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Advanced Quantum Mechanics

Exercise sheet 12

Winter term 2015/16

Homepage: http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml

Due date: Monday, **January 25th**, 2016 (10 am, i.e. before the lecture starts)

29. Classical scattering from a hard sphere (4 points)

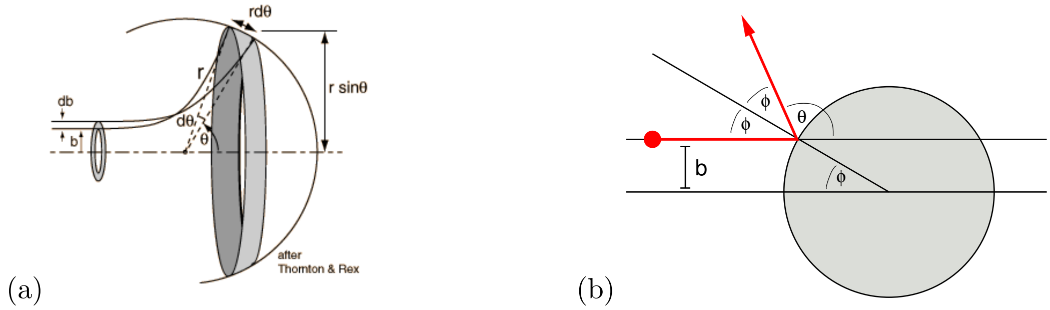

Abbildung 1: (a) Definition of the impact parameter $b(\theta)$ for classical scattering. (b) Classical scattering of a particle from a hard sphere.

a) A constant flux of classical particles, j, i.e., particles per area per unit time is hitting a spherical obstacle. The number of particles ν scattered into the solid angle $d\Omega = 2\pi \sin\theta d\theta$ per unit time per flux j defines the scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\nu}{j}.$$

From the classical trajectories one can relate the scattering angle θ to the impact parameter b, see Fig. 1(a), which defines the function $b(\theta)$. Using that the number of incident particles per unit time in the range b and $b + db$ equals the number of particles scattered between θ and $\theta + d\theta$ per unit time, show that the scattering cross section is given by

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{b(\theta)}{\sin\theta}\left|\frac{db(\theta)}{d\theta}\right|. \tag{1}$$

b) Evaluate the function $b(\theta)$ for the scattering of a particle from a hard sphere with radius R, see Fig. 1(b). Compute the differential cross section using Eq. (1). Verify that the total cross section $\sigma = \int d\Omega \frac{d\sigma(\theta)}{d\Omega}$ is just given by the projected area of the sphere πR^2.

30. Scattering from a central potential (8 points)

In scattering theory, we are usually interested in the behavior far away from the scattering center, where the wave function can be described by a superposition of incoming and outgoing spherical waves:

$$\psi_{l,m,k}(r \to \infty) = -Y_l^m(\theta, \phi) \left(\frac{e^{-ikr+il\pi/2}}{2ikr} - \frac{e^{ikr-il\pi/2+2i\delta_l(k)}}{2ikr} \right) \tag{2}$$

where we used spherical coordinates (r, θ, ϕ), the $Y_l^m(\theta, \phi)$ are the spherical harmonics, and l, m, k label the eigenvalues of $\mathbf{L}^2$, L_z and H respectively. In order to motivate equation (2), especially the choice of the phase factors, we will consider the case without any potential, where the solutions are plane waves.

a) We can expand the plane waves in terms of the angular momentum eigenstates

$$e^{i\mathbf{kr}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m} Y_l^m(\theta, \phi) j_l(kr) \tag{3}$$

with (yet) unknown coefficients $a_{l,m}$. The $j_l(kr)$ are spherical Bessel functions of the first kind that are defined by

$$j_l(x) = (-x)^l \left(\frac{1}{x} \frac{\partial}{\partial x} \right)^l \frac{\sin(x)}{x}. \tag{4}$$

Show that you can choose the coordinate system in such a way that only $m = 0$ terms contribute. Then use the orthogonality of the spherical harmonics to determine the coefficients $a_{l,0}$.

Hint: One way to do this is by using $Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos(\theta))$, where P_l is a Legendre polynomial defined by $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2 - 1)^l$. By doing a partial integration to shift the derivatives onto $e^{ikr\cos(\theta)}$ one recovers the spherical Bessel functions in a Taylor series form (proof this by Tayler expanding $\sin(x)/x$ and performing the derivatives in (4)). The identity

$$\sum_{m=0}^{l} \frac{(-1)^m l!}{m!(l-m)!)} \frac{2}{2m+2n+1} = 2(2l)!! \frac{(2n-1)!!}{2l+2n+1)!!}$$

might be useful.

b) Determine the asymptotic behavior of the spherical Bessel function for large arguments $|x| \to \infty$. This limit is determined by the contribution where the derivatives act on the trigonometric functions only. Show with help of (3) that this implies indeed that plane waves are of the form (2) with $\delta_l(k) = 0$.

Hint: $\left(\frac{d}{dx}\right)^\ell \sin x = (-1)^\ell \sin(x - \pi\ell/2)$.

31. Resonant scattering (8 points)

Consider the one-dimensional stationary Schrödinger equation

$$\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\Big)\psi=\varepsilon\psi,$$

where the potential barrier is only finite in a limited region of width $2x_0$ close to the origin, i.e., $V(x)=0$ for $|x|>x_0$ and $V(x)=V_0$ for $|x|<x_0$.
Scattering states propagating from the left to the right are defined as

$$\psi(x)=\begin{cases} e^{ikx}+r(k)e^{-ikx} & \text{for} \quad x<-x_0 \\ Ae^{i\tilde{k}x}+Be^{-i\tilde{k}x} & \text{for} \quad -x_0<x<x_0 \\ t(k)e^{ikx} & \text{für} \quad x>x_0 \end{cases}$$

where $k=\sqrt{2m\varepsilon}/\hbar$ and $\tilde{k}=\sqrt{2m(\varepsilon-V_0)}/\hbar$. Determine the transmission coefficient $|t|^2$ by requiring that both the wave function $\psi(x)$ and its derivative $\psi'(x)$ are continuous at $x=\pm x_0$. When does the transmission coefficient become maximal?