Collective states of interacting anyons, edge states and the nucleation of topological liquids

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**Anyons and computing**

**Abelian anyons**

\[ \psi(x_2, x_1) = e^{i\pi \theta} \cdot \psi(x_1, x_2) \]

fractional phase

**Non-Abelian anyons**

\[ \psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n) \]

\[ \psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n) \]

In general, \( M \) and \( N \) do not commute!

**Topological quantum computing**

Degenerate manifold = qubit

Employ **braiding** of non-Abelian anyons to perform computing (unitary transformations).

Matrix depends only on the topology of the braid swept out by quasiparticle world lines!

Robust quantum computation? (Kitaev ’97; Freedman, Larsen and Wang ’01)

Illustration N. Bonesteel
Non-Abelian anyons

Ising anyons = Majorana fermions
- p-wave superconductors
- Moore-Read state
- Kitaev’s honeycomb model

Fibonacci anyons
- Read-Rezayi state
- Levin-Wen model

Ordinary spins
- quantum magnets

$SU(2)_2$
$SU(2)_3$
$SU(2)_k$
$SU(2)_\infty$
$\textbf{SU}(2)_k$ = ‘deformations’ of SU(2)

Quantum numbers in $\textbf{SU}(2)_k$

$0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \frac{k}{2}$

cutoff level $k$
‘quantization’

Fusion rules

\[ j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots + \min(j_1 + j_2, k - j_1 - j_2) \]

for all $k \geq 2$

\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

for all $k \geq 4$

\[ 1 \times 1 = 0 + 1 + 2 \]
$p_x + ip_y$ superconductors

Possible realizations:
- $Sr_2RuO_4$
- $p$-wave superfluid of cold atoms
- $A_1$ phase of $^3$He films

Topological properties of $p_x + ip_y$ superconductors

- $\sigma$-vortices carry “half-flux” $\phi = \frac{hc}{2e}$
- Characteristic “zero mode” $\sigma \times \sigma = 1 + \psi$
- 2N vortices give degeneracy of $2^N$. 
Fractional quantum Hall liquids

“Pfaffian” state

Moore & Read (1994)
Charge $e/4$ quasiparticles
Ising anyons
Nayak & Wilzcek (1996)

“Parafermion” state

Read & Rezayi (1999)
Charge $e/5$ quasiparticles
Fibonacci anyons
Slingerland & Bais (2001)
A soup of anyons

What is the collective state of a set of interacting anyons?

Does this collective behavior somehow affect the character of the underlying parent liquid?
A soup of anyons

SU(2)\textsubscript{k} liquid

finite density of anyons
(anyons are at fixed positions or ‘pinned’)

\[ a \gg \xi_m \]

The ground state has a macroscopic degeneracy.

\[ a \ll \xi_m \]

Anyons approach each other and interact. The interactions will lift the degeneracy.
Collective states: possible scenarios

The collective state of anyons is **gapped**.

The parent liquid remains **unchanged**.
The collective state of anyons is a **gappless quantum liquid**.

A **gappless phase nucleates** within the parent liquid.
The collective state of anyons is a **gapped quantum liquid**.

A novel liquid is **nucleated** within the parent liquid.
A soup of anyons

\[ \text{SU}(2)_k \text{ liquid} \]

**SU(2)_k liquid**

finite density of anyons
(anyons are at fixed positions or ‘pinned’)

**SU(2)_k fusion rules**

\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

ergetically split
multiple fusion outcomes

**“Heisenberg” Hamiltonian**

\[ H = J \sum_{\langle ij \rangle} \prod_{ij}^0 \]

Anyonic Heisenberg model

**SU(2)_k fusion rules**

\( \frac{1}{2} \times \frac{1}{2} = 0 + 1 \)

**“Heisenberg” Hamiltonian**

\[ H = J \sum_{\langle ij \rangle} \prod_{ij}^0 \]

energetically split multiple fusion outcomes

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Which fusion channel is favored? – Non-universal

- **p-wave superconductor**
  
  M. Cheng *et al.*, arXiv:0905.0035
  
  \( 1/2 \times 1/2 \rightarrow 0 \)
  
  short distances, then oscillates

- **Moore-Read state**
  
  M. Baraban *et al.*, arXiv:0901.3502
  
  \( 1/2 \times 1/2 \rightarrow 1 \)
  
  short distances, then oscillates

- **Kitaev’s honeycomb model**
  
  
  \( 1/2 \times 1/2 \rightarrow 0 \)

