Interacting anyons in topological quantum liquids

Program on low dimensional electron systems  KITP 2009
Topological quantum liquids

**Spontaneous symmetry breaking**
- ground state has **less** symmetry than high-\(T\) phase
- Landau-Ginzburg-Wilson theory
- **local** order parameter

**Topological order**
- ground state has **more** symmetry than high-\(T\) phase
- degenerate ground states
- **non-local** order parameter
- quasiparticles have fractional statistics = **anyons**
Anyons and computing

**Abelian** anyons

\[ \psi(x_2, x_1) = e^{i\pi\theta} \cdot \psi(x_1, x_2) \]

fractional phase

**Non-Abelian** anyons

\[ \psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n) \]
\[ \psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n) \]

In general, \( M \) and \( N \) do not commute!

Topological quantum computing

Degenerate manifold = qubit

Employ **braiding** of non-Abelian anyons to perform computing (unitary transformations).

Illustration N. Bonesteel
Non-Abelian anyons

Ising anyons = Majorana fermions
- p-wave superconductors
- Moore-Read state
- Kitaev’s honeycomb model

\[ SU(2)_2 \]

Fibonacci anyons
- Read-Rezayi state
- Levin-Wen model

\[ SU(2)_3 \]

ordinary spins
- quantum magnets

\[ SU(2)_\infty \]
$SU(2)^k_k \; = \; \text{‘deformations’ of } SU(2)$

Quantum numbers in $SU(2)^k_k$

0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\ldots$, $\frac{k}{2}$

cutoff level $k$

“quantization”

Fusion rules

$j_1 \times j_2 \; = \; |j_1 - j_2| + (|j_1 - j_2| + 1)$

$+ \ldots + \min(j_1 + j_2, k - j_1 - j_2)$

for all $k \geq 2$

\[
\begin{array}{c}
\frac{1}{2} \times \frac{1}{2} = 0 + 1 \\
\end{array}
\]

for all $k \geq 4$

\[
\begin{array}{c}
1 \times 1 = 0 + 1 + 2 \\
\end{array}
\]
**p_x+i p_y superconductors**

Possible realizations:
- Sr$_2$RuO$_4$
- p-wave superfluid of cold atoms
- A$_1$ phase of $^3$He films

Topological properties of $p_x+i p_y$ superconductors

- $\sigma$-vortices carry “half-flux” $\phi = \frac{hc}{2e}$
- Characteristic “zero mode” $\sigma \times \sigma = 1 + \psi$

$2N$ vortices give degeneracy of $2^N$. 

---

© Simon Trebst
Fractional quantum Hall liquids

Charge $e/4$ quasiparticles
Ising anyons
“Pfaffian” state
Moore & Read (1994)
Nayak & Wilzcek (1996)

Charge $e/5$ quasiparticles
Fibonacci anyons
“Parafermion” state
Read & Rezayi (1999)
Slingerland & Bais (2001)
A soup of anyons

What is the collective state of a set of interacting anyons?

Does this collective behavior somehow affect the character of the underlying parent liquid?
A soup of anyons

SU(2)_k liquid

finite density of anyons
(anyons are at fixed positions or ‘pinned’)

\[ a \gg \xi_m \]

The ground state has a macroscopic degeneracy.

\[ a \ll \xi_m \]

Anyons approach each other and interact. The interactions will lift the degeneracy.
The collective state of anyons is \textbf{gapped}.

The parent liquid remains \textbf{unchanged}.
Collective states: possible scenarios

The collective state of anyons is a **gappless quantum liquid**. A **gappless phase nucleates** within the parent liquid.
The collective state of anyons is a **gapped quantum liquid**. A novel liquid is nucleated within the parent liquid.
A soup of anyons

SU(2)\textsubscript{k} liquid

\textbf{finite density of anyons}
(anyons are at fixed positions or ‘pinned’)

SU(2)\textsubscript{k} fusion rules

\[
\frac{1}{2} \times \frac{1}{2} = 0 + 1
\]

energetically split
multiple fusion outcomes

\begin{align*}
H &= J \sum_{\langle ij \rangle} \prod_{ij}^{0}
\end{align*}

Anyonic Heisenberg model

SU(2)_k fusion rules

\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

“Heisenberg” Hamiltonian

\[ H = J \sum_{\langle ij \rangle} \prod_{ij}^0 \]

energetically split multiple fusion outcomes

Which fusion channel is favored? – Non-universal

- **p-wave superconductor**
  - M. Cheng et al., arXiv:0905.0035
  - \( 1/2 \times 1/2 \rightarrow 0 \)
  - short distances, then oscillates

