Topological order

concepts
examples
numerics
prelude
(Quantum) matter

- Water
- Ice
- Superconductor
- Bose-Einstein condensate
Motivation – a paradigm

\[ \mathcal{H} = - \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z \]

interacting many-body system
Motivation – a paradigm

\[ \mathcal{H} = - \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z \]
Motivation – a paradigm

Spontaneous symmetry breaking

- ground state has **less** symmetry than Hamiltonian
- **local** order parameter
- **phase transition** / Landau-Ginzburg-Wilson theory

$$\mathcal{H} = - \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$
Every rule has an exception

Sometimes, the exact opposite happens

- ground state has **more** symmetry than Hamiltonian
- **non-local** order parameter
- **emergence** of degenerate ground states, exotic statistics, ...

$$
\mathcal{H} = \sum_{j=1}^{N} \left( \frac{1}{2m} \left( \mathbf{p}_j - \frac{e}{c} \mathbf{A}(\mathbf{x}_j) \right)^2 + eA_0(\mathbf{x}_j) \right) \\
+ \sum_{i<j} V(|\mathbf{x}_i - \mathbf{x}_j|) \\
\text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)
$$
Topological quantum matter

- **Spontaneous symmetry breaking**
  - ground state has *less* symmetry than Hamiltonian
  - Landau-Ginzburg-Wilson theory
  - *local* order parameter

- **Topological order**
  - ground state has *more* symmetry than Hamiltonian
  - degenerate ground states
  - *non-local* order parameter
When does this happen?

Some of the most intriguing phenomena in condensed matter physics arise from the splitting of ‘accidental’ degeneracies.

- interacting many-body system
- ‘accidental’ degeneracy
- residual effects select ground state

Phase diagram of cuprate superconductors

- non Fermi liquid
- pseudo gap
- Fermi liquid
- AF
- SC

T

hole doping
When does this happen?

Some of the most intriguing phenomena in condensed matter physics arise from the splitting of ‘accidental’ degeneracies.

- interacting many-body system
- ‘accidental’ degeneracy
- residual effects select ground state

But they are also notoriously difficult to handle, due to

- multiple energy scales
- complex energy landscapes / slow equilibration
- strong coupling
concepts
Topological quantum matter

- **Xiao-Gang Wen:** A ground state of a many-body system that *cannot* be fully characterized by a *local* order parameter.

- Often characterized by a variety of non-local “*topological properties*”.

- A topological phase can be positively identified by its *entanglement properties*. 
Knots & edge states

• Bringing a topological and a conventional state into spatial proximity will result in a gapless edge state – literally a knot in the wavefunction.

• We know this: “Counterintuitive states”
Knots & edge states

China

Hong Kong

Flipper bridge
the archetypal example

quantum Hall effect
Quantum Hall effect

Quantization of Hall conductivity for a two-dimensional electron gas at very low temperatures in a high magnetic field.

\[ \sigma = \nu \frac{e^2}{h} \]

Semiconductor heterostructure confines electron gas to two spatial dimensions.
Quantum Hall states

Landau levels

$$E_n = \hbar \frac{eB}{m} \left( n + \frac{1}{2} \right)$$

Landau level degeneracy
integer quantum Hall
fractional quantum Hall

2\(\Phi / \Phi_0\)
orbital states
filled level
incompressible liquid
incompressible liquid

B

edge states

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examples
the toric code

the drosophila
for lattice models
of topologically ordered phases
The toric code

\[ H_{TC} = -A \sum_v \prod_{j \in \text{vertex}(v)} \sigma^z_j - B \sum_p \prod_{j \in \text{plaquette}(p)} \sigma^x_j \]

Hamiltonian has only local terms.
All terms commute → exact solution!

