Topological order

Concepts, numerics, and examples

Fall school FOR1807
Würzburg, October 2013

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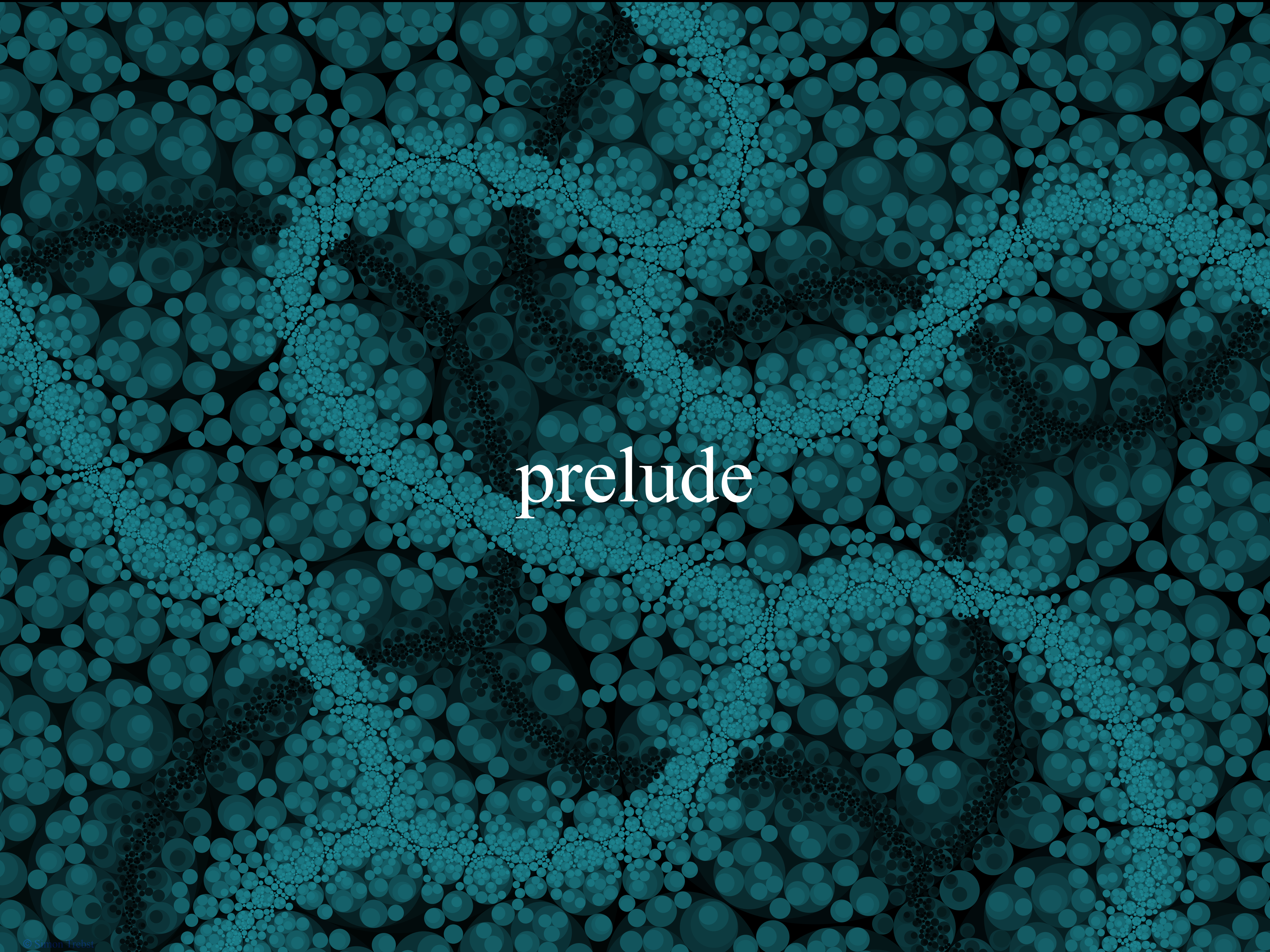
Topological order

concepts

examples

numerics

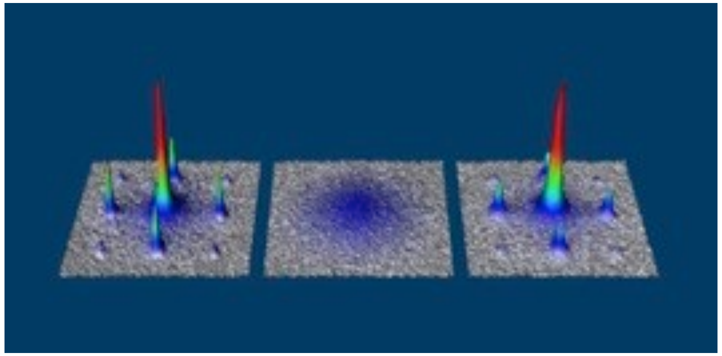
prelude



(Quantum) matter

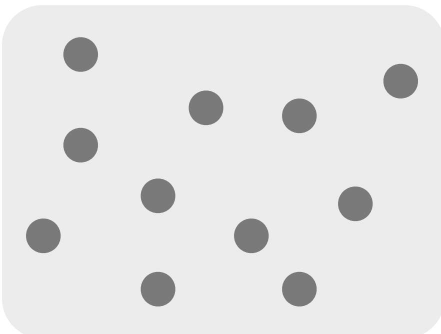
water ice

superconductor



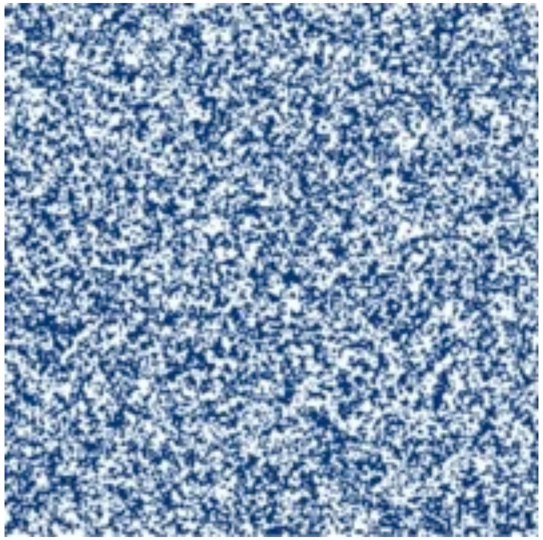
Bose-Einstein condensate

Motivation – a paradigm

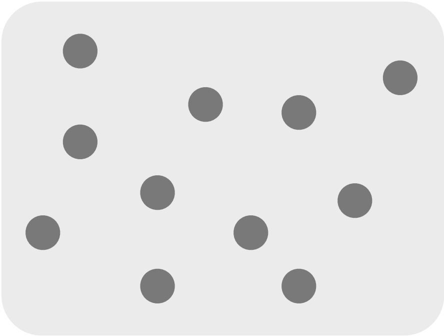


interacting
many-body system

$$\mathcal{H} = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$



Motivation – a paradigm



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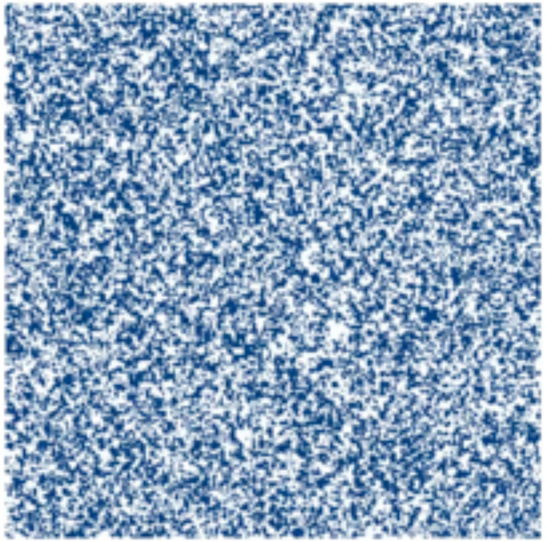
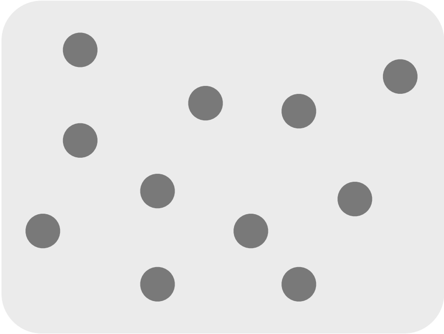


Motivation – a paradigm



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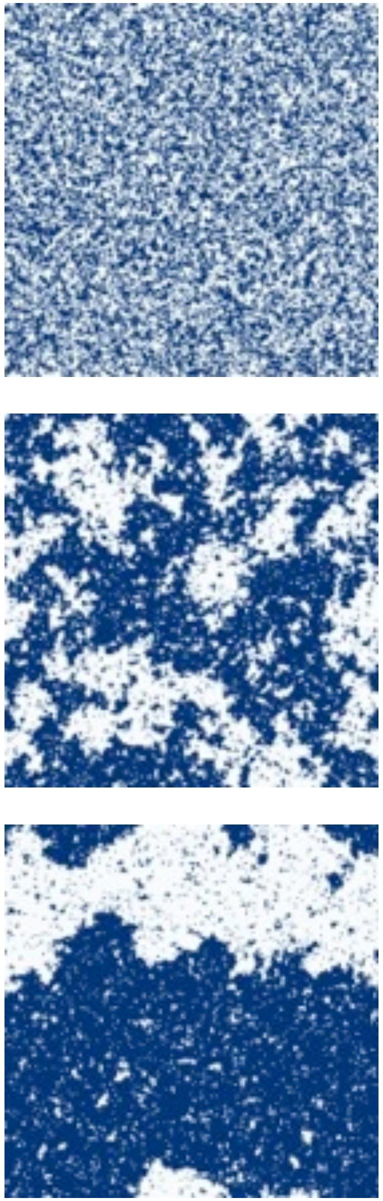
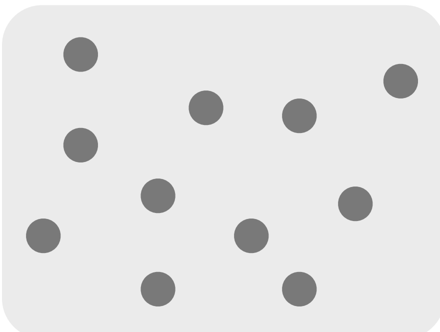


Spontaneous symmetry breaking

- ground state has **less** symmetry than Hamiltonian
- **local** order parameter
- **phase transition** / Landau-Ginzburg-Wilson theory

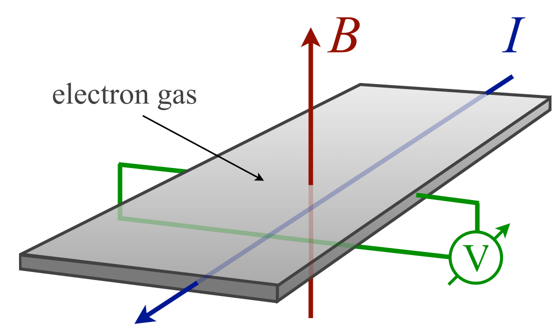
Every rule has an exception



interacting
many-body system

$$\mathcal{H} = \sum_{j=1}^{N} \left(\frac{1}{2m} \left(\mathbf{p}_j - \frac{e}{c}\mathbf{A}(\mathbf{x}_j) \right)^2 + e\mathbf{A}_0(\mathbf{x}_j) \right) + \sum_{i<j} V(|\mathbf{x}_i - \mathbf{x}_j|)$$

$$\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$



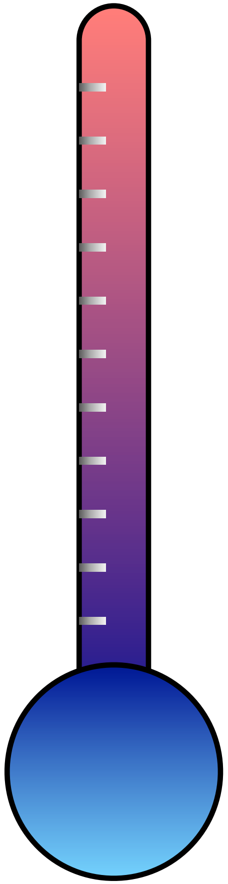
Sometimes, the exact opposite happens

- ground state has **more** symmetry than Hamiltonian
- **non-local** order parameter
- **emergence** of degenerate ground states, exotic statistics, ...

Topological quantum matter

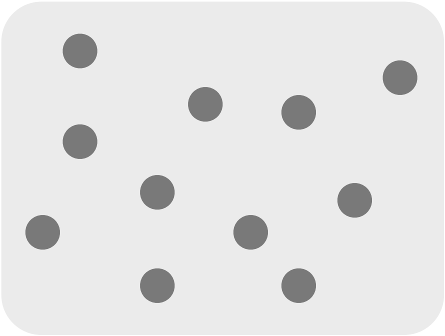
- **Spontaneous symmetry breaking**
  - ground state has **less** symmetry than Hamiltonian
  - Landau-Ginzburg-Wilson theory
  - **local** order parameter

- **Topological order**
  - ground state has **more** symmetry than Hamiltonian
  - degenerate ground states
  - **non-local** order parameter

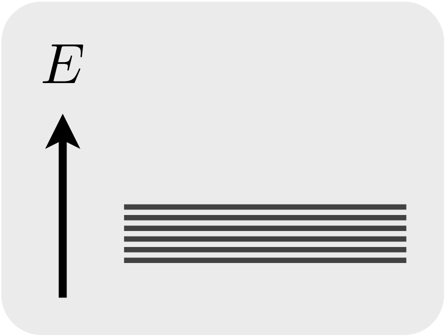


When does this happen?

Some of the most intriguing phenomena in condensed matter physics arise from the splitting of ‘accidental’ degeneracies.



