Berry Phase

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Introduction

The talk mostly comprises of an introductory lecture on the concept of Berry Phase, it's applicability to Condensed matter systems and, mostly to form an appetizer for the series of talks that follow. This report briefly summarizes the relevant concepts, equations, and is mostly constructed from multiple references. Most of the literature that I have used during the course of preparation for the talk(and the report) have been listed down in the bibliography and, it's suggestive that these are referred to for further details and concise understanding of the subject matter.

Berry Phase

We consider a Hamiltonian $\hat{H}$ which is a function of some time dependent parameter $\beta = \beta\,(t)$:

$$\hat{H} = \hat{H}(\beta(t)) \tag{1}$$

Upon adiabatic[1] variation of β in the parameter space, an initial eigen state of $\hat{H}$, say, $|n(\beta)\rangle$ in the particle's Hilbert space Ω, acquires a *Geometric phase*[2] also popularly known as the *Berry Phase* [1], apart from the usual dynamical phase term of course:

$$\left|n(\beta^{'})\right\rangle = |n(\beta)\rangle\, e^{i\alpha_n(C)}.e^{i\theta_n(t)} \tag{2}$$

where,

$$\mathbf{BerryPhase} : \alpha_n(C) = i\oint_C \langle n(\beta)|\,|\nabla_\beta n(\beta)\rangle.d\beta \tag{3}$$

$$DynamicalPhase : \theta_n(T) = \frac{-1}{\hbar}\int_0^T dt^{'}\, E_n(t^{'}) \tag{4}$$

C signifies the closed loop[3] path traversed in the β-space. Also, $\alpha(C)$[4] is gauge invariant over closed loops which follows directly from Stokes Theorem [2].

Relevant Example

This section is mostly dedicated to illustrate Berry phases in *Magnetic textured* materials, which is mostly the highlight of the Seminar. We can model the current system with the following Hamiltonian [2]:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m_e} + \vec{M}.\vec{\sigma} \tag{5}$$

Upon assuming a smooth texture for the *Magnetization vector* $\vec{M}$ or in other words, an adiabaticity of the process ensures that an e^- tossed into such a lattice[5] with an initial spin state that matches that of the lattice, the spin "sniffs" [6] the lattice magnetization and emerges accordingly. With this idea, we can make the following ansatz for the full e^- wavefunction $|\psi(\phi,t)\rangle$ for the Hamiltonian:

$$|\psi(\phi,t)\rangle = \tilde{\psi}(\phi,t)\,|\chi(\phi,t)\rangle \tag{6}$$

$\tilde{\psi}(\phi,t)$ and $|\chi(\phi,t)\rangle$ represents the system ground state amplitude and the eigen spinor of the $\vec{M}.\vec{\sigma}$ operators respectively(ϕ is just a spatial parameter). With an appropriate "*Vector potential*" $\vec{A}_{eff}$ and scalar potential V_{eff}[6] defined as,

$$\vec{A}_{eff} = i\hbar\,\langle\chi|\,\nabla_\phi\,|\chi\rangle \tag{7}$$

$$V_{eff} = \frac{\hbar^2}{2m_e}\left(\langle\nabla_\phi\chi|\,|\nabla_\phi\chi\rangle + |\langle\chi|\,\nabla_\phi\,|\chi\rangle|^2\right) \tag{8}$$

the *effective Hamiltonian* [2] $\hat{H}_{eff}$ and hence the Schrödinger equation reads:

$$\underbrace{\left(\frac{1}{2m_e}\left(\hat{P} - \vec{A}_{eff}\right)^2 + V_{eff} - \varphi_{eff} + \beta\right)}_{\hat{H}_{eff}}\tilde{\psi} = i\hbar\dot{\tilde{\psi}} \tag{9}$$

β and φ_{eff} correspond to eigen value of $\vec{M}.\vec{\sigma}$ operator and, effective potential($i\hbar\,\langle\chi|\,\partial_t\,|\chi\rangle$) terms respectively. Consequently, the e^- experiences a Lorentz force:

$$\vec{F}_L = -\left(\vec{E} + \vec{v}_e \times \left[\nabla \times \vec{A}_{eff}\right]\right) \tag{10}$$

$\vec{E}$ has it's origins in φ_{eff} and $\vec{A}_{eff}$. On camparing the $\vec{A}_{eff}$ term with equation(3), it is quite apparent that we

[1]Adiabaticity is not a serious restriction; Arhonov Bohm effect is one good example [1]

[2]Unlike most phenomenon in Physics that acquire certain terminology that aren't really connected to what they describe(*QCD* is a delightful example), Geometric phase is rightly named; [3]has a thorough discussion illuminating the fact.

[3]It is not required that the path be closed; a variation in β is all that's desired to acquire a *Berry Phase*. However, for some systems, $\alpha_n(t)$ could be zero over C; see for example [4]

[4]This is a purely Real quantity i.e, $\alpha(C) \in \Re$. This follows from the normalization of $|n(\beta)\rangle$'s [1]

[5]Skyrmionic lattices are one such example that presents "*Magnetic Texture*"/"*Magnetic Whirls*" as they are also called, in nature.

[6]The definition of $\vec{A}_{eff}$ fixes the functional form of V_{eff}. Noticeable is the fact, V_{eff} comprises of second order terms and these can be to some extent ignored($V_{eff} \cong 0$);this is because of the adiabaticity of $\vec{M}$. We then preserve the term only as a matter of logical consistency

have a Berry Phase term contributing to the Lorentz force through $\vec{B}_{eff}$ $(= \nabla \times \vec{A}_{eff})$, the "*effective Magnetic field*". This then means, we should have observable consequences of Berry Phase in such magnetic systems! In fact, it *is* observed and in a sense responsible for *Topological Hall effect* in such materials [7].

What is so Geometric about the Geometric phase?

Consider an e^- at the origin and a **B**-field that traces out a cone with inclination angle θ with it's magnitude(B) and, the inclination(θ) fixed at all times. One can construct a sphere with the e^- at the center. This would form our parameter space and, ϕ represents the parameter that's subject to an *adiabatic* variation. The Hamiltonian for this system reads:

$$\hat{H}(\theta(t)) = \frac{e}{m_e} B(\vec{\theta}(t)).\vec{S} \tag{11}$$

The Geometric phase corresponding to this system maybe written down as:

$$\begin{aligned} \alpha(C) = i \oint \langle \chi_+ | \, | \nabla \chi_+ \rangle . R \sin\theta . d\phi &= -\pi(1 - \cos\theta) \qquad (12) \\ &= \frac{-1}{2} \Omega(C) \end{aligned}$$

$\Omega(C)$ simply corresponds to the solid angle that the surface enclosed by the closed path(traced out by the **B** field) projects at the sphere center [4]. In other words, Geometric phase is a quantity that's proportional to the solid angle projected by the surface enclosed within the closed loop, in the parameter space.

References

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