- **Moore-Read state**
  - M. Baraban et al., arXiv:0901.3502
  - \( 1/2 \times 1/2 \rightarrow 1 \)
  - short distances, then oscillates

- **Kitaev’s honeycomb model**
  - \( 1/2 \times 1/2 \rightarrow 0 \)

Connection to topological charge tunneling: P. Bonderson, arXiv:0905.2726
Anyonic Heisenberg model

SU(2)_k fusion rules

\[ \frac{1}{2} \times \frac{1}{2} = 0 + 1 \]

“Heisenberg” Hamiltonian

\[ H = J \sum_{\langle ij \rangle} \prod_{ij}^0 \]

ergetically split
multiple fusion outcomes

SU(2)_k liquid

Anyonic Heisenberg model


**SU(2)\(_k\)** fusion rules

\[
\frac{1}{2} \times \frac{1}{2} = 0 + 1
\]

"Heisenberg" Hamiltonian

\[
H = J \sum_{\langle ij \rangle} \prod_{ij}^0
\]

energetically split multiple fusion outcomes

**SU(2)\(_k\)** liquid

chain of anyons

‘golden chain’ for SU(2)\(_3\)
**Anyonic Heisenberg model**


### SU(2)$_k$ fusion rules

\[
\frac{1}{2} \times \frac{1}{2} = 0 + 1
\]

energetically split multiple fusion outcomes

### “Heisenberg” Hamiltonian

\[
H = J \sum_{\langle ij \rangle} \prod_{ij}^0
\]

### Example: chains of anyons

\[
\tau \tau \tau \tau \tau \tau \tau (\tau = 1/2)
\]

### Hilbert space

\[
| \tau, x_1, x_2, x_3, \ldots \rangle
\]

### Hamiltonian

\[
H = \sum_i F_i \prod_i^0 F_i
\]

F-matrix = 6j-symbol
Critical ground state

Finite-size gap

\[ \Delta(L) \propto \left( \frac{1}{L} \right)^{z=1} \]

conformal field theory description

Entanglement entropy

\[ S_{\text{PBC}}(L) \propto \frac{c}{3} \log L \]

central charge

\[ c = \frac{7}{10} \]
Conformal energy spectra

\[ E = E_1 L + \frac{2\pi \nu}{L} \left( -\frac{c}{12} + h_L + h_R \right) \]

scaling dimension

\[ L = 36 \]

\[ Z_2 \text{ sublattice symmetry} \]
Conformal energy spectra

$L = 36$

primary fields descendants

rescaled energy $E(K)$

momentum $K \ [2\pi/L]$

primary fields

scaling dimensions

$I \quad \epsilon \quad \epsilon' \quad \epsilon'' \quad \sigma \quad \sigma'$

0 1/5 6/5 3 3/40 7/8

thermal operators $K = 0$

spin op. $K = \pi$

$Z_2$ sublattice symmetry
central charge $c = 7/10$
The operators $X_i = -d \, H_i$ form a representation of the Temperley-Lieb algebra

\[
\begin{align*}
(X_i)^2 &= d \cdot X_i \\
X_i X_{i \pm 1} X_i &= X_i \\
[X_i, X_j] &= 0 \\
\text{for } |i - j| \geq 2
\end{align*}
\]

The transfer matrix is an integrable representation of the RSOS model.
# Deformed spin-1/2 chains

<table>
<thead>
<tr>
<th>level $k$</th>
<th>$1/2 \times 1/2 \rightarrow 0$</th>
<th>$1/2 \times 1/2 \rightarrow 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Ising $c = 1/2$</td>
<td>Ising $c = 1/2$</td>
</tr>
<tr>
<td>3</td>
<td>tricritical Ising $c = 7/10$</td>
<td>3-state Potts $c = 4/5$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$</td>
<td>$\frac{SU(2)_k}{U(1)}$</td>
</tr>
<tr>
<td>5</td>
<td>$k$-critical Ising $c = 1-6/(k+1)(k+2)$</td>
<td>$Z_k$-parafermions $c = 2(k-1)/(k+2)$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Heisenberg AFM $c = 1$</td>
<td>Heisenberg FM $c = 2$</td>
</tr>
</tbody>
</table>
Topological symmetry

