Percolation

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I. Introduction

Percolation is a concept useful in understanding disordered systems. A percolating system in its simple form is visualised as a lattice with sites having an occupancy probability p. A percolating cluster is said to be formed when a cluster of occupied sites spans the system from one end to the other. The key feature of this system is the presence of a phase transition exhibiting critical behavior around a critical probability (p_c), above which one is certain to have a percolating cluster. This idea is applied to model conductivity of random resistor networks, where the occupied sites in the lattice are conductors and the unoccupied sites act as insulators.

II. Critical behavior

The critical probability at which the phase transition occurs depends on the lattice type, and the number of dimensions. In one dimension the critical probability is easily seen to be equal to 1. The critical behavior is however seen in the quantities average cluster size(S) and the correlation length(ξ). With $p_c = 1$ -

$$\mathrm{S} = \left(\frac{1+p}{p_c - p}\right) \propto (p_c - p)^{-1}$$

$$\xi = -\frac{1}{\ln p} \propto (p_c - p)^{-1}$$

In higher dimensions, the number of configurations of a cluster on the lattice sites increases rapidly with the number of dimensions. However, there are two simplifications that can be made. In d dimensions, the number of ways of forming a chain in a cluster of 4 occupied sites is proportional to $(2d-1)^3$, whereas the number of ways of forming a loop goes as $d(d-1)$. Clearly as $d \to \infty$, the contribution of loops becomes negligible when compared to that of the trees. Therefore we can think of a lattice that only has trees and no loops in it. Apart from this, in any given number of dimensions d, the surface area(S) is related to the volume(V) as $S \propto V^{(1-\frac{1}{d})}$. As $d \to \infty$ the surface goes as the volume. Following these two characteristics, a particular lattice that contains only trees could then be used to solve the problem of percolation in infinite dimensions. Such a lattice is called Bethe Lattice or Cayley tree.

The results for Bethe lattice are given below. The order of the Bethe lattice(z) is the number of bonds for each point in lattice. The strength of the percolating cluster(P) is the the probability that an arbitrarily chosen site is a part of an infinite cluster. The critical behavior of the strength and the average cluster size is seen based on their dependence on the occupation probability.

$$p_c = \frac{1}{z-1} = \frac{1}{2} \quad \text{for} \quad z = 3$$

$$\mathrm{P} = 1 - \left(\frac{1-p}{p}\right)^3 \propto (p - p_c)^{-1}$$

$$\mathrm{S} = \frac{1+p}{1-2p} \propto (p_c - p)^{-1}$$

After having obtained the results for simple but informative lattices, the critical exponents defined for important quantities are given below.

$$P \propto |p - p_c|^{\beta}\,, \;\; S \propto |p - p_c|^{-\gamma}\,, \;\; \xi \propto |p - p_c|^{-\nu}$$

III. Electrical Conductivity

The strength of the percolating cluster gives the fraction of occupied sites that are a part of the infinite cluster. It can be assumed that the conductivity(Σ) of a random resistor network also follows the same behavior and that it is proportional to the strength P. However, in a percolating cluster, a major part of the cluster is formed by "dead ends" that do no lead to the other end of the network. This makes the conductivity grow much slowly than the strength of percolating cluster after the critical probability. So we introduce a new critical exponent -

$$\Sigma \propto |p - p_c|^{\mu}$$

Hence, following the fact that the phase transition was actually observed in a random resistor network around the same critical probability, the conductivity can still be written using this new critical exponent(μ) that may be related to β in some way.

References

[1] Dietrich Stauffer, Amnon Aharony, *Introduction to Percolation Theory*, Taylor & Francis, 2nd Revised Edition, 1994.