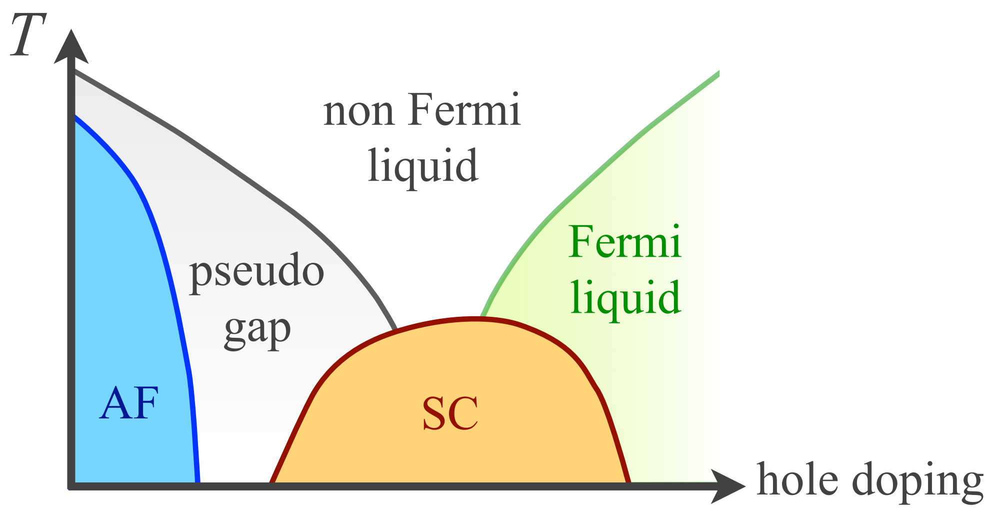
interacting
many-body system



‘accidental’
degeneracy

residual effects
select ground state

phase diagram of
cuprate superconductors



When does this happen?

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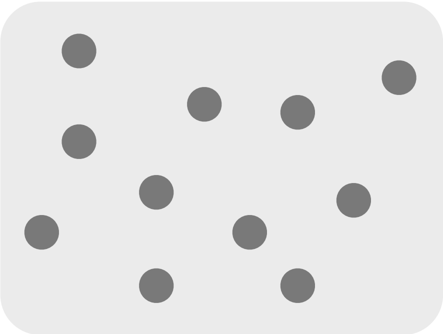
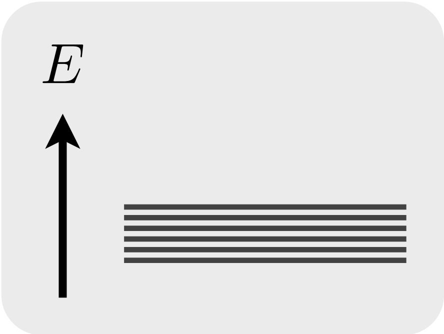


interacting
many-body system



‘accidental’
degeneracy

residual effects
select ground state

But they are also notoriously difficult to handle, due to

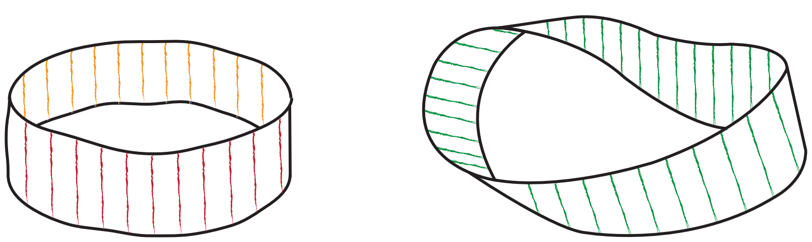
- multiple energy scales
- complex energy landscapes / slow equilibration
- strong coupling

concepts

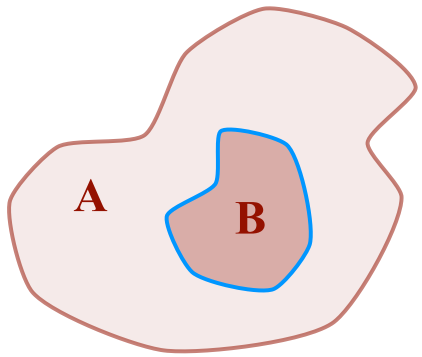
Topological quantum matter

- Xiao-Gang Wen: A ground state of a many-body system that *cannot* be fully characterized by a *local* order parameter.

- Often characterized by a variety of non-local "*topological properties*".



- A topological phase can be positively identified by its *entanglement properties*.



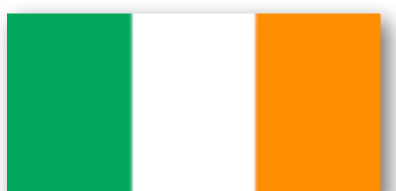
Knots & edge states

- Bringing a topological and a conventional state into spatial proximity will result in a *gapless edge state* – literally a *knot* in the wavefunction.

- We know this: "Counterintuitive states"

China

UK



Ireland

Australia

Japan

Hong Kong

Knots & edge states

China

Hong Kong

Flipper bridge

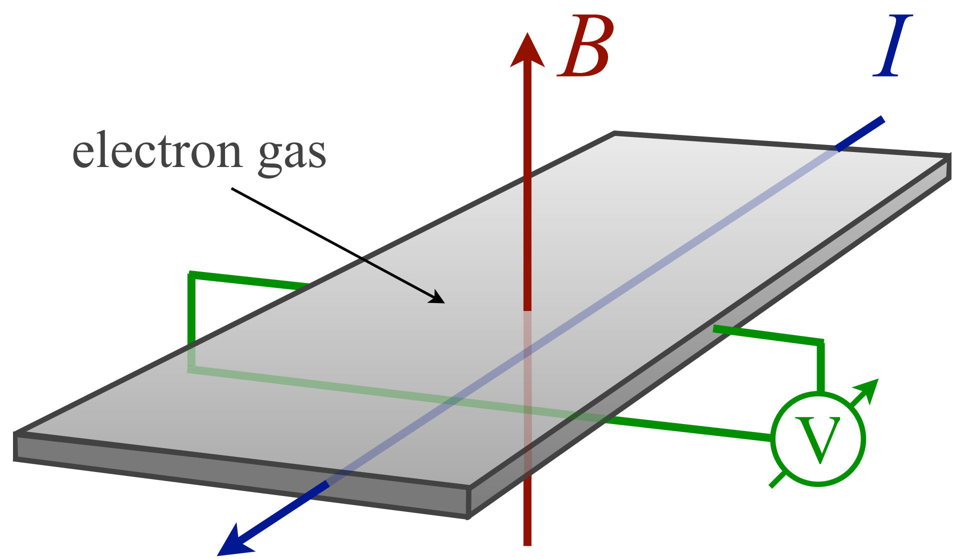
the archetypal example

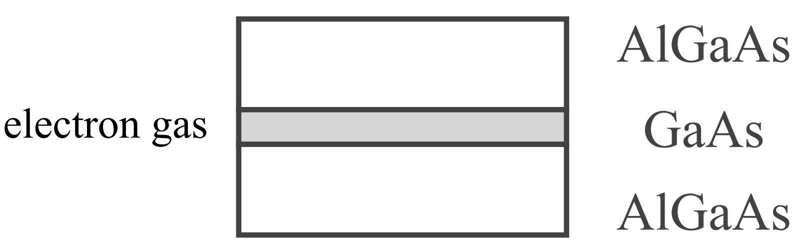
quantum Hall effect

Quantum Hall effect

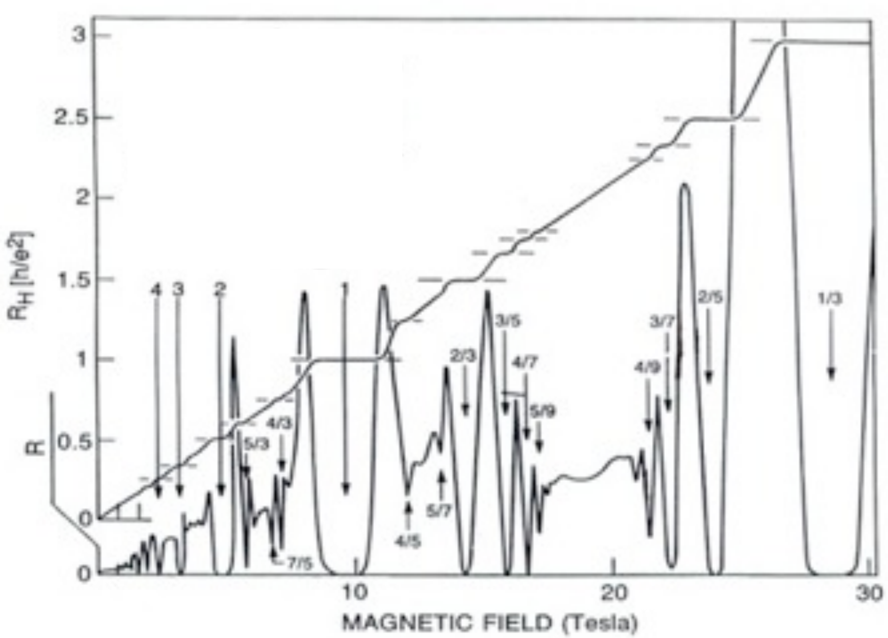
Quantization of Hall conductivity
for a two-dimensional electron gas
at very low temperatures
in a high magnetic field.

$$\sigma = \nu \frac{e^2}{h}$$





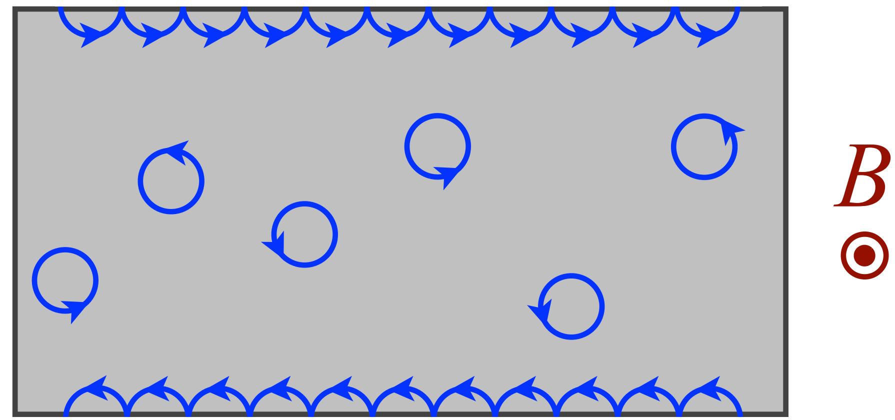
Semiconductor heterostructure confines
electron gas to two spatial dimensions.



Quantum Hall states

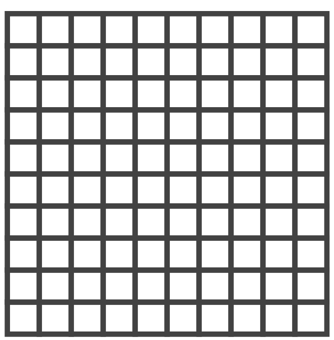
Landau levels

$$E_n = h\frac{eB}{m}\left(n + \frac{1}{2}\right)$$



edge states

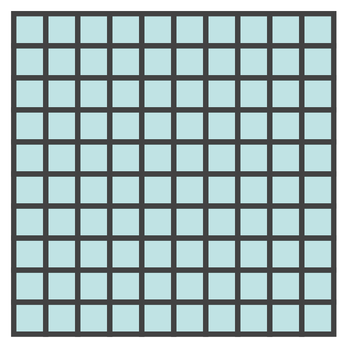
Landau level **degeneracy**



$$2\Phi/\Phi_0$$

orbital states

integer quantum Hall

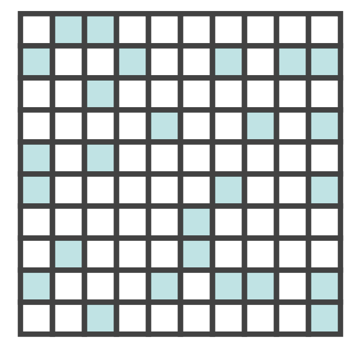


filled level

↓

incompressible liquid

fractional quantum Hall



partially filled level

↓ Coulomb repulsion

incompressible liquid

examples

the **toric code**

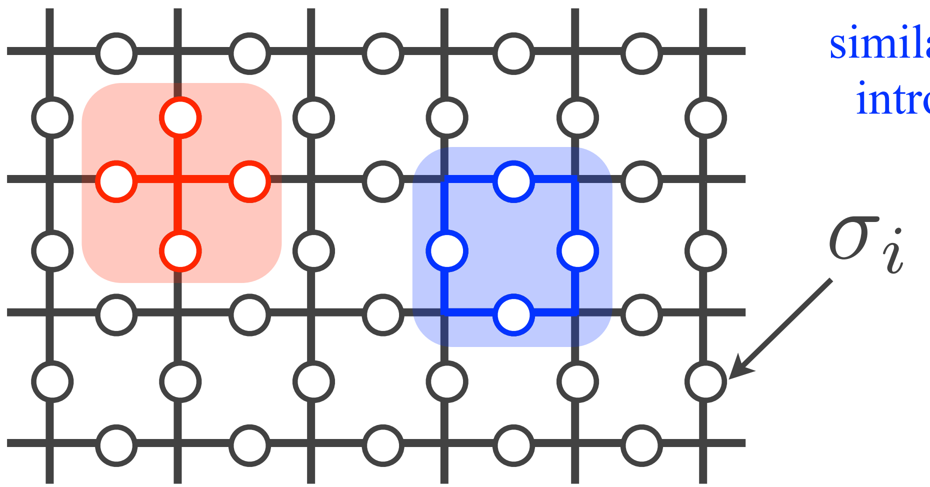
the drosophila
for lattice models
of topologically ordered phases

The toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A\sum_{v}\prod_{j\in\mathrm{vertex}(v)}\sigma_j^z - B\sum_{p}\prod_{j\in\mathrm{plaquette}(p)}\sigma_j^x$$

similar to ring exchange
introduces frustration



Hamiltonian has only local terms.

