
Advanced Quantum Mechanics

Exercise sheet 13

Winter term 2015/16

Homepage: <http://www.thp.uni-koeln.de/trebst/Lectures/2015-QM2.shtml>

Due date: Monday, **February 1st**, 2016 (10 am, i.e. before the lecture starts)

32. Scattering on a spherical potential well (10 points)

In this exercise, we consider a spherical well with $U(r) = -U_0\theta(R - r)$, where $\theta(x)$ is a step function with $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$ and $U_0 > 0$. The wave function is ϕ -independent and can be expanded as

$$\psi(r, \theta) = \sum_{l=0}^{\infty} \frac{u_l(r)}{r} P_l(\cos(\theta)),$$

where the $u_l(r)$ obey $u_l(r=0) = 0$ and

$$\left[\partial_r^2 - \frac{l(l+1)}{r^2} - V(r) + k^2 \right] u_l(r) = 0,$$

where $V(r) = \frac{2mU(r)}{\hbar^2}$ is the effective potential.

a) At low energies, the $l = 0$ (s-wave) channel dominates the scattering amplitude. Compute the $l = 0$ phase shift δ_0 by requiring that u_0 and its derivative are continuous at $r = R$. Show that the $l = 0$ partial cross section becomes

$$\begin{aligned} \sigma_0 &= \frac{4\pi}{k^2} \sin^2(\delta_0) \\ &\rightarrow 4\pi R^2 \left(\frac{\tan(K_0 R)}{K_0 R} - 1 \right)^2 \quad \text{in the limit } k \rightarrow 0, \end{aligned}$$

where $K_0 = \frac{\sqrt{2mU_0}}{\hbar}$, and compute the scattering length a_0 defined by

$$a_0 = - \lim_{k \rightarrow 0} \frac{\delta_0(k)}{k}.$$

b) For $K_0 R = \pi(n + \frac{1}{2})$ the partial cross section and the scattering length diverge and the scattering length also changes its sign. This phenomenon is called **resonant scattering** and originates from the presence of **bound states** in the system. In order to see this, consider bound states with $E < 0$, where the function $u_0(r)$ decays exponentially for $r > R$. Show that bound states can only exist for $K_0 R > \frac{\pi}{2}$ and that the condition $K_0 R = \pi(n + \frac{1}{2})$ marks the emergence of an additional bound state.

33. $1/r^2$ scattering potential (10 points)

Consider the scattering of a particle in three spatial dimensions from the potential

$$V(\mathbf{r}) = \frac{\lambda}{|\mathbf{r}|^2}. \quad (1)$$

The aim of this exercise is the computation of the differential scattering cross-section $\frac{d\sigma}{d\Omega}$.

- a) You can solve the scattering problem exactly. Write the Schrödinger equation for the radial part of the wavefunction $u_{k\ell}(r)$. Show that it assumes the form for a free particle

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \frac{\hbar^2 \ell'(\ell' + 1)}{2mr^2} \right) u_{k\ell}(r) = \frac{\hbar^2 k^2}{2m} u_{k\ell}(r). \quad (2)$$

Determine ℓ' that is a function of the angular momentum ℓ and the potential strength λ . We know that for large distances $r \rightarrow \infty$ the asymptotic behavior of $u_{k\ell}(r)$ is given by $u_{k\ell}(r) \propto \sin(kr - \frac{\pi\ell'}{2})$. This also holds for a non-integer ℓ' . Using this asymptotic behavior of the radial wavefunction, determine the phase shift $\delta_\ell(k)$! The differential scattering cross section in terms of the phase shift is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left| \sum_{\ell=0}^{\infty} (2\ell + 1) e^{i\delta_\ell(k)} \sin \delta_\ell(k) \mathcal{P}_\ell(\cos \theta) \right|^2. \quad (3)$$

- b) In the lectures, you derived the Born approximation for scattering from weak potentials. In the Born approximation the differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi\hbar^2} \right)^2 |V_{\mathbf{k}_f - \mathbf{k}_0}|^2, \quad (4)$$

where \mathbf{k}_f and \mathbf{k}_0 is the out- and in-going momentum, respectively, with $|\mathbf{k}_0| = |\mathbf{k}_f|$ and $\mathbf{k}_0 \mathbf{k}_f = |\mathbf{k}|^2 \cos \theta$, and $V_{\mathbf{k}}$ is the Fourier transform of the scattering potential.

Evaluate the differential scattering cross-section perturbatively using the Born approximation, Eq (4).

Hint #1: $\int_0^\infty dx \frac{\sin x}{x} = \pi/2$.

Now expand the result from part (a) in lowest order in λ and recover the result of the Born approximation.

Hint #2: Use that $\sum_{\ell=0}^{\infty} \mathcal{P}_\ell(x) = \frac{1}{\sqrt{2(1-x)}}$.