**Topological Phase Transitions and Gauge Theory**

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**INTRODUCTION**

Nematic crystals are composed of elongated molecules which can be modeled as **directors**, a vector without orientation. A naive picture consists of a 1st order phase transition separating two phases: a disordered **isotropic phase** where there is no preferred alignment of the directors, and an ordered **nematic phase** where there is a preferred alignment.

**Topological Defects**

The head-tail symmetry of the directors implies that their order parameter space can be characterized as $\mathbb{R}P^2$. The fundamental group $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$, which means that there exist line defects (called **disclinations** or **π-defects**).

To model these defects, a lattice structure is applied to the nematic crystal. The director at each lattice site is arbitrarily given an orientation. An additional variable, the comparator $U$, measures if there can be a systematic way of assigning orientations to the directors.

The comparator may be thought of as “parallel transporting” the oriented directors between the lattice sites. If the director can be parallel transported to each other, $U = +1$, if not, $U = -1$. The presence of a π-defect forces one $U = -1$.

**Lattice Gauge Theory**

Two conditions are required for the Hamiltonian:

1. Invariance under the transform $\vec{S}_i \rightarrow -\vec{S}_i$
2. Distinguish between defect/no defect

Condition 1 implies that the system should be invariant under arbitrary transformations $\phi_i \in \mathbb{Z}_2 = \{-1, +1\}$ performed at any lattice point $i$. The oriented directors and comparators therefore transform as

$$\vec{S}_i \rightarrow \vec{S}_i' = \phi_i \vec{S}_i$$  \hfill (1a)

The Hamiltonian can be constructed as

$$\mathcal{H} = -J \sum_{i,j} \vec{S}_i \cdot \vec{S}_j U_{ij} - K \sum_{\Box} U_{ij} U_{jk} U_{kl} U_{li}. \hfill (2)$$

The first term measures the relative alignment of the directors. The $U_{ij}$ term is needed to make this term gauge invariant. The second term measures the energy cost of a defect. Its form is the simplest gauge invariant form measuring this cost.

**Phase Diagram**

The phase diagram can be constructed from the limiting cases of the parameters $J$ and $K$ in $\mathcal{H}$:

1. $K = \infty$: Can choose a gauge in which every $U_{\Box} = +1$. This yields a 2nd order Heisenberg phase transition.
2. $K = 0$: This is the naive nematic picture described by a 1st order phase transition.
3. $J = 0$: Pure gauge theory situation described by an Ising model. This is a 2nd order phase transition.

The lattice gauge model reveals an additional topologically ordered phase and 2nd order phase transitions not contained in the naive model.

**References**