The vertex term

\[ H_{TC} = -A \sum_v \prod_{j \in \text{vertex}(v)} \sigma^z_j - B \sum_p \prod_{j \in \text{plaquette}(p)} \sigma^x_j \]

- is minimized by an **even** number of down-spins around a vertex.
- Replacing down-spins by loop segments maps ground state to closed loops.
- Open ends are (charge) excitations costing energy \(2A\).
The plaquette term

\[ H_{TC} = -A \sum_v \prod_{j \in \text{vertex}(v)} \sigma_j^z - B \sum_p \prod_{j \in \text{plaquette}(p)} \sigma_j^x \]

- flips all spins on a plaquette.
- favors equal amplitude superposition of all loop configurations.
- Sign changes upon flip (vortices) cost energy $2B$.

fugacity = 1
free loop creation

isotopy
free local distortions

surgery
reconnections

Toric code – quantum loop gases

Ground-state manifold is a quantum loop gas.

Ground-state wavefunction is equal superposition of loop configurations.

Toric code – quantum loop gases

Ground-state manifold is a quantum loop gas.

Topological sectors defined by winding number parity $P_y/x = \prod_{i \in C_{x/y}} \sigma^z_i$. 

Toric code – quantum loop gases

Ground-state manifold is a quantum loop gas.

Topological sectors defined by winding number parity $P_y/x = \prod_{i \in c_{x/y}} \sigma_i^z$.

Anyons in the toric code

Two types of excitations

• electric charges: open loop ends violate vertex constraint

• magnetic vortices: plaquettes giving -1 when flipped

We get a minus sign taking one around the other
electric charges and magnetic vortices are mutual anyons

The toric code with a perturbation

\[ H_{TC} = -J_e \sum_{s} A_s - J_m \sum_{p} B_p + \sum_{i} \left( h_x \sigma^x_i + h_z \sigma^z_i \right) \]

\( A_s = \prod_{j \in \text{star}(s)} \sigma^x_j \)

\( B_p = \prod_{j \in \partial p} \sigma^z_j \)

topological phase

paramagnet
Toric code and the transverse field Ising model

\[ B_p = \prod_{j \in \partial p} \sigma^z_j \]

\[ B_p = 2\mu^z_p \]

\[ \sigma^x_i = \mu^x_p \mu^x_q \]

\[ \mathcal{H}_{TC} = -J_m \sum_p B_p + h_x \sum_i \sigma^x_i \]

\[ \mathcal{H}_{TFIM} = -2J_m \sum_p \mu^z_p + h_x \sum_{\langle p,q \rangle} \mu^x_p \mu^x_q \]
Mapping to 3D Ising model

\[ \mathcal{H}_{cl} = -K_{\tau} \sum_{\tau,p} S_p(\tau) S_p(\tau + \Delta\tau) - \sum_{\tau,\langle p,q \rangle} K S_p(\tau) S_q(\tau) \]

\[ K_{\tau} = -\frac{1}{2} \ln [\tanh(\Delta\tau \cdot B)] \]

plaquette flips

\[ K = \frac{1}{2} \Delta\tau \cdot h \]

magnetic field

\[ K_c = 0.2216595(26) \]


\[ B = 1 \]

\[ K_{\tau} = K \]

gives

\[ \Delta\tau = 0.761403 \]

\[ h_c = 0.58224 \]

→ numerical simulation
Quantum phase transitions

toric code

\[ \mathcal{H}_{TC} = -J_m \sum_p B_p + h_x \sum_i \sigma_i^x \]

topological phase \( \lambda_c \) \rightarrow \lambda \rightarrow \text{paramagnet}

transverse field Ising model

\[ \mathcal{H}_{TFIM} = -2J_m \sum_p \mu_p^z + h_x \sum_{\langle p,q \rangle} \mu_p^x \mu_q^x \]

disordered state \( \lambda_c \) \rightarrow \lambda \rightarrow \text{paramagnet}
Phase diagram

Excitations: condensation vs. confinement

vertex excitations
“electric charges”

plaquette excitations
“magnetic vortices”

magnetic vortices

electric charges

\[ \frac{\Delta}{2B} \]

\[ \xi_c^2/(L^2 + 2) \]

\[ L = 8 \quad L = 12 \quad L = 16 \]

topological phase
paramagnet
Toric code: phase transitions

Hamiltonian deformations $\mathcal{H}$

3D Ising universality

$z = 1$

RG flow

$z \sim 2$

2D Ising universality

“conformal QCP”

Fradkin & Shenker (1979)
ST et al. (2007)
Tupitsyn et al. (2008)
Vidal, Dusuel, Schmidt (2009)

Castelnovo & Chamon (2008)
Fendley (2008)

toric code

wavefunction deformations $|\psi\rangle$

paramagnet
quantum double models

looking beyond the drosophila
Quantum double models form a larger family of lattice models harboring non-trivial topological order, e.g. non-Abelian string nets.

