INTRODUCTION TO RANDOM MATRIX THEORY

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I. Introduction

Random matrix theory deals with the statistical properties of large random generated matrices. Possible fields of application in physics are disordered and chaotic systems. The position of energy levels in such systems appear to be random, therefore the Hamiltonian can be regarded as a random matrix with certain statistical properties. In the absence of any symmetry, all matrix entries are assumed to be Gaussian distributed independent random variables.

II. Symmetry Classes

Wigner and Dyson introduced three symmetry classes in order to classify a Hamiltonian H, depending on the presence or absence of time-reversal (TRS) and spin-rotation (SRS) symmetry. These symmetry classes can be characterized by an index β, which counts the number of degrees of freedom in the matrix elements. As the transformation $H \to UHU^{-1}$, with U an orthogonal, unitary or symplectic matrix leaves the ensemble invariant, it is called orthogonal, unitary or symplectic. In the case of $\beta = 2$ time-reversal symmetry is broken by a magnetic field or magnetic impurities. In the presence of time reversal-symmetry, $\beta = 1$ if spin is conserved, and $\beta = 4$ if spin-rotation symmetry is broken by spin-orbit scattering.

β	TRS	SRS	H
1	Yes	Yes	real symmetric
2	No	No	Hermitian
4	Yes	No	quaternion self-dual [1]

[1] $H_{ij} = h_{ij}^{(0)} + ih_{ij}^{(1)}\sigma_x + ih_{ij}^{(2)}\sigma_y + ih_{ij}^{(3)}\sigma_z$
$(h_{ij}^{(\mu)} \in \mathbb{R}, h_{ij}^{(0)} = h_{ji}^{(0)}, h_{ij}^{(k)} = -h_{ji}^{(k)})$

III. Random Matrix Theory

Due to the Gaussian probability distribution, the ensemble is called Gaussian orthogonal ensemble (GOE) for $\beta = 1$, Gaussian unitary ensemble (GUE) for $\beta = 2$ and Gaussian symplectic ensemble (GSE) for $\beta = 4$.
The joint probability of eigenvalues ($\{E_n\}$) of a NxN random matrix is given by

$$P(E_1, ..., E_N) \propto \prod_{n>m} |E_n - E_m|^\beta \exp\left(- \frac{\beta}{2\delta_s^2} \sum_n E_n^2 \right).$$

Here δ_s denotes the mean level spacing between two adjacent eigenvalues. The probability for a small spacing is small and the probability for a degeneracy is zero. Therefore eigenvalues of a random matrix repel each other. The approximation for the distribution function of spacing δE between two adjacent eigenvalues is called *Wigner-Dyson distribution*.

$$P(\delta E) = a_\beta \left(\frac{\delta E}{\delta_s} \right)^\beta \exp\left(- b_\beta \left(\frac{\delta E}{\delta_s} \right)^2 \right)$$

The expectation value of the density of eigenvalues has in all three symmetry cases the shape of a semicircle. This is called *Wigner semicircle law*.

References

[1] Yuli V. Nazarov and Yaroslav M. Blanter, *Quantum Transport*, Cambridge University Press, 2009.

[2] Madan Lal Mehta, *Random Matrices*, Academic Press, second edition, 1991 .