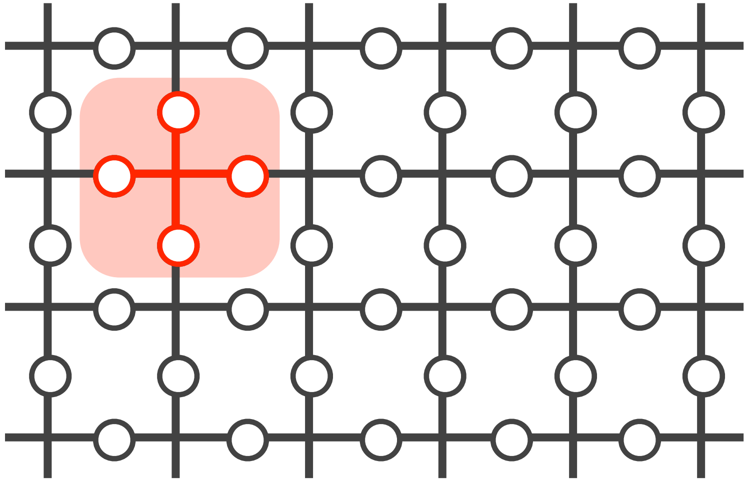
All terms commute → **exact solution!**

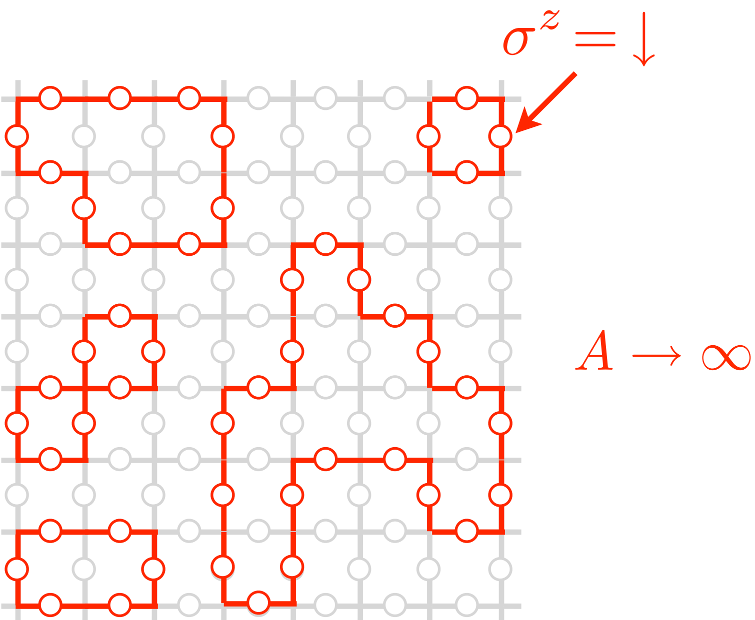
The vertex term

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = \boxed{-A\sum_{v}\prod_{j\in\mathrm{vertex}(v)}\sigma_j^z} - B\sum_{p}\prod_{j\in\mathrm{plaquette}(p)}\sigma_j^x$$

- is minimized by an **even** number of down-spins around a vertex.
- Replacing down-spins by loop segments maps ground state to closed loops.
- Open ends are (charge) excitations costing energy $2A$.



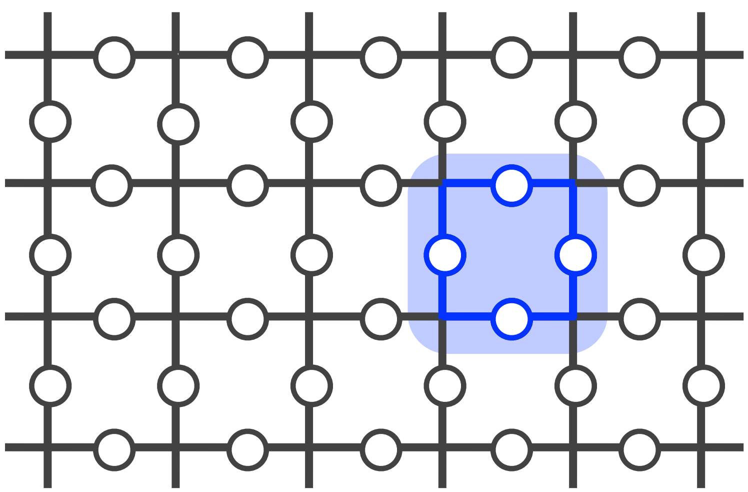


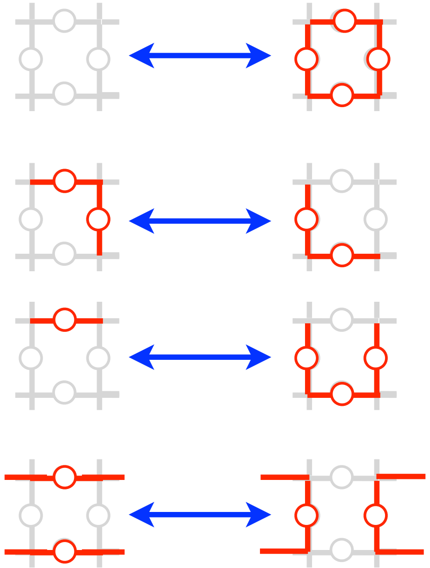
The plaquette term

A. Kitaev, Ann. Phys. **303**, 2 (2003).

$$H_{\mathrm{TC}} = -A \sum_{v} \prod_{j \in \mathrm{vertex}(v)} \sigma_j^z - \boxed{B \sum_{p} \prod_{j \in \mathrm{plaquette}(p)} \sigma_j^x}$$

- flips all spins on a plaquette.
- favors equal amplitude superposition of all loop configurations.
- Sign changes upon flip (vortices) cost energy $2B$.





fugacity = 1
free loop creation

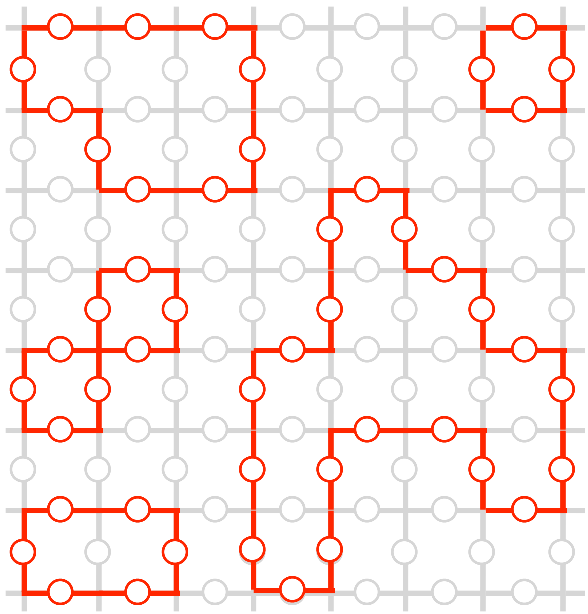
isotopy
free local distortions

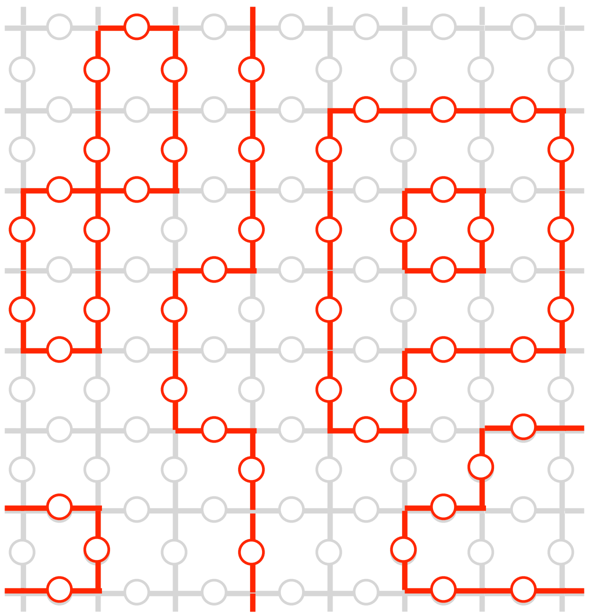
surgery
reconnections

Toric code – quantum loop gases

A. Kitaev, Ann. Phys. **303**, 2 (2003).

Ground-state manifold is a **quantum loop gas**.





Ground-state wavefunction is **equal superposition** of loop configurations.

Toric code – quantum loop gases

A. Kitaev, Ann. Phys. **303**, 2 (2003).

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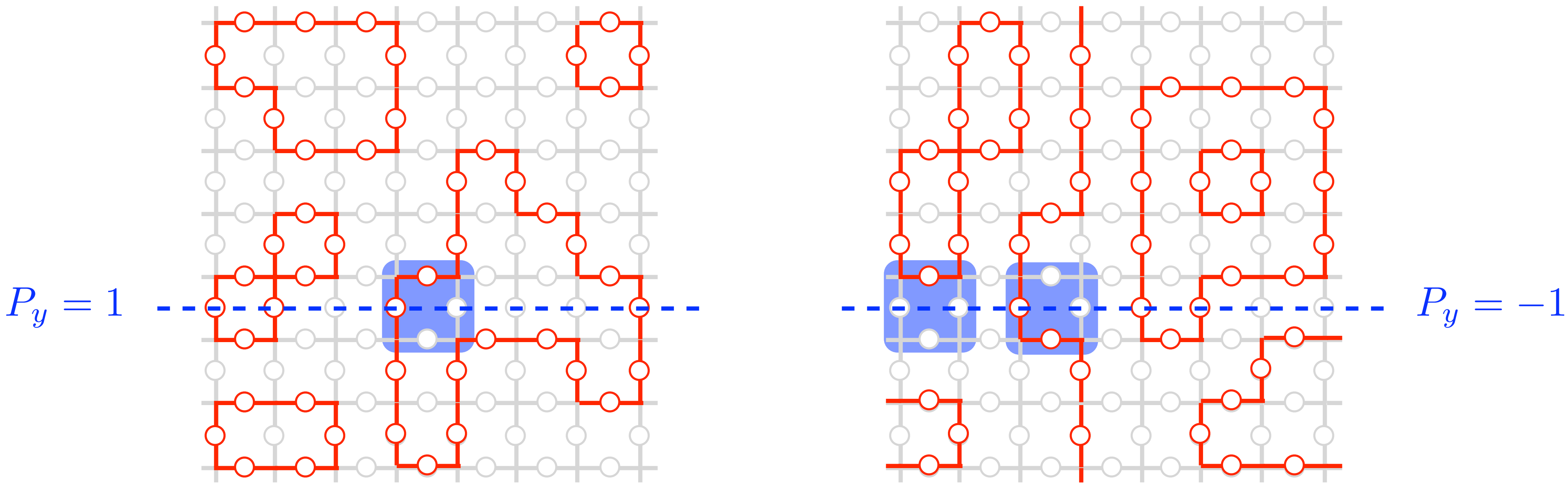

Topological sectors defined by winding number parity $P_{y/x} = \prod_{i \in c_{x/y}} \sigma_i^z$

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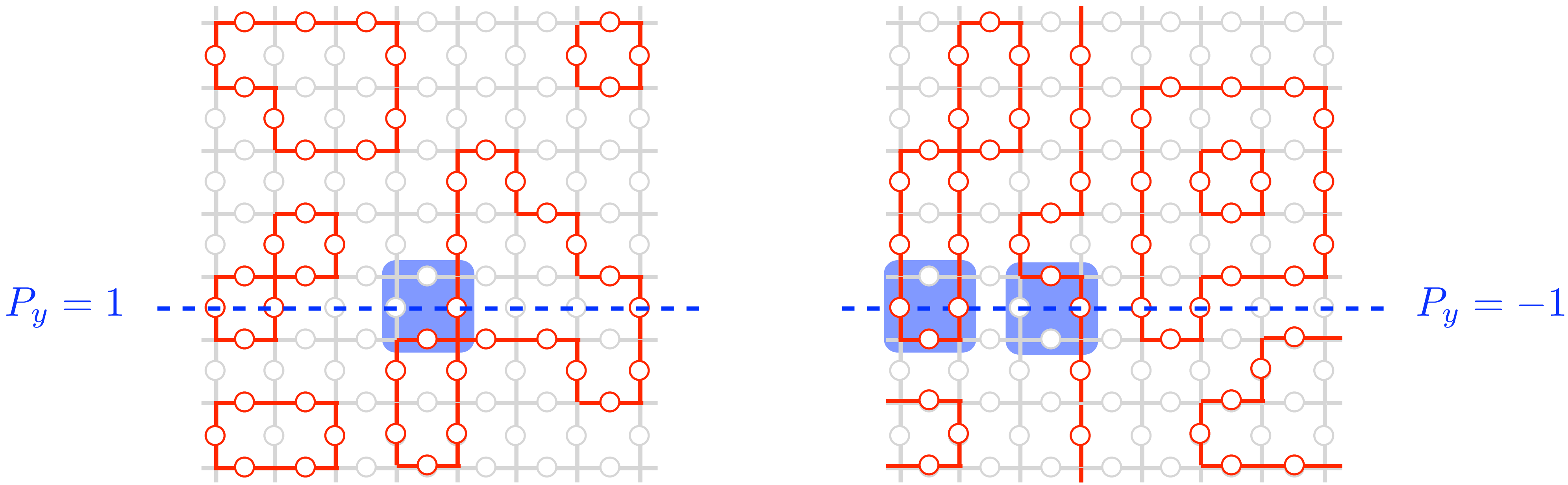


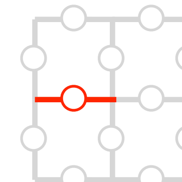
Topological sectors defined by winding number parity $P_{y/x} = \prod_{i \in c_{x/y}} \sigma_i^z$

Anyons in the toric code

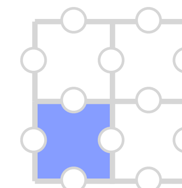
A. Kitaev, Ann. Phys. **303**, 2 (2003).

