Topological Defects

Alessandro Fasse - afasse@ketbra.net

Introduction

This talk gives the starting point in a series of talks about quantum knots it deals more technical aspects of the subject. Instead of treating physical problems I will introduce mathematical concepts, we will be using in the subsequent talks. Regarding to the excellent work of N.D. Mermin this talk is closely related to his publication at the Review of Modern Physics [1].

The order parameter

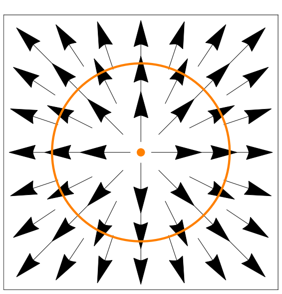
In order to treat some medium we have to introduce a function

$$f : A \longrightarrow R, \quad r \longmapsto f(r) \tag{1}$$

which is called the **order parameter**, A is the **ordered medium** and R is the **order-parameter space**. Also we assume that f is continuous except of some points P, which are the **topological defects** of that ordered medium. In order to give one simple example for an order-parameter space one can think of spin in a planar field. Every directional pointing of one these spins is characterized by a point on a circle, i.e. we have $R = S^1$.

Illustration of topological defects in the case of planar spins

In the following we will introduce the concept of the so-called **winding number**, which gives us the ability to study a singular point (topological defect) of an order parameter without being even close to that point. As an example, the following figure shows radial outwards orientated planar spins.



The point in the center is such a topological effect. We can see this by following the behavior of the order parameter f around the circle. The vector rotates one time within the mathematical positive sense around itself, therefore we say this point has the winding number $+1$. It is very important to mention that this value of the winding number does not change under continuous transformations of f and therefore we can define the so-called homotopy as:

Homotopy

Let $f(r) : A \to R$ and $g(r) : A \to R$ be two order parameter. Then we will call f homotop to g if there exists a function

$$h : A \times [0, 1] \longrightarrow R, \quad (r, t) \longmapsto h_t(r), \tag{2}$$

such that $h_0(r) = f(r)$, $h_1(r) = g(r)$ and h_t is continuous for all $t \in [0, 1]$. h is called a homotopy. This is somehow just a mathematical definition but gets important to define the fundamental group of an order parameter space:

The Fundamental Group

Now we have all the tools we need, in order to say that all order parameter with the same winding number are homotop to each other and form therefore a set. The next step is to define from this set the so-called fundamental group, which contains all of these classes. This group is completely defined by all the possible states that $f(r)$ can achieve, so therefore by the topology of R. In the talk we introduced loops, i.e. curves in R which start and end at a point x. For these loops it is easy to define a homotopy and a product between two of them. The product is given by a new loop which passes through the first and then the second one. As shown in the talk these properties imply a group structure, which is then called the fundamental group at a point x. Without any proof we can assume that all fundamental groups at all points $x \in R$ are isomorphic to each other. Therefore it is sufficient to calculate the fundamental group at one point as shown in the case of two examples. The first example considers the circle S^1. As we have seen before every loop is characterized by a winding number which is an element of $\mathbb{Z}$. Winding numbers are also additive under the product of two loops and therefore we find that

$$\pi_1(S^1) = \mathbb{Z}. \tag{3}$$

Another example is the sphere S^2: Any loop on a sphere can be shrunk continuous to a single point, therefore

$$\pi_1(S^2) = 0. \tag{4}$$

References

[1] N. D. Mermin. The topological theory of defects in ordered media. *Rev. Mod. Phys.*, 51:591–648, Jul 1979.