---

Connection to topological charge tunneling:  

P. Bonderson, arXiv:0905.2726
Anyonic Heisenberg model


\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

SU(2)_k fusion rules

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SU(2)_k liquid
Anyonic Heisenberg model


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\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

“Heisenberg” Hamiltonian

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energetically split multiple fusion outcomes

SU(2)_k liquid

chain of anyons

‘golden chain’ for SU(2)_3
Anyonic Heisenberg model

SU(2)_k fusion rules

\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

Heisenberg” Hamiltonian

\[ H = J \sum_{ij} \prod_{ij}^0 \]

ergetically split multiple fusion outcomes

Example: chains of anyons

Hilbert space

\[ |x_1, x_2, x_3, \ldots \rangle \]

Hamiltonian

\[ H = \sum_i F_i \prod_i^0 F_i \]

F-matrix = 6j-symbol

Critical ground state

Finite-size gap

$$\Delta(L) \propto \left(\frac{1}{L}\right)^\frac{z=1}{2}$$

description

Entanglement entropy

$$S_{\text{PBC}}(L) \propto \frac{c}{3} \log L$$

central charge

$$c = \frac{7}{10}$$

Graphs showing:

- Energy gap $$\Delta$$ as a function of inverse system size $$1/L$$
- Entropy $$S(L)$$ as a function of system size $$L$$

Legend:

- Lanczos
- DMRG

- Even length chain
- Odd length chain

- Periodic boundary conditions
- Open boundary conditions
The operators $X_i = -d H_i$ form a representation of the Temperley-Lieb algebra

\[(X_i)^2 = d \cdot X_i\]

\[X_i X_{i \pm 1} X_i = X_i\]

\[[X_i, X_j] = 0\]

for $|i - j| \geq 2$

The transfer matrix is an **integrable representation** of the RSOS model.
# Deformed spin-1/2 chains

<table>
<thead>
<tr>
<th>level $k$</th>
<th>1/2 $\times$ 1/2 $\rightarrow$ 0 (‘antiferromagnetic’)</th>
<th>1/2 $\times$ 1/2 $\rightarrow$ 1 (‘ferromagnetic’)</th>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>Ising</td>
<td>Ising</td>
</tr>
<tr>
<td></td>
<td>$c = 1/2$</td>
<td>$c = 1/2$</td>
</tr>
<tr>
<td>3</td>
<td>tricritical Ising</td>
<td>3-state Potts</td>
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<tr>
<td></td>
<td>$c = 7/10$</td>
<td>$c = 4/5$</td>
</tr>
<tr>
<td>4</td>
<td>$SU(2)_{k-1} \times SU(2)_1$</td>
<td>$SU(2)_k$</td>
</tr>
<tr>
<td>5</td>
<td>$SU(2)_k$</td>
<td>$U(1)$</td>
</tr>
<tr>
<td>$k$</td>
<td>k-critical Ising</td>
<td>$Z_k$-parafermions</td>
</tr>
<tr>
<td></td>
<td>$c = 1-6/(k+1)(k+2)$</td>
<td>$c = 2(k-1)/(k+2)$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Heisenberg AFM</td>
<td>Heisenberg FM</td>
</tr>
<tr>
<td></td>
<td>$c = 1$</td>
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## Topological protection

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<td></td>
</tr>
<tr>
<td>$5$</td>
<td>pentacritical Ising</td>
<td>c = 8/7</td>
</tr>
<tr>
<td></td>
<td>$c = 6/7$</td>
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- ✓: Topological protection
- ✘: No topological protection
Gapless modes & edge states

SU(2) liquid

finite density interactions

SU(2) liquid

critical theory
(AFM couplings)

\[
\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}
\]

gapless modes = edge states

\[
\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}
\]

nucleated liquid

arXiv:0810.2277
Example: Ising meets Fibonacci

$SU(2)_3$ liquid

gapless modes = edge states

$\frac{SU(2)_2 \times SU(2)_1}{SU(2)_3}$

$c = 7/10$

nucleated liquid

$SU(2)_2 \times SU(2)_1$

When Ising meets Fibonacci:

a tricritical Ising edge ($c = 7/10$)

$c = 7/10 \times U(1)$
Gapless modes & edge states

SU(2)_k liquid

finite density interactions

SU(2)_k liquid

critical theory \( \frac{SU(2)_k}{U(1)} \)

(FM couplings)

gapless modes = edge states

\( \frac{SU(2)_k}{U(1)} \)

nucleated liquid \( U(1) \)

(Abelian)

arXiv:0810.2277
The 2D collective state

A gapped topological liquid that is distinct from the parent liquid.

