Topological Phase Transitions and Gauge Theory

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Introduction

Nematic crystals are composed of elongated molecules which can be modeled as **directors**, a vector without orientation. A naîve picture consists of a 1^{st} order phase transition separating two phases: a disordered **isotropic phase** where there is no preferred alignment of the directors, and an ordered **nematic phase** where there *is* a preferred alignment.

Topological Defects

The head-tail symmetry of the directors implies that their order parameter space can be characterized as $\mathbb{RP}^2$. The fundamental group $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$, which means that there exist line defects (called **disclinations** or **π-defects**).

To model these defects, a lattice structure is applied to the nematic crystal. The director at each lattice site is arbitrarily given an orientation. An additional variable, the **comparator** U, measures if there can be a systematic way of assigning orientations to the directors.

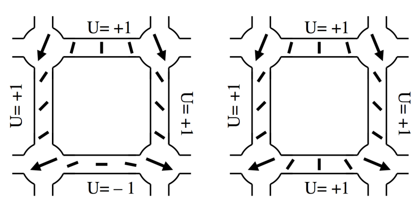

Figure 1: *The presence of a defect (left) or no defect (right) can be seen from analyzing e.g. the bottom link. [1]*

The comparator may be thought of as "parallel transporting" the oriented directors between the lattice sites. If the director can be parallel transported to each other, $U = +1$, if not, $U = -1$. The presence of a π-defect forces one $U = -1$.

Lattice Gauge Theory

Two conditions are required for the Hamiltonian:

1. Invariance under the transform $\vec{S}_i \to -\vec{S}_i$
2. Distinguish between defect/no defect

Condition 1. implies that the system should be invariant under arbitrary transformations $\phi_i \in \mathbb{Z}_2 = \{-1, +1\}$ performed at any lattice point i. The oriented directors and comparators therefore transform as

$$\vec{S}_i \to \vec{S}'_i = \phi_i \vec{S}_i \tag{1a}$$

$$U_{ij} \to U'_{ij} = \phi_i U_{ij} \phi_j \tag{1b}$$

The Hamiltonian can be constructed as

$$\mathcal{H} = -J \sum_{i,j} \vec{S}_i \cdot \vec{S}_j U_{ij} - K \sum_{\square} U_{ij} U_{jk} U_{kl} U_{li}. \tag{2}$$

The first term measures the relative alignment of the directors. The U_{ij} term is needed to make this term gauge invariant. The second term measures the energy cost of a defect. Its form is the simplest gauge invariant form measuring this cost.

Phase Diagram

The phase diagram can be constructed from the limiting cases of the parameters J and K in $\mathcal{H}$:

1. $K = \infty$: Can choose a gauge in which every $U_\square = +1$. This yields a 2^{nd} order Heisenberg phase transition.
2. $K = 0$: This is the naîve nematic picture described by a 1^{st} order phase transition.
3. $J = 0$: Pure gauge theory situation described by an Ising model. This is a 2^{nd} order phase transition.

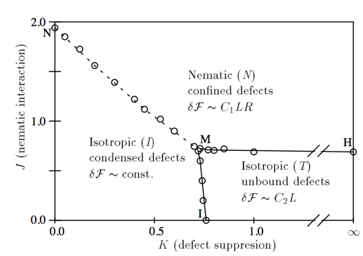

Figure 2: *Isotropic phase I (left), nematic phase N (top), topological isotropic phase T (right).[1]*

The lattice gauge model reveals an additional topologically ordered phase and 2^{nd} order phase transitions not contained in the naîve model.

References

[1] Paul Lamert, Daniel Rokhsar, and John Toner. *Phys. Rev E.*, 52:1778, 1995.