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# Advanced Quantum Mechanics

Exercise sheet 8

*Winter term 2015/16*

**Homepage**: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

**Due date**: Monday, **December 14th**, 2015 (10 am, i.e. before the lecture starts)

## 19. Two fermionic states in a photon-cavity (10 points)

Consider photons in a cavity that are excited with a frequency $\omega$ with creation operator $\mathbf{a}^\dagger$. The photons couple to two fermionic states with creation operators $\mathbf{c}_i^\dagger$ with $i = 0, 1$. The Hamiltonian that describes the interaction between the photons and the fermionic states via their dipole moment is given by

$$\mathcal{H}_{\text{dipole}} = \hbar\omega\mathbf{a}^\dagger\mathbf{a} \; + \frac{1}{2}\hbar\Omega_0(\mathbf{c}_1^\dagger\mathbf{c}_1 - \mathbf{c}_0^\dagger\mathbf{c}_0) + \hbar g(\mathbf{a} + \mathbf{a}^\dagger)\left(\mathbf{c}_1^\dagger\mathbf{c}_0 + \mathbf{c}_0^\dagger\mathbf{c}_1\right). \tag{1}$$

$\hbar\Omega_0$ is the energy difference between the two fermion states.

**a)** Explain, why for $\omega \approx \Omega_0$ it is a good approximation to consider instead the following Hamiltonian:

$$\mathcal{H} = \hbar\omega\mathbf{a}^\dagger\mathbf{a} \; + \frac{1}{2}\hbar\Omega_0(\mathbf{c}_1^\dagger\mathbf{c}_1 - \mathbf{c}_0^\dagger\mathbf{c}_0) + \hbar g\left(\mathbf{a}\,\mathbf{c}_1^\dagger\mathbf{c}_0 + \mathbf{a}^\dagger\,\mathbf{c}_0^\dagger\mathbf{c}_1\right). \tag{2}$$

**b)** Consider the subspace with two fermions and the subspace containing zero fermions. Why do the eigenenergies of $\mathcal{H}$ (and $\mathcal{H}_{\text{dipole}}$) not depend on $\Omega_0$ and $g$ in these two cases?

**c)** Show that the operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a} \; + \mathbf{c}_1^\dagger\mathbf{c}_1$ commutes with the Hamiltonian $\mathcal{H}$.

**d)** Consider the subspace containing a single fermion. Why does it follow from **b)** that the Hamiltonian $\mathcal{H}$ only couples the two states $|0\rangle|n+1\rangle$ and $|1\rangle|n\rangle$ with $n = 0, 1, 2, 3, ...$? Show that within this $2 \times 2$ subspace for a given $n$ the Hamiltonian $\mathcal{H}$ reduces to

$$H = \hbar\omega(n + \frac{1}{2})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \hbar\begin{pmatrix} \delta/2 & g\sqrt{n+1} \\ g\sqrt{n+1} & -\delta/2 \end{pmatrix} \tag{3}$$

with the detuning $\delta = \Omega_0 - \omega$. Determine the eigenenergies of the Hamiltonian $H$.

## 20. Lorentz transformations (6 points)

Consider the Lorentz transformation to a system that moves with relative velocity $v$ in $x$-direction. The corresponding matrix is given by

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

where the rapidity $\eta$ is given by $\tanh\eta = v/c$.

**a)** Determine the coordinates in the moving system: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ where $(x^\nu) = (ct, \vec{r})$ and express the result explicitly in terms of the velocity $v$. The coordinates of the origin in the moving frame are given by $(x_0'^\mu) = (c\,t', 0, 0, 0)$. What are the coordinates in the rest frame?

**b)** In the system at rest, there is a finite electric field pointing along the $z$-direction, $\vec{E} = E_z \hat{e}^z$. What is the corresponding electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$? With the help of $F^{\mu\nu}$ one obtains the electromagnetic tensor in the moving frame as $F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\mu{}_\beta F^{\alpha\beta}$. What are the corresponding electric and magnetic fields in the moving system, $\vec{E}'$ and $\vec{B}'$, respectively.

**c)** Consider two consecutive Lorentz boosts along the $x$-axis first with rapidity $\eta_1$ and afterwards with rapidity $\eta_2$. Show that this corresponds to a single boost with rapidity $\eta = \eta_1 + \eta_2$.

**d)** Consider two consecutive Lorentz boosts first along the $x$-axis with rapidity $\eta_1$ and afterwards along the $y$-axis with rapidity $\eta_2$. Derive the addition formula for adding the two relativistic velocities $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{u} = \mathbf{v}_1 + \sqrt{1 - \frac{\mathbf{v}_1^2}{c^2}}\,\mathbf{v}_2 .$$

## 21. Relativistic electrons (4 points)

In this exercise we consider a relativistic electron. Using Lorentz transformations, we can change the inertial frame that the electron moves in. This will change some of properties of the electron, while leaving other characteristics invariant. Argue why the electric charge as well as the rest mass are identically the same in any inertial frame, while in contrast the mass depends on the choice of inertial frame.