1 Density matrices and entanglement

Density operator:
\[ \rho = \sum_i p_i |\psi_i><\psi_i| \]

- \( p_i \) - probability, that the system is in \( |\psi_i> \), with \( \sum_i p_i = 1 \)
- \( |\psi_i>s \) need not be orthonormal, and decomposition is not unique.

Conditions for density operator:
1. \( \text{tr} \rho = 1 \)
2. \( \rho \) is positive.

\( \rho \) is a Hermitian operator with an orthonormal basis
\( \rho_{mn} = <m | \rho | n> \) is called as density matrix.

1.1 Pure and mixed states
- pure quantum state: a vector state in Hilbert space
- mixed quantum state: a probabilistic mixture of pure states

Density matrices for
- pure state: \( \rho = |\psi_i><\psi_i| \)
- mixed state: \( \rho = \sum_i p_i |\psi_i><\psi_i| \)

If \( \text{tr}(\rho^2) = 1 \) \( \iff \) pure state, and
if \( \text{tr}(\rho^2) < 1 \) \( \iff \) mixed state

1.2 Entanglement

For a quantum system which consists of two subsystems A,B,C,... the \( \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes ... \) is the Hilbert space of the composite system.
If \( |\psi>_{ABC...} = |\psi>_A \otimes |\psi>_B \otimes |\psi>_C \otimes ... \) then \( |\psi>_{ABC...} \) is separable or product state.
Otherwise \( |\psi>_{ABC...} \) is entangled.
1.3 Reduced density matrix

\[ \rho_A = \text{tr}_B \rho \quad \text{and} \quad \rho_B = \text{tr}_A \rho \]

for two-party system with subsystems A and B.

If \( \rho_A \) is pure \((\text{tr}(\rho_A)^2) = 1)\) then \( |\psi >_{AB} \) is separable, and
if \( \rho_A \) is mixed \((\text{tr}(\rho_A)^2) < 1)\) then \( |\psi >_{AB} \) is entangled.

2 Entropy as entanglement measure

2.1 Von Neumann entropy

\[ S(\rho) = -\text{tr}(\rho \ln \rho) \]

If \( \lambda_i \)- eigenvalue of \( \rho \) then

\[ S(\lambda_i) = -\sum_i \lambda_i \ln \lambda_i \]

with \( \rho = \sum_i \lambda_i |i><i| \)

\[ S(\rho = |\psi ><\psi|) = 0 \]

Entropy of a pure state is zero.

\[ S(\rho_A) \]

measures entanglement in \( \rho \), if \( \rho \) is a pure state.

If \( S(\rho_A) \) is pure \( S(\rho_A) = 0 \) \( \Rightarrow \) no entanglement, and
if \( S(\rho_A) \) is mixed \( \Rightarrow \) \( S(\rho) \) is entangled.

2.2 Rényi entropy

\[ S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln \text{tr}(\rho_A^\alpha) , \quad \alpha \in \mathbb{N} \]

Rényi entropy can be calculated in a Monte Carlo simulation.

3 Entanglement measures for mixed states

3.1 Requirements

Here some possible requirements for entanglement measure.

1. If \( \rho \) is a product state then \( E(\rho) = 0 \).
2. There are maximally entangled states for subsystems of equal distribution

\[ |\psi_d > = \frac{|0,0> + |1,1> + ... + |d-1,d-1>}{\sqrt{d}} \]

which will be used for the normalization of \( E \).

\[ E(|\psi_d >) = \ln d \]

3. Entanglement can not increase under LOCC operations.

LOCC - local quantum operation classical communication
4. For a pure state $S(\rho = |\psi><\psi|)$ the quantity $E$ reduces to the entropy of entanglement again.

$$E(|\psi><\psi|) = (S \circ tr_B)(|\psi><\psi|) = S(tr_B(|\psi><\psi|)) = S(\rho_A)$$

As following two example methods for entanglement measure for mixed state.

### 3.2 Relative entropy of entanglement

$$E_R(\rho) = \inf_{\sigma \in S} tr[\rho(\log \rho - \log \sigma)]$$

as “distance” of the entangled state $\rho$ to the closest separable state $\sigma$

### 3.3 Entanglement of formation

$$E_F(\rho) = \inf_{\text{decomp},\rho} \sum_i p_i S(tr_B(|\psi_i><\psi_i|))$$

The entanglement of formation represents the minimal possible average entropy of all pure state decompositions.

References:

- J. Helmes, Masterthesis, Ch. 3, (2012)