- Quantum double models are generally constructed from an underlying anyon theory.
- Key ingredient are so-called fusion rules of anyons.

M. Levin and X.-G. Wen, Phys. Rev. B 71, 045110 (2005);
Quantum double models form a larger family of lattice models harboring non-trivial topological order, e.g. non-Abelian string nets.

- **Loop gas configuration**
- **String net configuration**

**Z₂ anyon theory**
- $0 \times 0 = 0$
- $0 \times 1 = 1$
- $1 \times 1 = 0$

**Fibonacci anyon theory**
- $0 \times 0 = 0$
- $0 \times 2 = 2$
- $2 \times 2 = 0 + 2$
numerics
Numerical identification of topological order

Which observables can we measure numerically to identify topological order?

- **bulk** properties
  - entanglement entropy & spectrum
  - ground-state degeneracy
- **edge** properties
  - entanglement entropy
  - energy spectra
bulk properties

entanglement entropy
If two quantum mechanical objects are interwoven in such a way that their collective state cannot be described as a product state we say they are entangled.

\[ |\psi\rangle = \cos \alpha |\uparrow\rangle_A |\downarrow\rangle_B + \sin \alpha |\downarrow\rangle_A |\uparrow\rangle_B \]

- Calculate the reduced density matrix for one of the two parts

\[ \rho_A = |\psi\rangle_A \langle \psi |_A \] (traced over subsystem B)

\[ \rho_A = \cos^2 \alpha |\uparrow\rangle_A \langle \uparrow |_A + \sin^2 \alpha |\downarrow\rangle_A \langle \downarrow |_A \]

- One quantitative measure of entanglement is the entanglement entropy

\[
S(\rho_A) = - \text{Tr}(\rho_A \log \rho_A)
\]

tvon Neumann entropy = first Renyi entropy

\[
S_n(\rho_A) = \frac{1}{1-n} \ln [\text{Tr}(\rho_A^n)]
\]

Renyi entropy

\[
S_2(\rho_A) = - \ln [\text{Tr}(\rho_A^2)]
\]

second Renyi entropy
Entanglement

If two quantum mechanical objects are \textbf{interwoven} in such a way that their \textbf{collective state} cannot be described as a product state we say they are \textbf{entangled}.

\[
|\psi\rangle = \cos \alpha |\uparrow\rangle_A |\downarrow\rangle_B + \sin \alpha |\downarrow\rangle_A |\uparrow\rangle_B
\]

\[
S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A)
\]

\[
S_2(\rho_A) = -\ln \left[ \text{Tr}(\rho_A^2) \right]
\]

\[
S_n(\rho_A) = \frac{1}{1 - n} \ln \left[ \text{Tr}(\rho_A^n) \right]
\]

\(S = \ln 2\)
Entanglement

- **Entanglement in quantum many-body systems**

  - Consider bipartition of system into two parts A and B, and calculate the **reduced density matrix** for the two parts

    \[ \rho_A = |\psi\rangle_A \langle \psi|_A \]  
    (traced over subsystem B)

  - One quantitative measure of entanglement is the **entanglement entropy**

    \[ S(\rho_A) = -\text{Tr}(\rho_A \log \rho_A) \]

    von Neumann entropy = first Renyi entropy

    \[ S_n(\rho_A) = \frac{1}{1-n} \ln \left[ \text{Tr}(\rho_A^n) \right] \]

    Renyi entropy

    \[ S_2(\rho_A) = -\ln \left[ \text{Tr}(\rho_A^2) \right] \]

    second Renyi entropy
Boundary law

Entanglement scales with the length $L$ of the boundary of the bipartition

$S \propto aL$
Corrections to the boundary law

Corrections to the boundary law can arise from

- **geometric aspects** of the bipartition

\[ S = aL + b \cdot (\# \text{ of corners}) \]
(for 2D gapped state)

- **topological aspects** of the bipartition

\[ S = aL - \gamma \cdot (\# \text{ of disconnected parts}) \]
(for 2D gapped state)
Corrections to the boundary law

Corrections to the boundary law also originate from the specific character of the underlying quantum many-body state!