Results for N-leg ladders give some supporting evidence for this.
Coupling two chains

The relevant operator couples the inner two edges.
Earlier work for Majorana fermions

Read & Ludwig \textit{PRB} (2000)

SU(2)$_2$ liquid

U(1) liquid

Grosfeld & Stern \textit{PRB} (2006)

weak pairing SC

strong pairing SC


SU(3)$_2$ liquid

SU(2)$_2$ liquid

Kitaev unpublished (2006)

Levin & Halperin \textit{PRB} (2009)

2D anyon systems

All of these previous results fit into our more general framework.
Recent work for Fibonacci anyons

All of these previous results fit into our more general framework.
A powerful correspondence

SU(2)\(_k\) liquid

finite density interactions

SU(2)\(_k\) liquid

collective states of anyonic spin chains

edge states of topological liquids

nucleation of novel topological liquids
Which liquid is nucleated?

Which fusion channel is favored? – Non-universal

**p-wave superconductor**

M. Cheng *et al.*, arXiv:0905.0035

\[ \mathbb{Z}_2 \times U(1) \rightarrow U(1) \]

331 Halperin

**Moore-Read state**

M. Baraban *et al.*, arXiv:0901.3502

\[ \mathbb{Z}_2 \times U(1) \rightarrow U(1) \]

Laughlin

**Kitaev’s honeycomb model**


\[ \mathbb{Z}_2 \times U(1) \rightarrow U(1) \]
Which liquid is nucleated?

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<tr>
<td><strong>bosonic quantum Hall</strong></td>
<td>$SU(2)_2$</td>
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<tr>
<td></td>
<td>$SU(2)_1 \times SU(2)_1$</td>
<td>$U(1)$</td>
</tr>
<tr>
<td></td>
<td>$220$ Halperin state</td>
<td></td>
</tr>
<tr>
<td><strong>fermionic quantum Hall</strong></td>
<td>$Z_2 \times U(1)$</td>
<td>$Z_2 \times U(1)$</td>
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<td>$\emptyset$</td>
</tr>
</tbody>
</table>

$SU(2)_k = Z_k \times U(1)$
Quantum Hall plateaus

\[ a \gg \xi_m \]

\[ \sigma \times \sigma \rightarrow 1 \]

quasiholes
Quantum Hall plateaus

\[ a \approx \xi_m \]

middle of plateau

\[ \sigma \times \sigma \rightarrow 1 \]

quasiholes
Quantum Hall plateaus

\[ a \approx \xi_m \]

\[ \sigma \times \sigma \rightarrow 1 \]

quasiholes

middle of plateau
Quantum Hall plateaus

\[ a \approx \xi_m \]

\[ a \gg \xi_m \]

\( \sigma \times \sigma \rightarrow 1 \)

\( \sigma \times \sigma \rightarrow \psi \)

quasiholes

quasiparticles
Quantum Hall plateaus

\[ a \approx \xi_m \]

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quasiholes

quasiparticles
Quantum Hall plateaus

\[ a \approx \xi_m \]

\[ \sigma \times \sigma \rightarrow 1 \]

\[ \sigma \times \sigma \rightarrow \psi \]

quasiholes

quasiparticles

middle of plateau
What changes (experimentally) as we move on the plateau?

- electrical transport: **unchanged** – remain on the plateau
- heat transport (neutral modes): **changes** – evidence of the new liquid
Conclusions

• Interacting non-Abelian anyons can support a wide variety of collective states:

  stable gapless states, gapped states, quasiparticles, ...

• In a topological liquid a finite density of interacting anyons nucleates a new topological liquid

  gapless states = edge states between top. liquids

arXiv:0810.2277