Two types of excitations

- **electric charges**: open loop ends violate vertex constraint

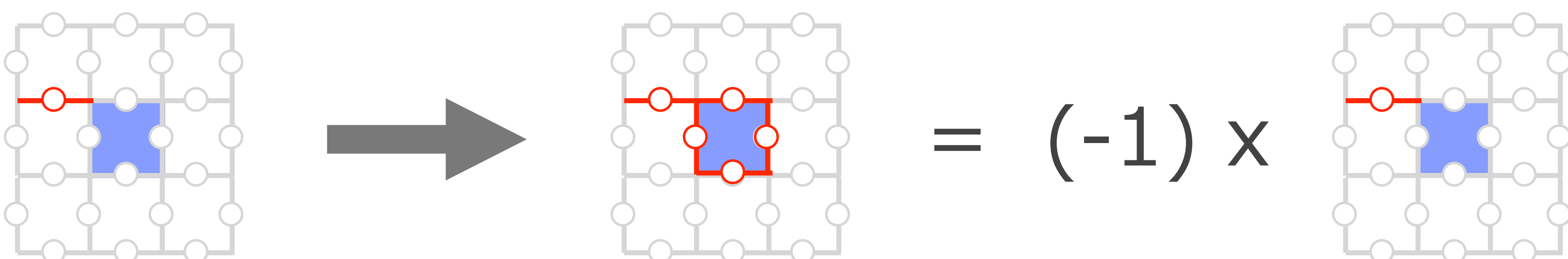


- **magnetic vortices**: plaquettes giving -1 when flipped

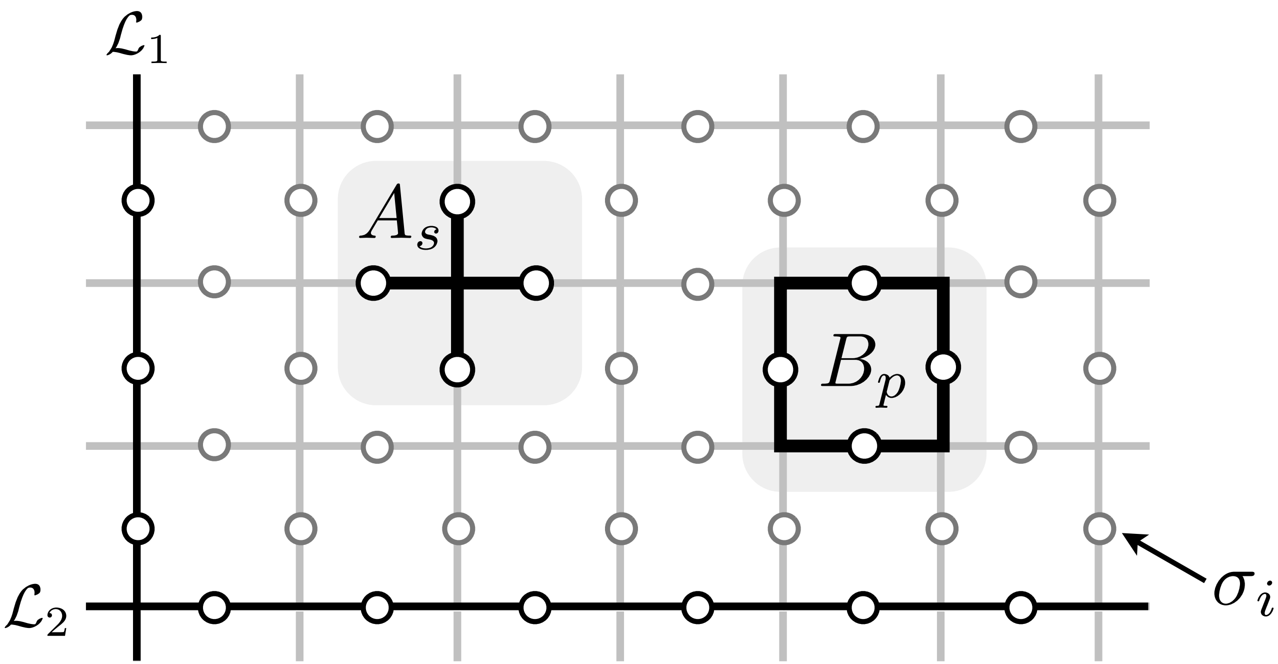


We get a minus sign taking one around the other

electric charges and magnetic vortices are **mutual anyons**



The toric code with a perturbation



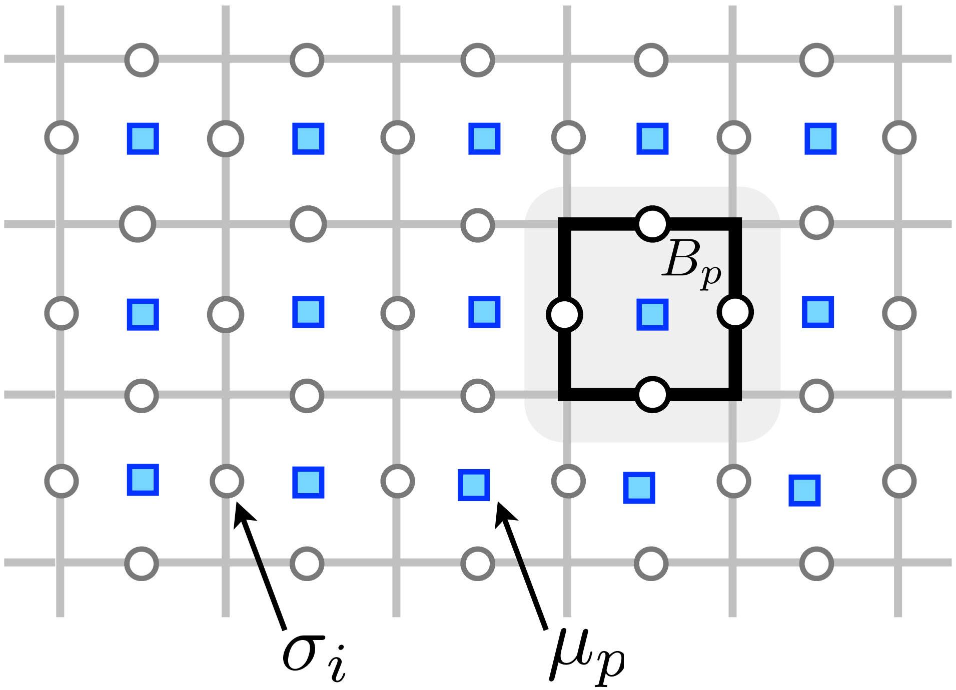
$$A_s = \prod_{j \in \mathrm{star}(s)} \sigma_j^x$$

$$B_p = \prod_{j \in \partial p} \sigma_j^z$$

$$\mathcal{H}_{\mathrm{TC}} = -J_e \sum_s A_s - J_m \sum_p B_p + \sum_i \left(h_x \sigma_i^x + h_z \sigma_i^z \right)$$

topological phase **paramagnet**

Toric code and the transverse field Ising model



$$B_p = \prod_{j \in \partial p} \sigma_j^z$$

$$B_p = 2\mu_p^z$$

$$\sigma_i^x = \mu_p^x \mu_q^x$$

$$\mathcal{H}_{\mathrm{TC}} = -J_m \sum_p B_p + h_x \sum_i \sigma_i^x$$

$$\Downarrow$$

$$\mathcal{H}_{\mathrm{TFIM}} = -2J_m \sum_p \mu_p^z + h_x \sum_{\langle p,q \rangle} \mu_p^x \mu_q^x$$

Mapping to 3D Ising model

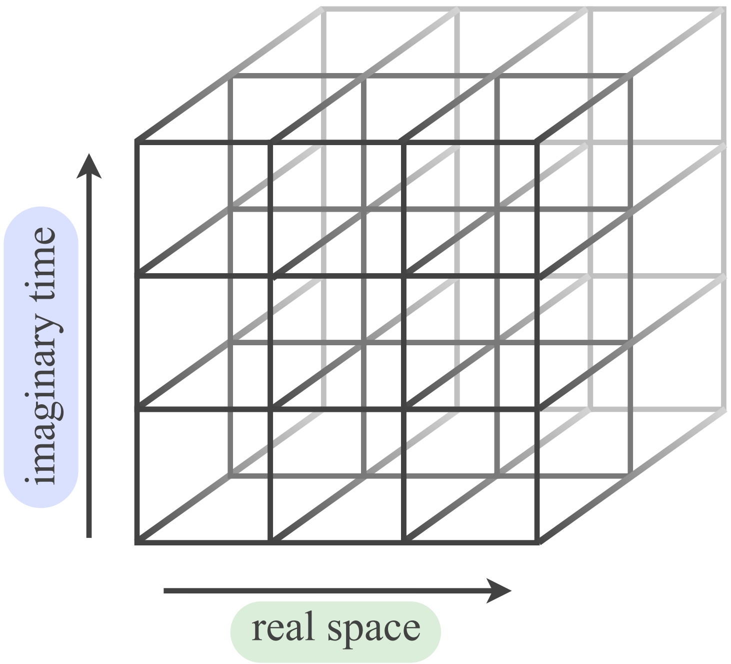
$$\mathcal{H}_{\text{cl}} = -K_\tau \sum_{\tau,p} S_p(\tau) S_p(\tau + \Delta\tau) - K \sum_{\tau,\langle p,q \rangle} S_p(\tau) S_q(\tau)$$

$$K_\tau = -\frac{1}{2} \ln\left[\tanh(\Delta\tau \cdot B)\right]$$

plaquette flips

$$K = \frac{1}{2} \Delta\tau \cdot h$$

magnetic field



$$K_c = 0.2216595(26)$$

A. M. Ferrenberg and D. P. Landau,
Phys. Rev. B **44**, 5081 (1991).

$$\begin{aligned} B &= 1 \\ K_\tau &= K \end{aligned} \quad \text{gives} \quad \begin{aligned} \Delta\tau &= 0.761403 \\ h_c &= 0.58224 \end{aligned}$$

→ numerical simulation

Quantum phase transitions

toric code

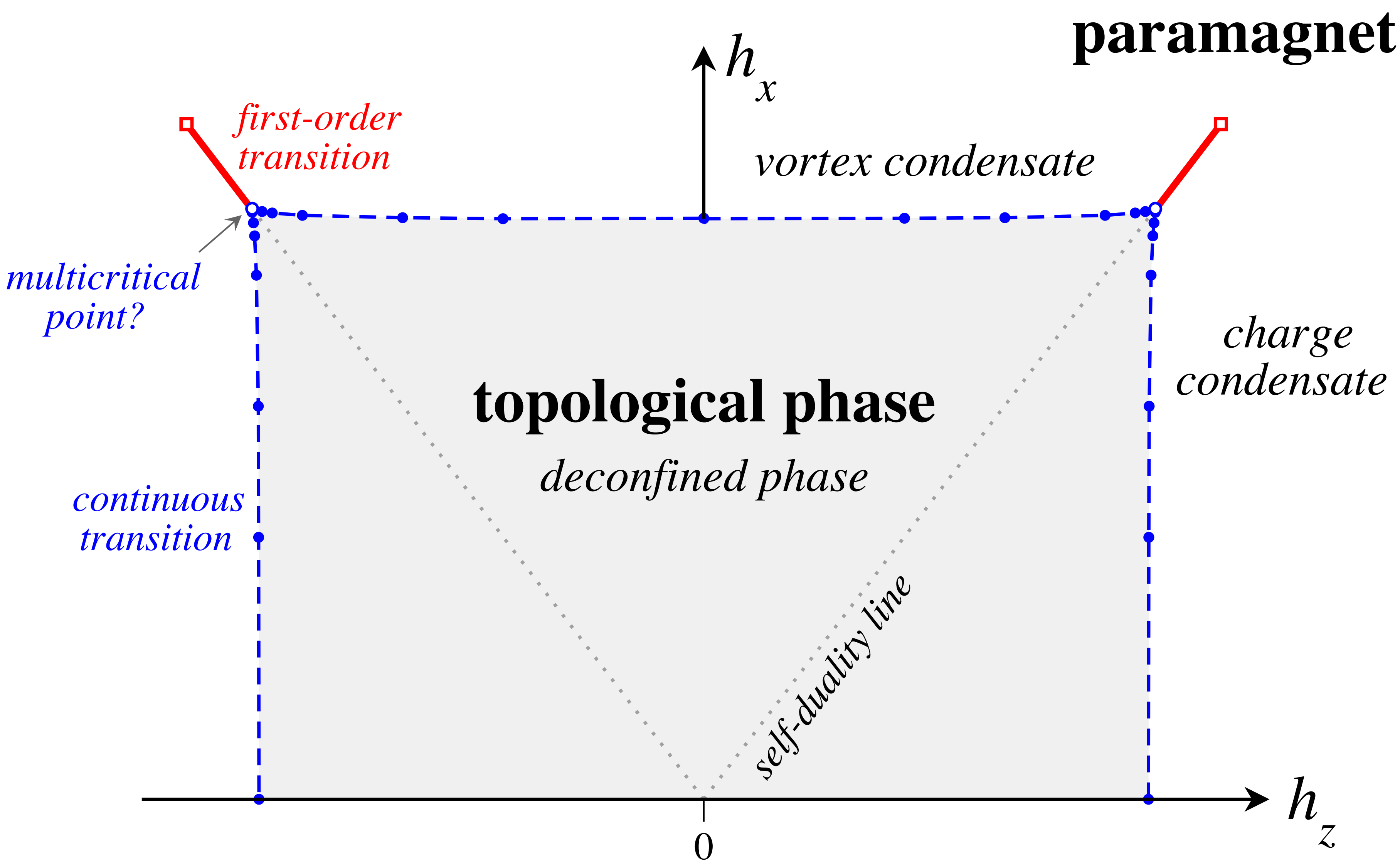
$$\mathcal{H}_{\mathrm{TC}} = -J_m \sum_p B_p + h_x \sum_i \sigma_i^x$$

transverse field Ising model

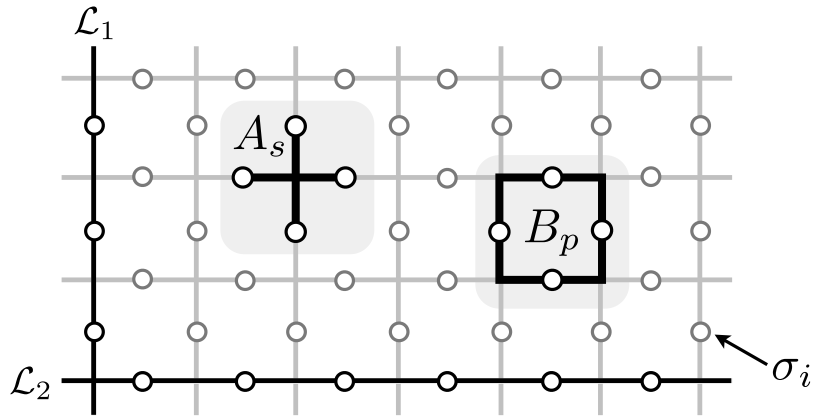
$$\mathcal{H}_{\mathrm{TFIM}} = -2J_m \sum_p \mu_p^z + h_x \sum_{\langle p,q \rangle} \mu_p^x \mu_q^x$$

Phase diagram

I.S. Tupitsyn *et al.*, Phys. Rev. B **82**, 085114 (2010)



