, z$ component of the polarization vector $\hat{e}_{\mathbf{k},\lambda}$. You are not required to prove this identity.

18. Coherent states of the electromagnetic field (8 points)

In this exercise we consider coherent states of the electromagnetic field and derive some of their important properties. Coherent states are eigenstates of the annihilation operator $a_{\mathbf{k},\lambda}$ for fixed $\mathbf{k}$ and λ:

$$a_{\mathbf{k},\lambda}|\phi\rangle = \phi|\phi\rangle$$

for an arbitrary complex number ϕ.

a) Show that $|\phi\rangle$ is a coherent state:

$$|\phi\rangle = c \exp\left(\phi\, a^{\dagger}_{\mathbf{k},\lambda}\right)|\Omega\rangle,$$

where $|\Omega\rangle$ is the photon vacuum. Determine the normalization constant c by requiring the coherent state to be normalized. Compute the overlap of two different coherent states $\langle\theta|\phi\rangle$.

b) Show that the action of the creation operator on a coherent state $|\phi\rangle$ is given by

$$a^{\dagger}_{\mathbf{k},\lambda}|\phi\rangle = (\partial_\phi + \phi^\star/2)|\phi\rangle.$$

c) Show that coherent states form a complete set, i.e. show that

$$\frac{1}{\pi}\int d\phi \int d\bar{\phi}\; |\phi\rangle\langle\phi| = \mathbb{1},$$

where $\int d\phi \int d\bar{\phi}$ denotes the integration over the complex plane and $\mathbb{1}$ is the identity operator.
Hint: One possible (though not the most elegant) way of proofing this identity is to show that the left-hand-side acts as the identity operator on the eigenbasis of the number operator.

d) One important property of coherent states is that they are the closest possible analog to classical states. Compute the expectation values $\langle\phi|\mathbf{E}(\mathbf{r},t)|\phi\rangle$ and $\langle\phi|\mathbf{B}(\mathbf{r},t)|\phi\rangle$ and compare to the behavior of classical electromagnetic fields.