- **topological** spin liquids
  \[ S = aL - \gamma \]

- **gapless** spin liquids
  - gapless modes at **singular point** in momentum space
    \[ S = aL + c\gamma(L_x, L_y) \]
  - gapless modes on **surface** in momentum space
    \[ S = cL \ln(L) \]

- **critical points, conformal critical points, Goldstone modes, ...**
  \[ S_{\text{QCP}} = aL + c\gamma(L_x, L_y) \]
  \[ S_{\text{cQCP}} = \mu L + \gamma_{\text{cQCP}} \]
  \[ S_{\text{G}} = aL + b\ln(L) + \gamma(L_x, L_y) \]
Topological entanglement entropy

Distilling the **topological correction** by using a clever sequence of partitions

\[
S = aL + b \cdot (\text{# of corners}) - \gamma \cdot (\text{# of disconnected parts})
\]

\[
S_{\text{topo}} = -S_{A_1} + S_{A_2} + S_{A_3} - S_{A_4} = -2\gamma
\]

Topological entanglement entropy

The topological correction is universal.

\[ \gamma = \ln \sqrt{\sum_{i=1}^{n} d_i^2} \]

quantum dimension of excitation

Examples:

- toric code (loop gas)
  
  \[
  \begin{array}{cccc}
  1 & e & m & em \\
  d_i & 1 & 1 & 1 & 1
  \end{array}
  \]
  \[ \gamma = \ln \sqrt{1 + 1 + 1 + 1} = \ln 2 \approx 0.693 \]

- Fibonacci theory (string net)
  
  \[
  \begin{array}{c}
  1 \\
  d_i & 1 & \tau \\
  \phi = \frac{1+\sqrt{5}}{2}
  \end{array}
  \]
  \[ \gamma = \ln \sqrt{1 + \phi^2} \approx 0.643 \]

A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006);
Let’s try this on the toric code

Examples:  

<table>
<thead>
<tr>
<th>1</th>
<th>e</th>
<th>m</th>
<th>em</th>
</tr>
</thead>
<tbody>
<tr>
<td>di</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ \gamma = \ln \sqrt{1 + 1 + 1 + 1} = \ln 2 \approx 0.693 \]

Identifying topological order by entanglement entropy

Hong-Chen Jiang\(^1\), Zhenghan Wang\(^2\) and Leon Balents\(^1\)*


**Figure 2** | The von Neumann entropy \( S(L_y) \) for the toric-code model in magnetic fields. a. \( S(L_y) \) with \( L_x = 4 - 16 \) at \( L_z = \infty \) for symmetric magnetic fields at \( h_x = h_y = h = 0.2, 0.3 \) and 0.4. By fitting \( S(L_y) = \alpha L_y - \gamma \), we get \( \gamma = 0.693(1), 0.691(4) \) and 0.001(5), respectively. b. The pure electric case, \( h_x = 0.3, h_y = 0.0 \), and comparison of \( S(L_y) \) in the MES obtained in the large \( L_y \) limit (black squares) with that of the absolute ground state from systems of dimensions \( L_x \times L_y = 20 \times 4, 24 \times 6, 24 \times 8, 24 \times 10 \) (red circles). Extrapolation shows that the MES has the universal TEE, whereas the absolute ground state has zero TEE.
Kagomé antiferromagnet

Heisenberg model

\[ H = \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \]

on kagomé lattice

FIG. 6 (color online). Renyi entropies \( S_\alpha \) of infinitely long cylinders for various \( \alpha \) versus circumference \( c \), extrapolated to \( c = 0 \). The negative intercept is the topological entanglement entropy \( \gamma \).