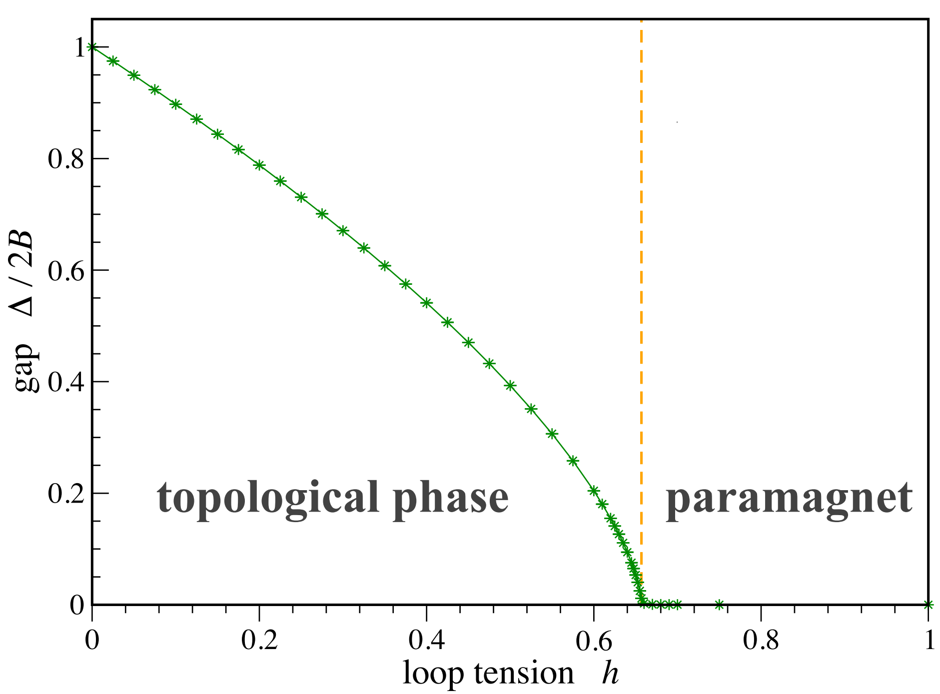
Excitations: condensation vs. confinement



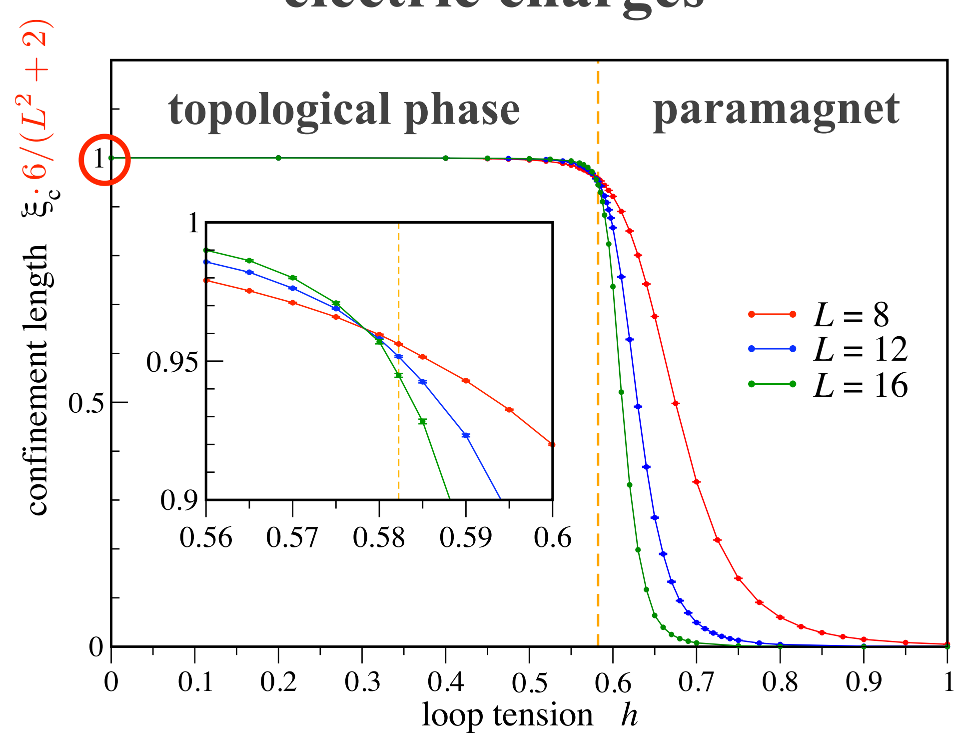
vertex excitations
"electric charges"

plaquette excitations
"magnetic vortices"

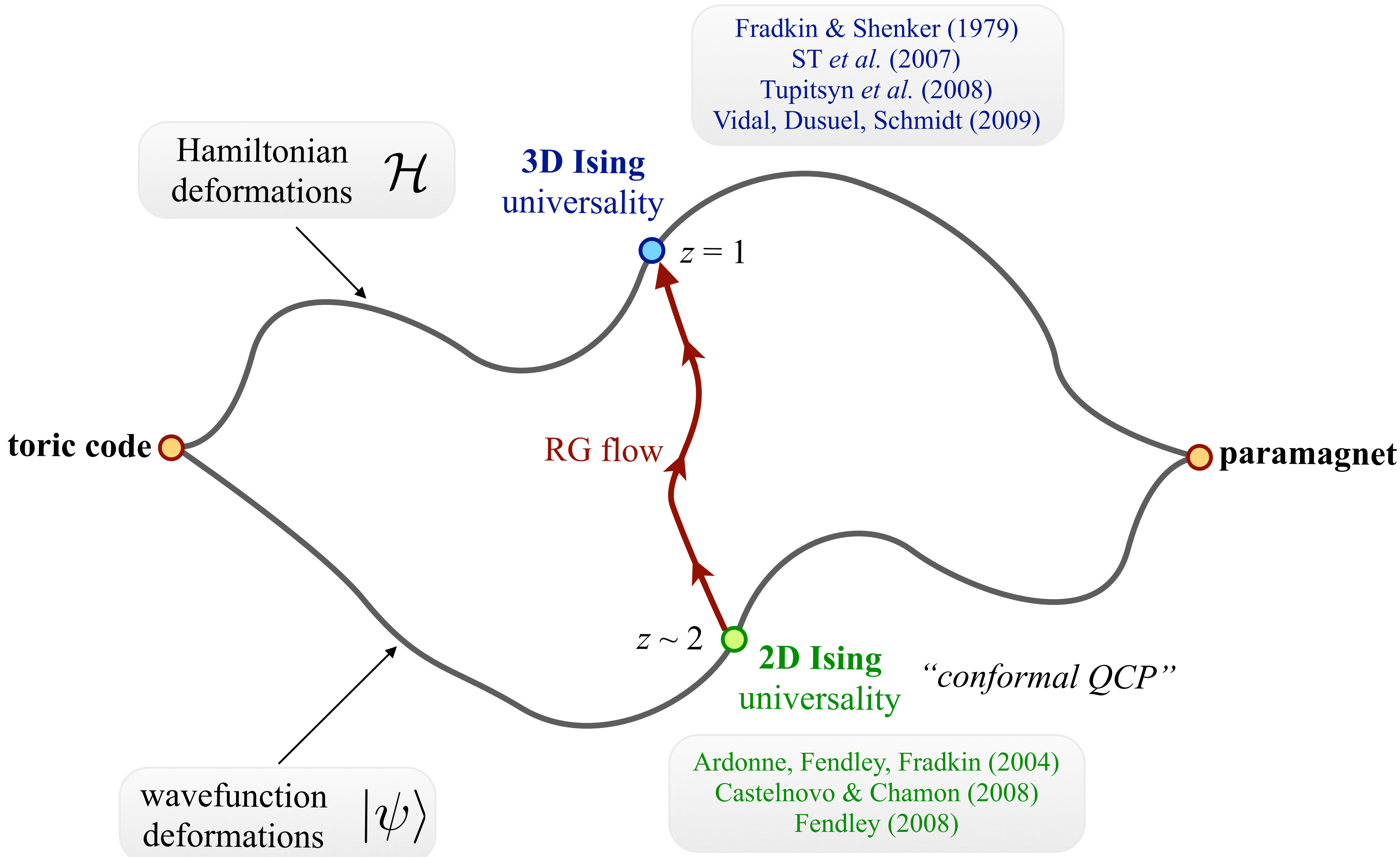
magnetic vortices



electric charges



Toric code: phase transitions

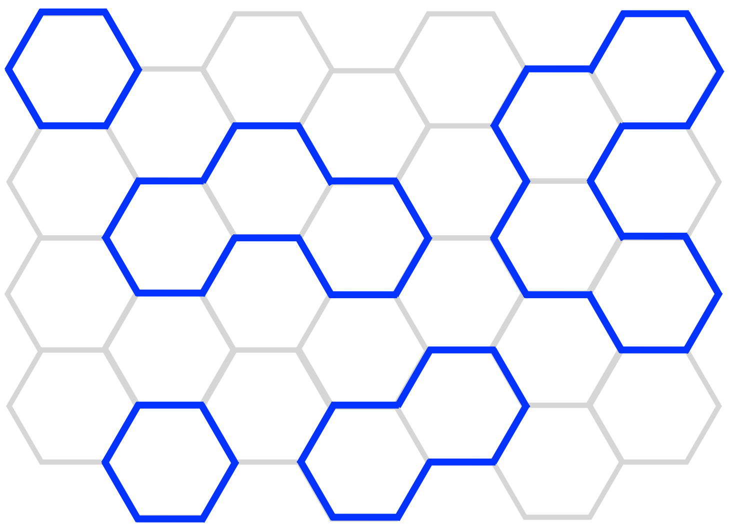


quantum double models

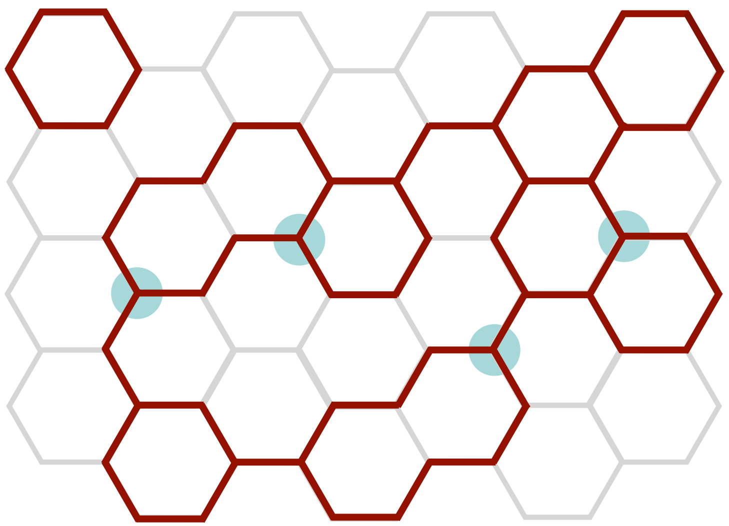
looking beyond the drosophila

quantum double models

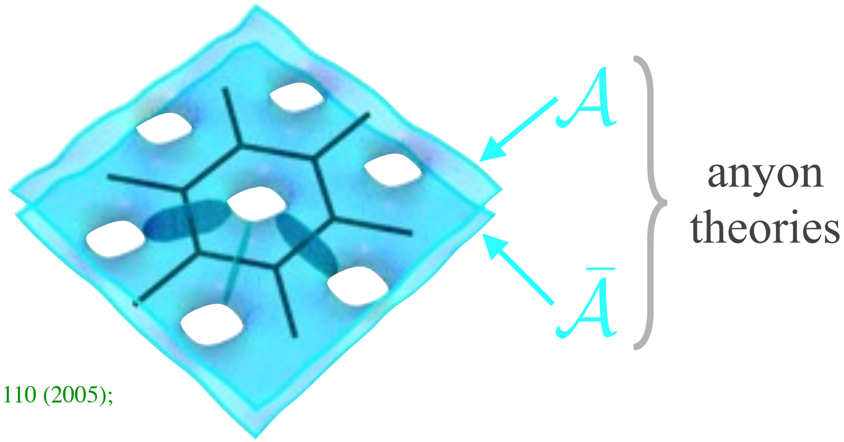
Quantum double models form a larger family of lattice models harboring non-trivial topological order, e.g. non-Abelian **string nets**.



loop gas configuration
(toric code)



string net configuration



- Quantum double models are generally constructed from an underlying **anyon theory**.
- Key ingredient are so-called **fusion rules** of anyons.

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005);
C. Gils et al., Nature Physics **5**, 834 (2009).

quantum double models

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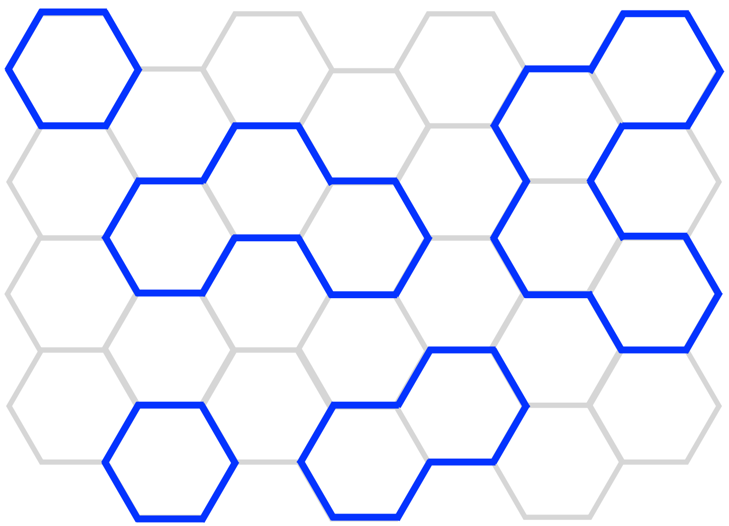
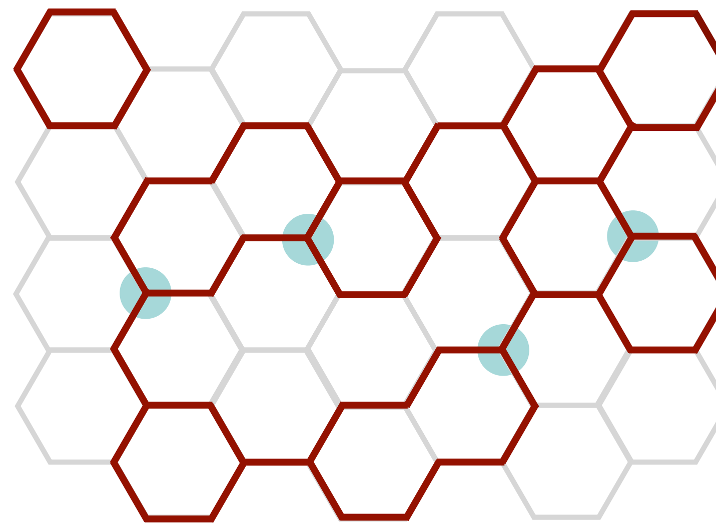


loop gas configuration



string net configuration

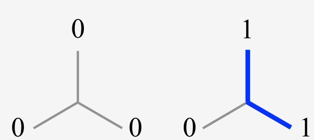
Z_2 anyon theory

$0 \times 0 = 0$

$0 \times 1 = 1$

$1 \times 1 = 0$

toric code



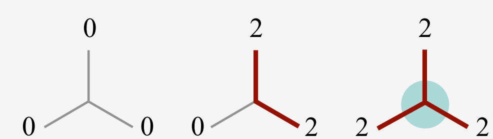
Fibonacci anyon theory

$0 \times 0 = 0$

$0 \times 2 = 2$

$2 \times 2 = 0 + 2$

non-Abelian



numerics

Numerical identification of topological order

Which **observables** can we measure numerically to identify topological order?

