Entanglement measures

1 Density matrices and entanglement

Density operator :

$$\rho \equiv \sum_i p_i \mid \psi_i >< \psi_i \mid$$

p_i - probability, that the system is in $\mid \psi_i >$,   with $\sum_i p_i = 1$
$\mid \psi_i >$-s   need not be orthonormal,
and decomposition is not unique.

Conditions for density operator :
1. tr$\rho = 1$
2. ρ is positive.

ρ is a Hermitian operator with an orthonormal basis
$\rho_{mn} =< m \mid \rho \mid n >$ is called as density matrix.

1.1 Pure and mixed states

- pure quantum state : a vector state in Hilbert space
- mixed quantum state : a probabilistic mixture of pure states

Density matrices for

- pure state : $\rho =\mid \psi_i >< \psi_i \mid$
- mixed state : $\rho = \sum_i p_i \mid \psi_i >< \psi_i \mid$

If $\mathrm{tr}(\rho^2) = 1$ <=>   pure state, and
if $\mathrm{tr}(\rho^2) < 1$ <=>   mixed state

1.2 Entanglement

For a quantum system which consists of two subsystems A,B,C,... the $\mathrm{H}_A \bigotimes \mathrm{H}_B \bigotimes \mathrm{H}_C \bigotimes$... is the Hilbert space of the composite system.
If $\mid \psi >_{ABC...} =\mid \psi >_A \bigotimes \mid \psi >_B \bigotimes \mid \psi >_C \bigotimes$ then $\mid \psi >_{ABC...}$ is separable or product state.
Otherwise $\mid \psi >_{ABC...}$ is entangled.

1.3 Reduced density matrix

$\rho_A = \text{tr}_B\rho$ and $\rho_B = \text{tr}_A\rho$
for two-party system with subsystems A and B.

If ρ_A is pure $(\text{tr}(\rho_A)^2) = 1)$ then $\mid \psi >_{AB}$ is separable, and
if ρ_A is mixed $(\text{tr}(\rho_A)^2) < 1)$ then $\mid \psi >_{AB}$ is entangled.

2 Entropy as entanglement measure

2.1 Von Neumann entropy

$$S(\rho) = -tr(\rho ln \rho)$$

If λ_i- eigenvalue of ρ then

$$S(\lambda_i) = -\sum_i \lambda_i \ln \lambda_i$$

with $\rho = \sum_i \lambda_i \mid i >< i \mid$

$S(\rho =\mid \psi >< \psi \mid) = 0$ Entropy of a pure state is zero.

$S(\rho_A)$ measures entanglement in ρ, if ρ is a pure state.
If $S(\rho_A)$ is pure $S(\rho_A) = 0$ => no entanglement, and
if $S(\rho_A)$ is mixed => $S(\rho)$ is entangled.

2.2 Rényi entropy

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln tr(\rho_A^\alpha) \quad , \quad \alpha \in \mathbb{N}$$

Rényi entropy can be calculated in a Monte Carlo simulation.

3 Entanglement measures for mixed states

3.1 Requirements

Here some possible requirements for entanglement measure.
1. If ρ is a product state then $E(\rho) = 0$.
2. There are maximally entangled states for subsystems of equal distribution

$$\mid \psi_d >= \frac{\mid 0,0 > + \mid 1,1 > + ... + \mid d-1, d-1 >}{\sqrt{d}}$$

which will be used for the normalization of E.

$$E(\mid \psi_d >) = \ln d$$

3. Entanglement can not increase under LOCC operations.
LOCC - local quantum operation classical communication

4. For a pure state $S(\rho = \mid \psi >< \psi \mid)$ the quantity E reduces to the entropy of entanglement again.

$$E(\mid \psi >< \psi \mid) = (S \circ tr_B)(\mid \psi >< \psi \mid) = S(tr_B(\mid \psi >< \psi \mid)) = S(\rho_A)$$

As following two example methods for entanglement measure for mixed state.

3.2 Relative entropy of entanglement

$$E_R(\rho) = \inf_{\sigma \in S} tr[\rho(\log \rho - \log \sigma)]$$

as "distance" of the entangled state ρ to the closest separable state σ

3.3 Entanglement of formation

$$E_F(\rho) = \inf_{decomp.\rho} \sum_i p_i S(tr_B(\mid \psi_i >< \psi_i \mid))$$

The entanglement of formation represents the minimal possible average entropy of all pure state decompositions.

References:

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