Figure 3 | The entanglement entropy \( S(L_y) \) of the kagome \( J_1-J_2 \) model in equation (2), with \( L_y = 4-12 \) at \( L_x = \infty \). By fitting \( S(L_y) = aL_y-y \), we get \( y = 0.698(8) \) at \( J_2 = 0.10 \) and \( y = 0.694(6) \) at \( J_2 = 0.15 \). Inset: kagome lattice with \( L_y = 12 \) and \( L_y = 8 \).

bulk properties

entanglement spectrum
The entanglement entropy is one quantitative measure of entanglement.

\[ \rho_A = |\psi\rangle_A \langle \psi|_A \]  
(traced over subsystem B)

\[ S(\rho_A) = - \text{Tr}(\rho_A \log \rho_A) \]

However, a lot of information possibly contained in the density matrix is discarded in this simple measure.

The entanglement spectrum aims at unraveling some of this information.

\[ \rho_A = |\psi\rangle_A \langle \psi|_A = \sum_{\alpha} \lambda_{\alpha}^2 |\phi_{\alpha}\rangle_A \langle \phi_{\alpha}|_A \]  
(Schmidt decomposition)

\[ S(\rho_A) = - \sum_{\alpha} \lambda_{\alpha}^2 \ln \lambda_{\alpha}^2 \]  
\[ \lambda_{\alpha} = e^{-\xi_{\alpha}/2} \]  
\[ \rho_A = e^{-H_E} \]  
\[ (H_E = - \ln \rho_A) \]

Entanglement spectrum

- The entanglement spectrum aims at unraveling some of the information contained in the density matrix

\[ \rho_A = |\psi\rangle_A \langle \psi|_A = \sum_{\alpha} \lambda_{\alpha}^2 |\phi_{\alpha}\rangle_A \langle \phi_{\alpha}|_A \]

\[ \lambda_{\alpha} = e^{-\xi_{\alpha}/2} \quad \rho_A = e^{-H_E} \]

Entanglement summary

- The **entanglement entropy** and **entanglement spectrum** are powerful bulk “observables” that allow
  - to **unambiguously distinguish topological order** from conventional order
  - to characterize the type of topological order to a good extent*

* in some measures, e.g. the topological entanglement entropy, some unambiguities might remain and require additional work

- The **reduced density matrix** is **readily available** in some numerical methods such as
  - exact diagonalization
  - density matrix renormalization group
    \[ \rho_A = |\psi\rangle_A \langle\psi|_A \]

- Other numerical techniques have caught up, in particular
  - quantum Monte Carlo → replica trick → Renyi entropies / entanglement entropy
    You will hear about this in Peter Bröcker’s talk tomorrow morning.
  - numerical linked cluster expansion → mutual information
bulk properties

ground-state degeneracy
Ground-state degeneracy

$\delta E \propto \exp(-\alpha L)$

$\delta E \propto L$

edge properties

entanglement entropy
Edge states

- **Edge states** correspond to the **gapless modes** of a critical one-dimensional system, which are typically described by a **conformal field theory** (CFT).

- The conformal field theory can again be identified via entanglement properties

\[
S = \frac{c}{3} \log \left( L \sin \left( \frac{\pi \ell}{L} \right) \right) \quad \ell = L/2 \quad S = \frac{c}{3} \log L
\]

C. Holzhey et al., Nucl. Phys. B 424, 44 (1994);
Edge states

- **Edge states** correspond to the gapless modes of a critical one-dimensional system, which are typically described by a conformal field theory (CFT).

- The CFT can be identified via entanglement properties

  \[ S = \frac{c}{3} \log L \]

- Even more information about the CFT reveals itself in the energy spectrum

  \[ E = E_1 L + \frac{2\pi \nu}{L} \left( -\frac{c}{12} + x_s \right) \]

![FIG. 2](image.png)  

![FIG. 3](image.png)  
We are done!

So what did we learn?
Summary

• The formation of **topological order** in an interacting quantum many-body system is one of the **most fascinating** phenomena in condensed matter physics.

• **Numerical investigations** of topological order build on concepts from
  - statistical physics
  - quantum information theory
  - mathematical physics
  - van Neumann & Renyi entropies
  - entanglement
  - boundary laws, anyon theories

• The exploration of topological order is a **rich and quickly evolving research field** – just at its beginning.

*All slides* of this presentation will become available on the **FOR1807 website** and our group webpage at [www.thp.uni-koeln.de/trebst](http://www.thp.uni-koeln.de/trebst)