- **bulk** properties
  - entanglement entropy & spectrum
  - ground-state degeneracy
- **edge** properties
  - entanglement entropy
  - energy spectra

bulk properties

entanglement entropy

Entanglement

If two quantum mechanical objects are **interwoven** in such a way that their **collective state** cannot be described as a product state we say they are **entangled.**

$$|\psi\rangle = \cos\alpha |\underset{A}{\uparrow}\rangle|\underset{B}{\downarrow}\rangle + \sin\alpha|\underset{A}{\downarrow}\rangle|\underset{B}{\uparrow}\rangle$$

- Calculate the **reduced density matrix** for one of the two parts

$$\rho_A = |\psi\rangle_A\langle\psi|_A$$

(traced over subsystem B)

$$\rho_A = \cos^2\alpha|\underset{A}{\uparrow}\rangle\langle\underset{A}{\uparrow}| + \sin^2\alpha|\underset{A}{\downarrow}\rangle\langle\underset{A}{\downarrow}|$$

- One quantitative measure of entanglement is the **entanglement entropy**

$$S(\rho_A) = -\mathrm{Tr}(\,\rho_A \log \rho_A\,)$$

von Neumann entropy = first Renyi entropy

$$S_n(\rho_A) = \frac{1}{1-n}\ln\left[\mathrm{Tr}(\,\rho_A^n\,)\right]$$

Renyi entropy

$$S_2(\rho_A) = -\ln\left[\mathrm{Tr}(\,\rho_A^2\,)\right]$$

second Renyi entropy

Entanglement

If two quantum mechanical objects are **interwoven** in such a way that their **collective state** cannot be described as a product state we say they are **entangled.**

$$|\psi\rangle = \cos\alpha |\uparrow\rangle|\downarrow\rangle + \sin\alpha|\downarrow\rangle|\uparrow\rangle$$

A B A B

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

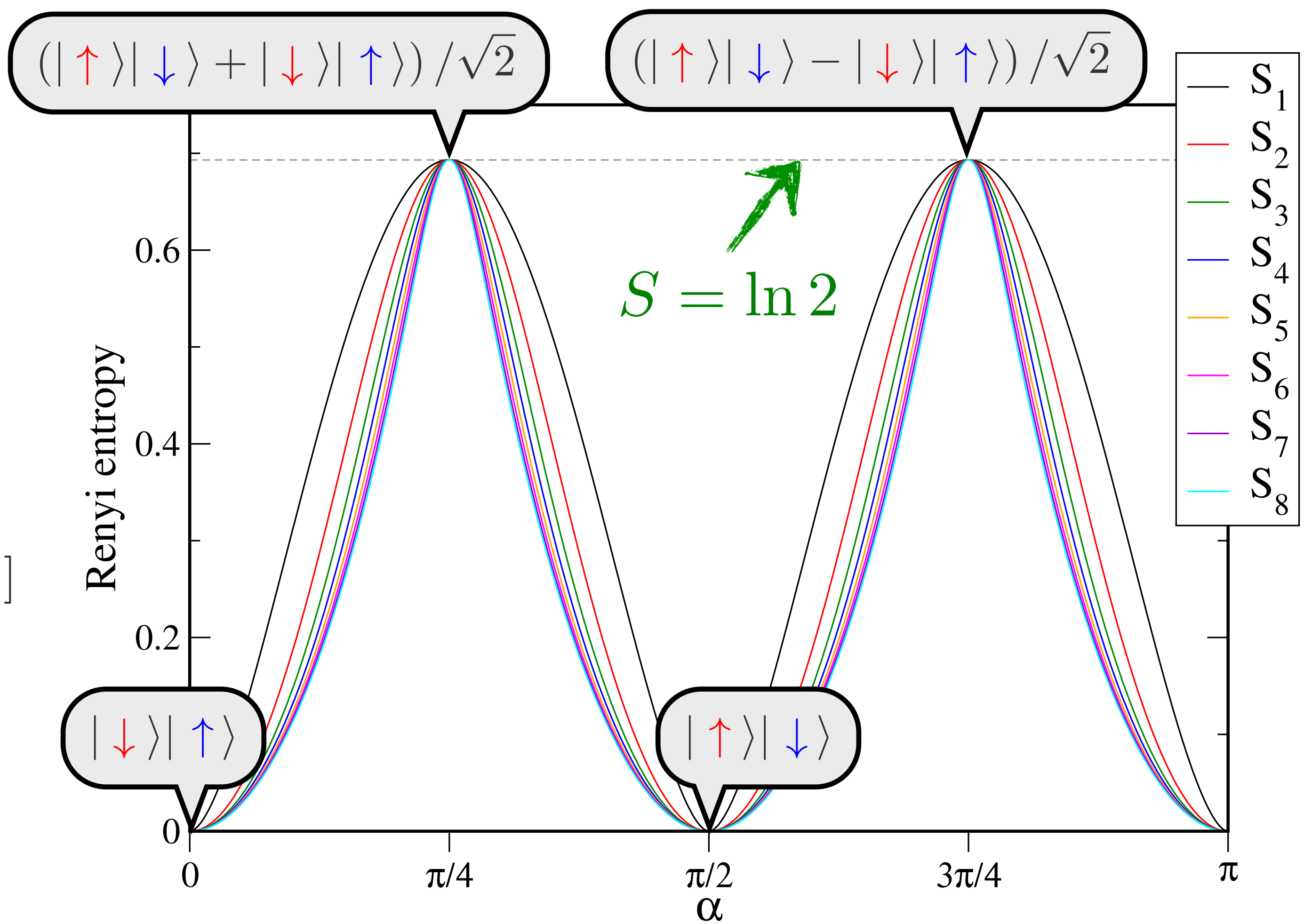
von Neumann entropy = first Renyi entropy

$$S_2(\rho_A) = -\ln\left[\mathrm{Tr}(\rho_A^2)\right]$$

second Renyi entropy

$$S_n(\rho_A) = \frac{1}{1-n}\ln\left[\mathrm{Tr}(\rho_A^n)\right]$$

Renyi entropy



Entanglement

- **Entanglement in quantum many-body systems**
  - Consider bipartition of system into two parts A and B, and calculate the **reduced density matrix** for the two parts

$$\rho_A = |\psi\rangle_A \langle\psi|_A$$ (traced over subsystem B)

  - One quantitative measure of entanglement is the **entanglement entropy**

$$S(\rho_A) = -\mathrm{Tr}(\, \rho_A \log \rho_A \,)$$

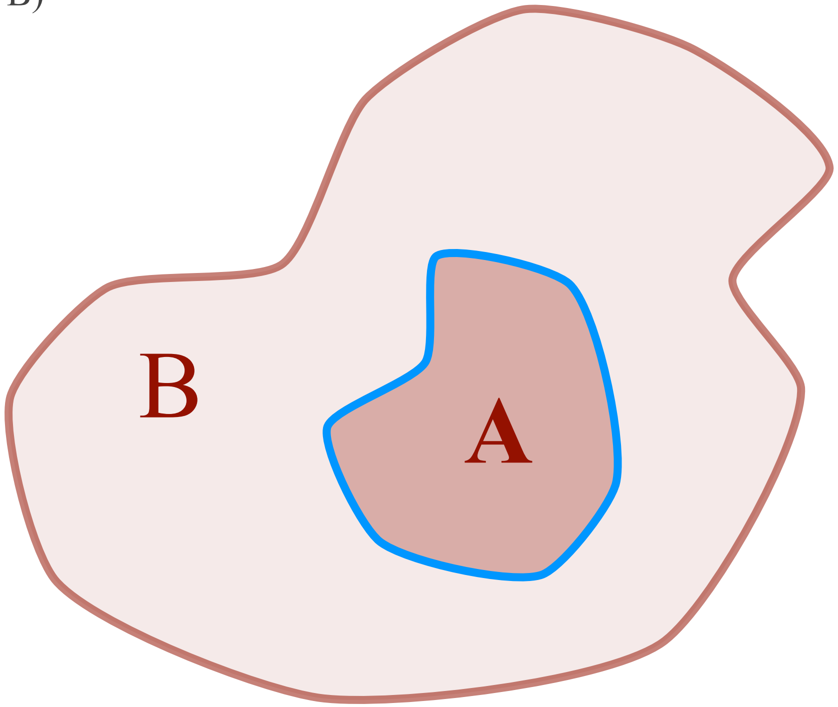
von Neumann entropy = first Renyi entropy

$$S_n(\rho_A) = \frac{1}{1-n} \ln \left[\mathrm{Tr}(\, \rho_A^n \,)\right]$$

Renyi entropy

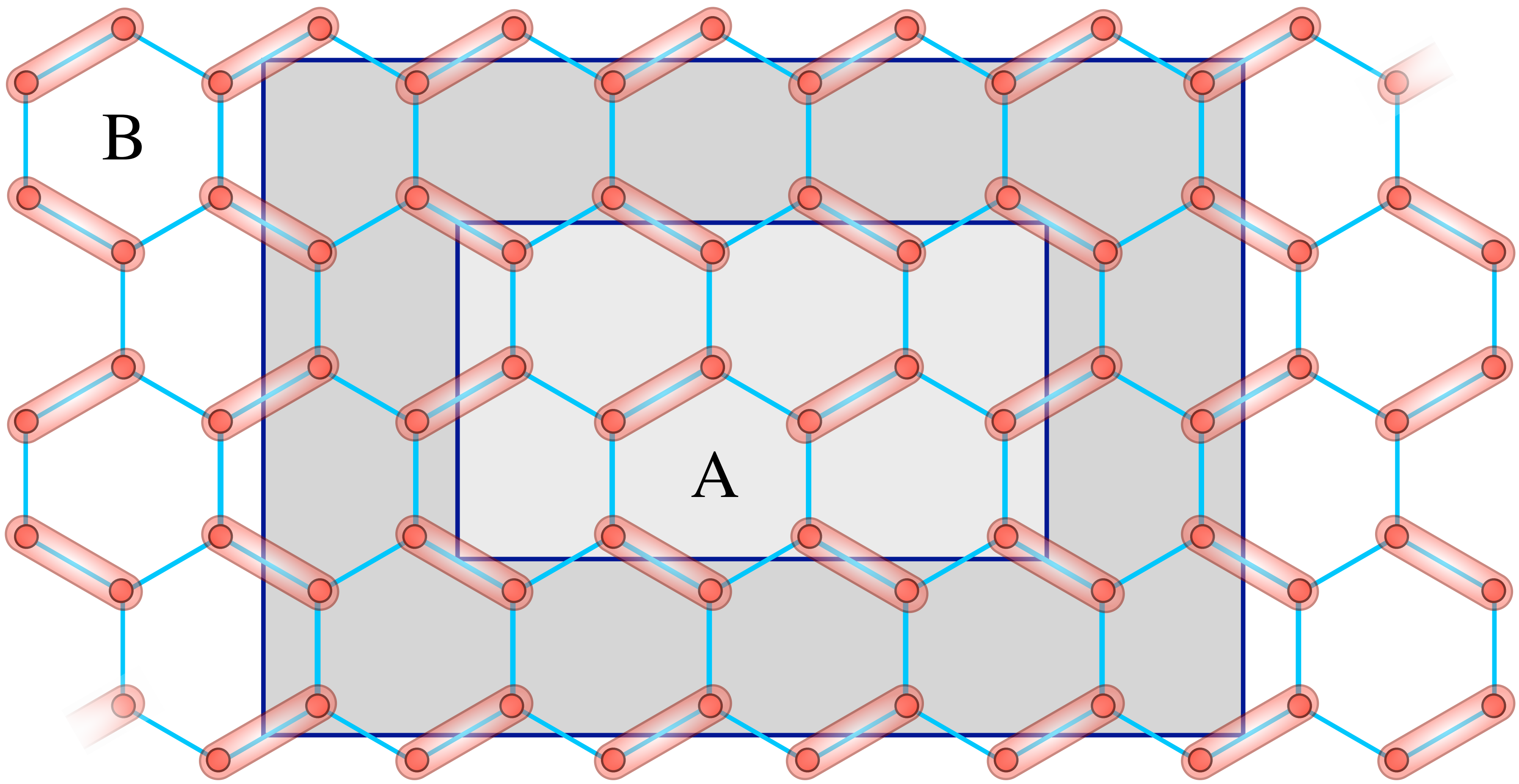
$$S_2(\rho_A) = -\ln \left[\mathrm{Tr}(\, \rho_A^2 \,)\right]$$

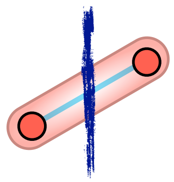
second Renyi entropy



Boundary law

also called **area law**





$S = \ln 2$

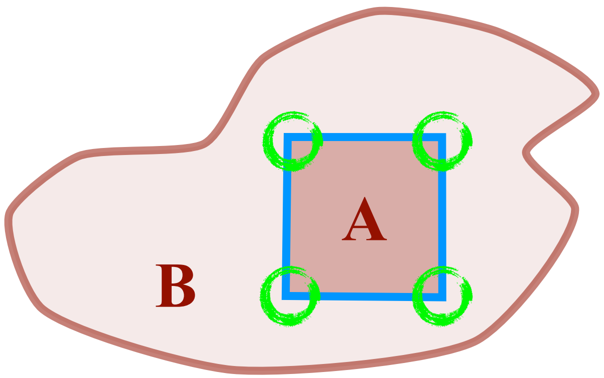
Entanglement scales with the length L of the **boundary** of the bipartition

$$S \propto aL$$

Corrections to the boundary law

Corrections to the boundary law can arise from

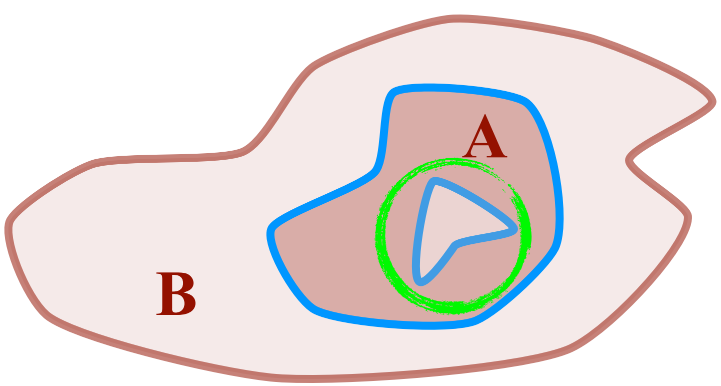
- **geometric aspects** of the bipartition



$$S = aL + b \cdot (\# \text{ of corners})$$

(for 2D gapped state)

- **topological aspects** of the bipartition



$$S = aL - \gamma \cdot (\# \text{ of disconnected parts})$$

(for 2D gapped state)

Corrections to the boundary law

Corrections to the boundary law
also originate from the specific
character of the underlying quantum many-body state!

- **topological** spin liquids

this talk

$$S = aL - \gamma$$

quantum double models
Levin-Wen / Kitaev models

- **gapless** spin liquids
  - gapless modes at **singular point** in momentum space

$$S = aL + c\,\gamma(L_x, L_y)$$

Kitaev model
(honeycomb)

  - gapless modes on **surface** in momentum space

$$S = cL\ln(L)$$

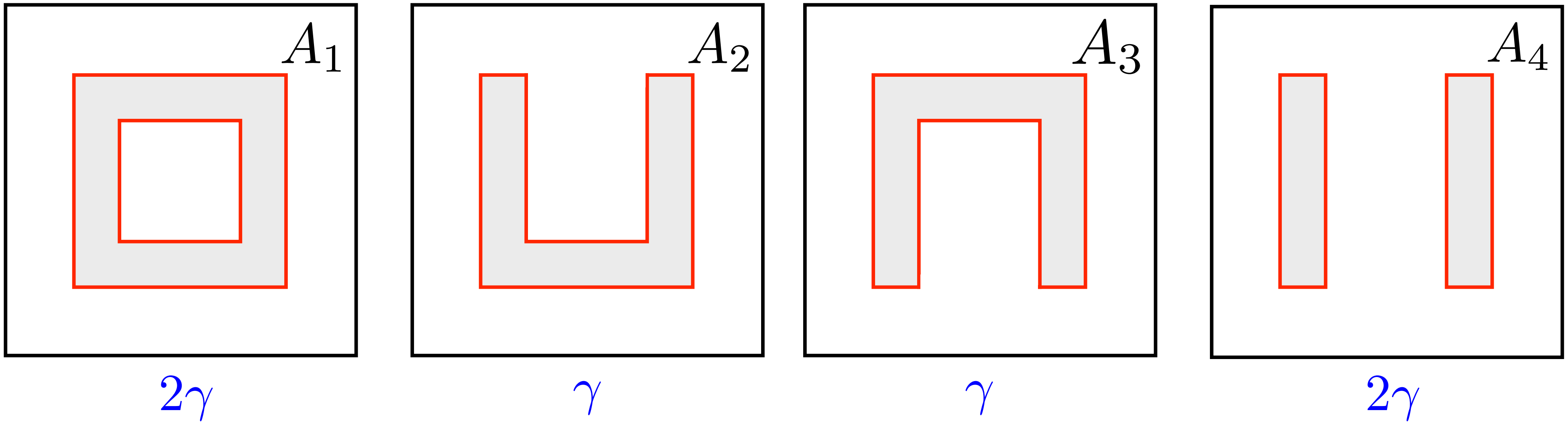
"Bose metals"
(Motrunich, Sheng & Fisher)

- **critical points, conformal critical points, Goldstone modes, ...**

$S_{\mathrm{QCP}} = aL + c\,\gamma(L_x, L_y)$ $\quad$ $S_{\mathrm{cQCP}} = \mu L + \gamma_{\mathrm{cQCP}}$ $\quad$ $S_{\mathrm{G}} = aL + b\ln(L) + \gamma(L_x, L_y)$

Topological entanglement entropy

Distilling the **topological correction**
by using a clever sequence of partitions



$$S = aL + b \cdot (\#\text{ of corners}) - \gamma \cdot (\#\text{ of disconnected parts})$$

$$S_{\text{topo}} = -S_{A_1} + S_{A_2} + S_{A_3} - S_{A_4} = -2\gamma$$

A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006);
M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).

Topological entanglement entropy

The **topological correction** is **universal.**

$$\gamma = \ln \sqrt{\sum_{i=1}^{n} d_i^2}$$

quantum dimension
of excitation

Examples:

- toric code (loop gas)

	$\mathbf{1}$	e	m	em
d_i	1	1	1	1

$$\gamma = \ln \sqrt{1+1+1+1} = \ln 2 \approx 0.693$$

- Fibonacci theory (string net)

	$\mathbf{1}$	τ
d_i	1	$\phi = \frac{1+\sqrt{5}}{2}$

$$\gamma = \ln \sqrt{1+\phi^2} \approx 0.643$$

A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006);
M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).

Let's try this on the toric code

Examples: • toric code (loop gas)

	$\mathbf{1}$	e	m	em
d_i	1	1	1	1

$$\gamma = \ln\sqrt{1+1+1+1} = \ln 2 \approx 0.693$$

ARTICLES
PUBLISHED ONLINE: 11 NOVEMBER 2012 | DOI: 10.1038/NPHYS2465
nature physics

Identifying topological order by entanglement entropy

Hong-Chen Jiang[1], Zhenghan Wang[2] and Leon Balents[1]*

Nature Physics **8**, 902 (2012).

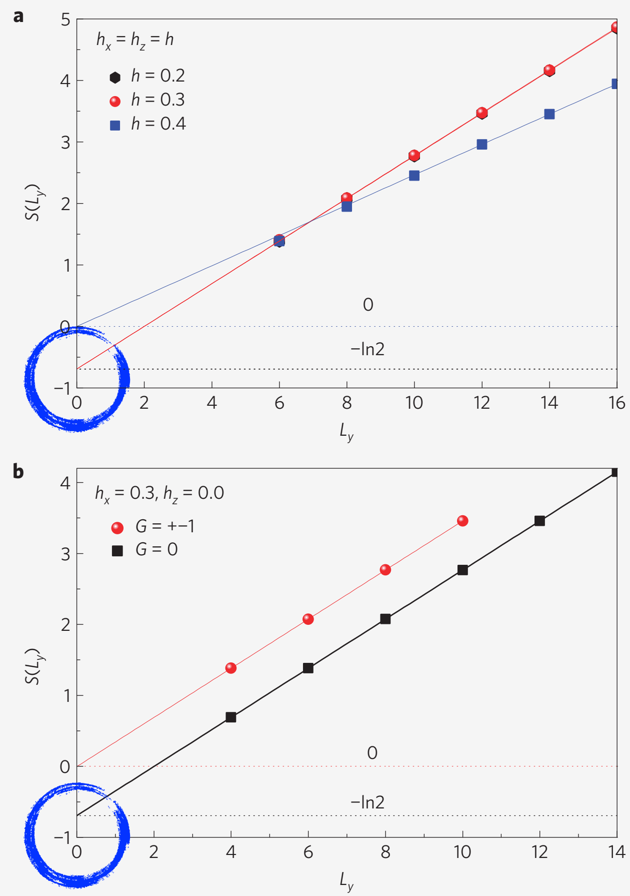
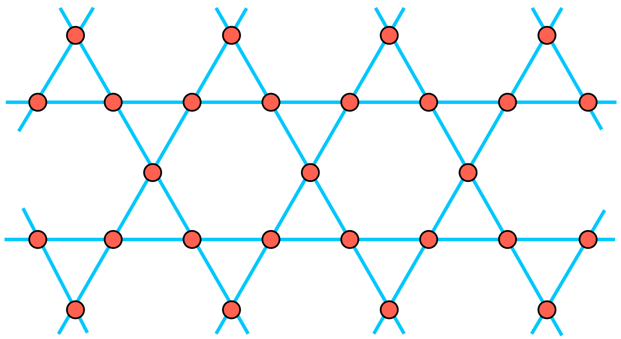


Figure 2 | The von Neumann entropy $S(L_y)$ for the toric-code model in magnetic fields. a, $S(L_y)$ with $L_y = 4$–16 at $L_x = \infty$ for symmetric magnetic fields at $h_x = h_z = h = 0.2$, 0.3 and 0.4. By fitting $S(L_y) = aL_y - \gamma$, we get $\gamma = 0.693(1)$, 0.691(4) and 0.001(5), respectively. **b**, The pure electric case, $h_x = 0.3, h_z = 0$, and comparison of $S(L_y)$ in the MES obtained in the large L_x limit (black squares) with that of the absolute ground state from systems of dimensions $L_x \times L_y = 20\times4, 24\times6, 24\times8, 24\times10$ (red circles). Extrapolation shows that the MES has the universal TEE, whereas the absolute ground state has zero TEE.

Kagomé antiferromagnet



Heisenberg model

$$H = \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

on kagomé lattice

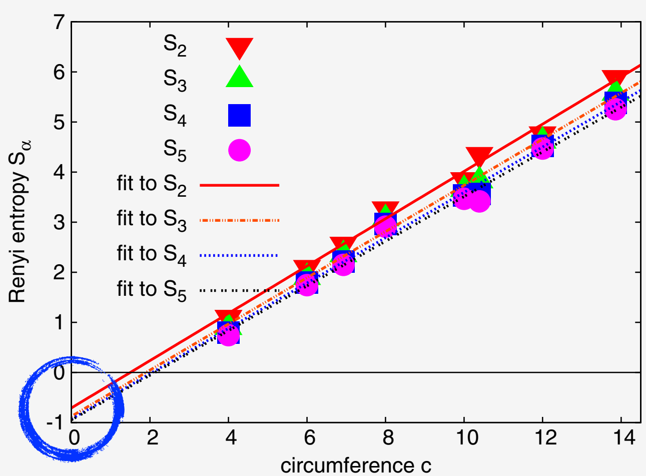


FIG. 6 (color online). Renyi entropies $\mathcal{S}_\alpha$ of infinitely long cylinders for various α versus circumference c, extrapolated to $c = 0$. The negative intercept is the topological entanglement entropy γ.

S. Depenbrock, I.P. McCulloch, and U. Schollwöck,
Phys. Rev. Lett. **109**, 067201 (2012).

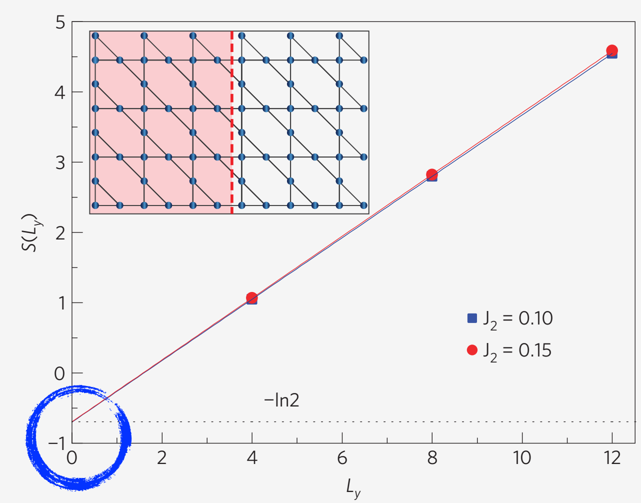


Figure 3 | The entanglement entropy $S(L_y)$ of the kagome J_1–J_2 model in equation (2), with $L_y = 4$–12 at $L_x = \infty$. By fitting $S(L_y) = aL_y - \gamma$, we get $\gamma = 0.698(8)$ at $J_2 = 0.10$ and $\gamma = 0.694(6)$ at $J_2 = 0.15$. Inset: kagome lattice with $L_x = 12$ and $L_y = 8$.

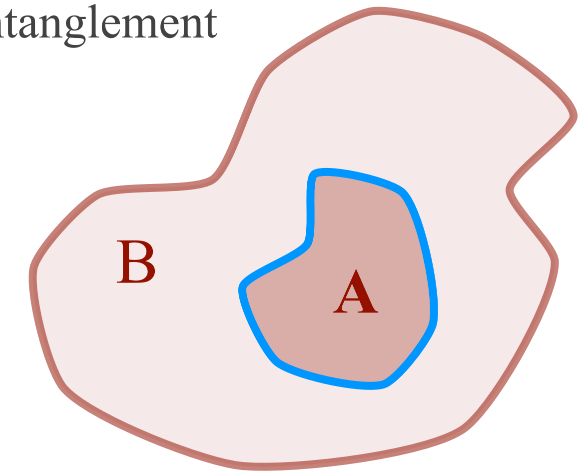
H.-C. Jiang, Z. Wang, and L. Balents,
Nature Physics **8**, 902 (2012).

bulk properties

entanglement spectrum

Entanglement spectrum

- The entanglement entropy is one quantitative measure of entanglement



$$\rho_A = |\psi\rangle_A \langle\psi|_A$$ (traced over subsystem B)

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

- However, **a lot of information** possibly contained in the density matrix **is discarded** in this simple measure.

- The **entanglement spectrum** aims at unraveling some of this information

$$\rho_A = |\psi\rangle_A \langle\psi|_A = \sum_\alpha \lambda_\alpha^2 |\phi_\alpha\rangle_A \langle\phi_\alpha|_A$$ (Schmidt decomposition)

$$S(\rho_A) = -\sum_\alpha \lambda_\alpha^2 \ln \lambda_\alpha^2$$

$$\lambda_\alpha = e^{-\xi_\alpha/2}$$

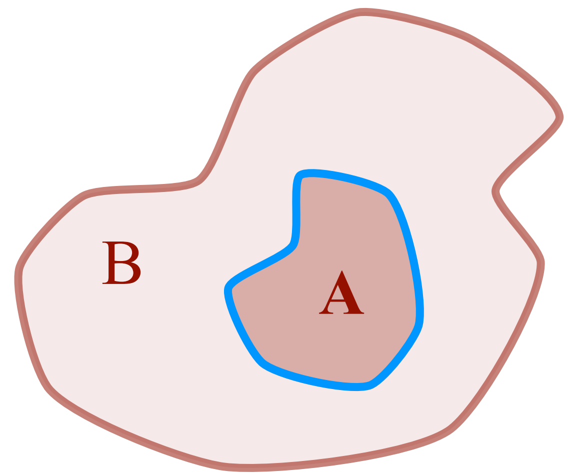
$$\rho_A = e^{-H_E}$$ "entanglement Hamiltonian"

$$(H_E = -\ln \rho_A)$$

H. Li and F.D.M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).

Entanglement spectrum

H. Li and F.D.M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).

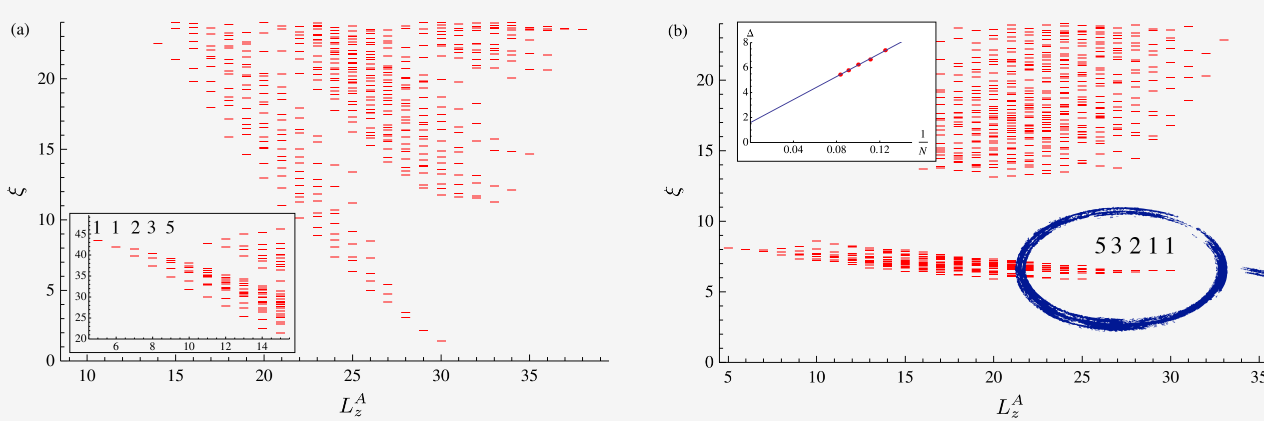


- The **entanglement spectrum** aims at unraveling some of the information contained in the density matrix

$$\rho_A = |\psi\rangle_A\langle\psi|_A = \sum_\alpha \lambda_\alpha^2 |\phi_\alpha\rangle_A\langle\phi_\alpha|_A$$

$$\lambda_\alpha = e^{-\xi_\alpha/2} \qquad \rho_A = e^{-H_E}$$

"entanglement Hamiltonian"



counting related to number of modes of the gapless edge theory (CFT)

R. Thomale, A. Sterdyniak, N. Regnault, and B.A. Bernevig, Phys. Rev. Lett. **104**, 180502 (2010).

Entanglement summary

- The **entanglement entropy** and **entanglement spectrum** are powerful bulk "observables" that allow
  - to **unambiguously distinguish topological order** from conventional order
  - to characterize the type of topological order to a good extent*

  * in some measures, e.g. the topological entanglement entropy, some unambiguities might remain and require additional work

- The **reduced density matrix** is **readily available** in some numerical methods such as
  - exact diagonalization
  - density matrix renormalization group

$$\rho_A = |\psi\rangle_A \langle\psi|_A$$

- Other numerical techniques have caught up, in particular
  - quantum Monte Carlo → replica trick → Renyi entropies / entanglement entropy

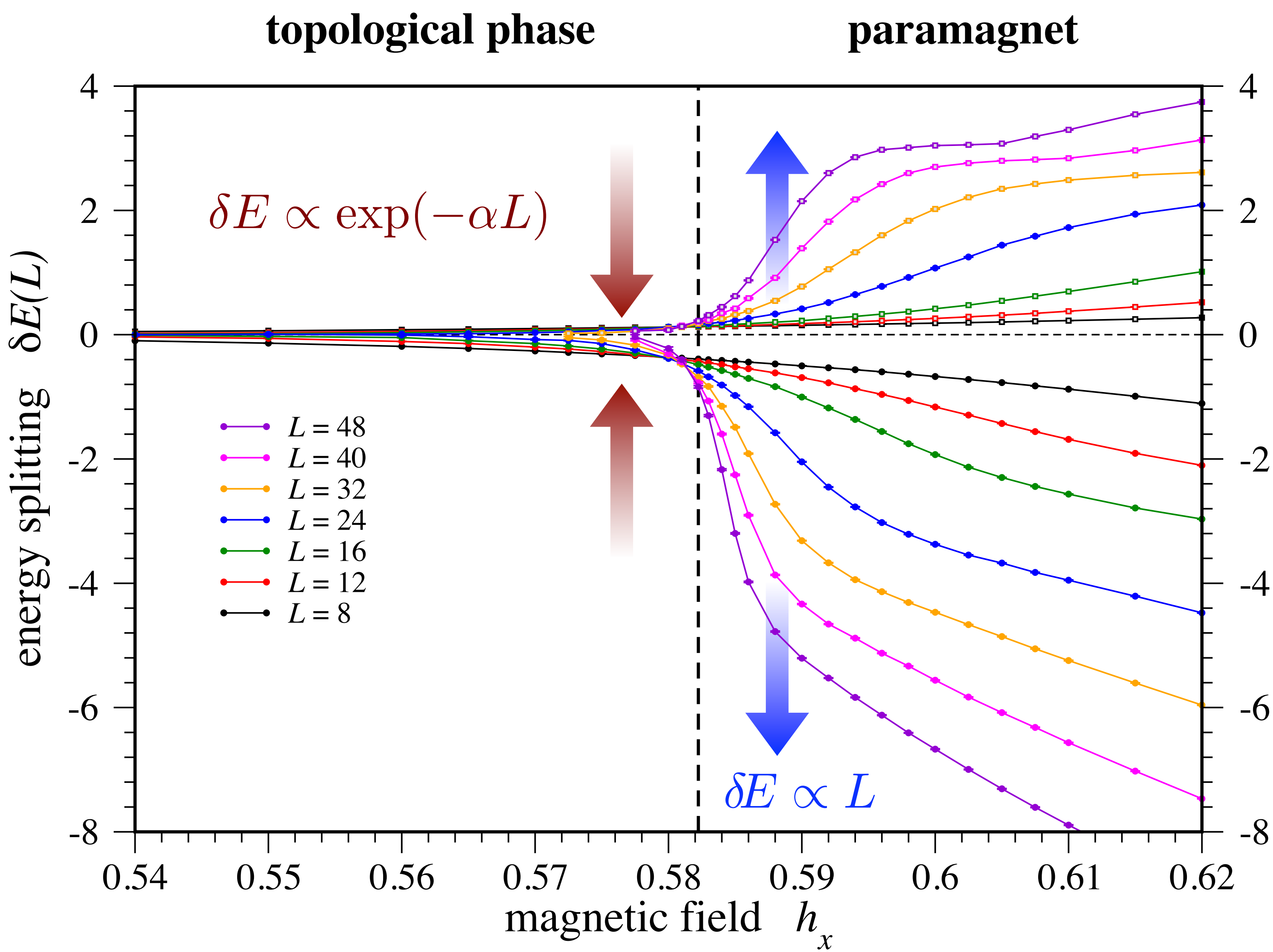
    You will hear about this in Peter Bröcker's talk tomorrow morning.
  - numerical linked cluster expansion → mutual information

bulk properties

ground-state degeneracy

Ground-state degeneracy

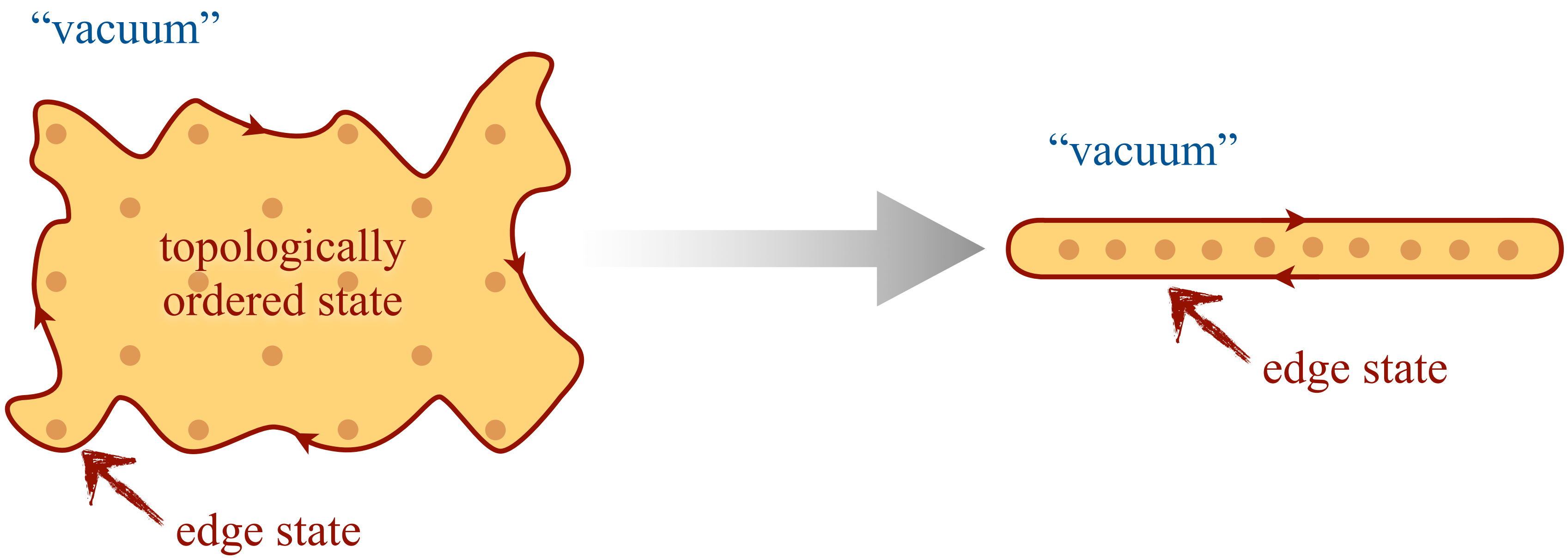
ST *et al.*, Phys. Rev. Lett. **98**, 070602 (2007).



edge properties

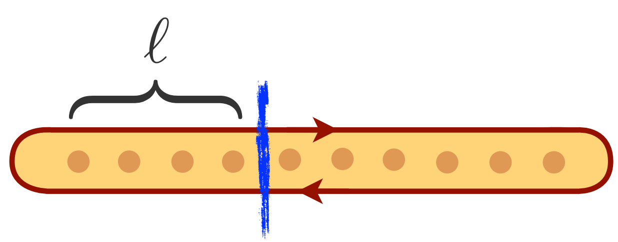
entanglement entropy

Edge states



- **Edge states** correspond to the **gapless modes** of a critical one-dimensional system, which are typically described by a **conformal field theory** (CFT).

- The conformal field theory can again be identified via entanglement properties



central charge

$$S = \frac{c}{3}\log\left(L\sin\left(\frac{\pi\ell}{L}\right)\right) \xrightarrow{\ell = L/2} S = \frac{c}{3}\log L$$

C. Holzhey et al., Nucl. Phys. B **424**, 44 (1994);
P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004).

Edge states

- **Edge states** correspond to the **gapless modes** of a critical one-dimensional system, which are typically described by a **conformal field theory** (CFT).

- The CFT can be identified via **entanglement properties**

$$S = \frac{c}{3} \log L$$

- Even more information about the CFT reveals itself in the **energy spectrum**

$$E = E_1 L + \frac{2\pi v}{L}\left(-\frac{c}{12} + x_s\right)$$

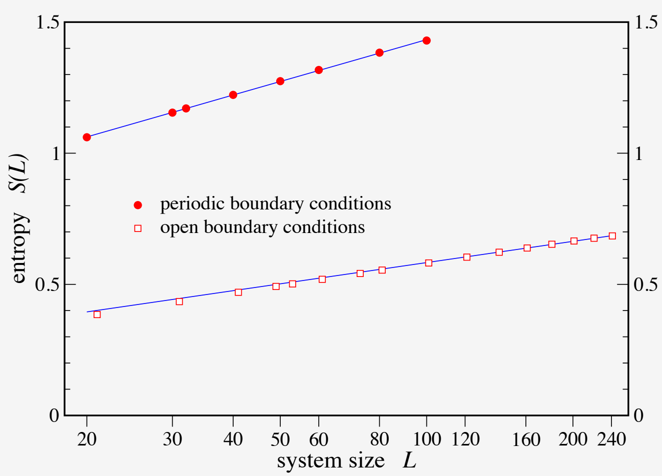


FIG. 2: (Color online) Entropy scaling for interacting Fibonacci anyons arranged along an open (open squares) or periodic chain (closed circles) versus the system size L.

A. Feiguin et al., Phys. Rev. Lett. **98**, 160409 (2007).

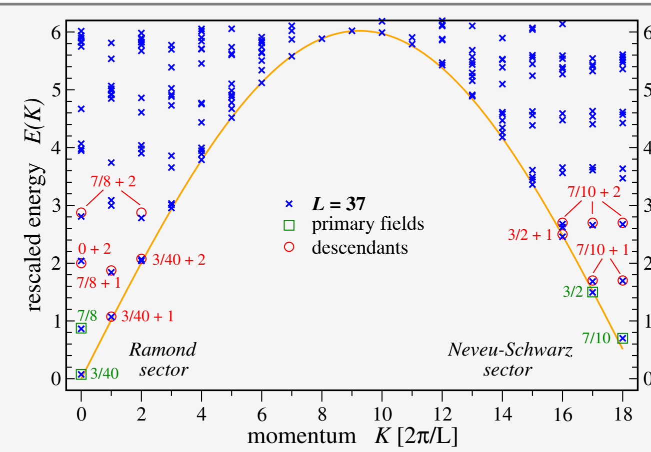


FIG. 3 (color online). Energy spectra for periodic chains of size L. Energies are rescaled and shifted such that the two lowest eigenvalues match the CFT assignments. Open boxes indicate positions of primary fields of the $c = \frac{7}{10}$ CFT. Open circles give positions of descendant fields as indicated

A. Feiguin et al., Phys. Rev. Lett. **98**, 160409 (2007).

We are done!

So what did we learn?

Summary

- The formation of **topological order** in an interacting quantum many-body system is one of the **most fascinating** phenomena in condensed matter physics
- **Numerical investigations** of topological order build on concepts from
  - statistical physics — van Neumann & Renyi entropies
  - quantum information theory — entanglement
  - mathematical physics — boundary laws, anyon theories
- The exploration of topological order is a **rich and quickly evolving research field** – just at its beginning.

All slides of this presentation will become available on the **FOR1807 website** and our group webpage at **www.thp.uni-koeln.de/trebst**