Relevant perturbations

- $\sigma_L \sigma_R$
- $\sigma'_L \sigma'_R$

prohibited by translational symmetry

- $\epsilon_L \epsilon_R$
- $\epsilon'_L \epsilon'_R$

prohibited by topological symmetry

Symmetry operator

$$\langle x'_1, \ldots, x'_L | Y | x_1, \ldots, x_L \rangle = \prod_{i=1}^{L} \left(F_{\tau x'_i \tau} x'_i \right) x'_{i+1}$$

with eigenvalues

- $S_{\tau - \text{flux}} = \phi$
- $S_{\text{no flux}} = -\phi^{-1}$

$$[H, Y] = 0$$
## Topological protection

<table>
<thead>
<tr>
<th>level $k$</th>
<th>$1/2 \times 1/2 \rightarrow 0$ 'antiferromagnetic'</th>
<th>$1/2 \times 1/2 \rightarrow 1$ 'ferromagnetic'</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Ising $c = 1/2$ ✓</td>
<td>Ising $c = 1/2$ ✓</td>
</tr>
<tr>
<td>3</td>
<td>tricritical Ising $c = 7/10$ ✓</td>
<td>3-state Potts $c = 4/5$ ✓</td>
</tr>
<tr>
<td>4</td>
<td>tetracritical Ising $c = 4/5$ ✓</td>
<td>$c = 1$ ✓</td>
</tr>
<tr>
<td>5</td>
<td>pentacritical Ising $c = 6/7$ ✓</td>
<td>$c = 8/7$ ✓</td>
</tr>
<tr>
<td>$k$</td>
<td>$k$-critical Ising $c = 1-6/(k+1)(k+2)$ ✓</td>
<td>$Z_k$-parafermions $c = 2(k-1)/(k+2)$ ✓</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Heisenberg AFM $c = 1$ ✗</td>
<td>Heisenberg FM $c = 2$ ✗</td>
</tr>
</tbody>
</table>
Gapless modes & edge states

\[ SU(2)^k \text{ liquid} \]

Critical theory (AFM couplings)

\[ \frac{SU(2)^{k-1} \times SU(2)^1}{SU(2)^k} \]

Finite density interactions

Gapless modes = edge states

\[ \frac{SU(2)^{k-1} \times SU(2)^1}{SU(2)^k} \]

Nucleated liquid
Example: Ising meets Fibonacci

SU(2)\_3 liquid → nucleated liquid

\[ \frac{SU(2)_2 \times SU(2)_1}{SU(2)_3} \]

\[ c = \frac{7}{10} \]

gapless modes = edge states

When Ising meets Fibonacci:

a tricritical Ising edge (c = 7/10)
Gapless modes & edge states

SU(2)\(_k\) liquid

finite density interactions

SU(2)\(_k\) liquid

Critical theory
(FM couplings)

\[
\frac{SU(2)}{U(1)}
\]

gapless modes = edge states

\[
\frac{SU(2)}{U(1)}
\]

nucleated liquid \(U(1)\)

(Abelian)

arXiv:0810.2277
Approaching two dimensions

The 2D collective state

A gapped topological liquid that is distinct from the parent liquid.

Results for N-leg ladders give some supporting evidence for this.

SU(2)_k liquid

SU(2)_k liquid

SU(2)_k liquid
Coupling two chains

SU(2)_k liquid

relevant operator couples \textbf{inner} two edges

SU(2)_k liquid
Earlier work for Majorana fermions

**Read & Ludwig** [PRB (2000)]
- SU(2)$_2$ liquid
- U(1) liquid

**Grosfeld & Stern** [PRB (2006)]
- strong pairing SC
- weak pairing SC

**Grosfeld & Schoutens** [arXiv:0810.1955]
- SU(3)$_2$ liquid

**Kitaev** unpublished (2006)
- Levin & Halperin [PRB (2009)]

---

2D anyon systems

All of these previous results fit into our more general framework.
Recent work for Fibonacci anyons

Read & Ludwig PRB (2000)
SU(2)$_2$ liquid
U(1) liquid$

SU(3)$_2$ liquid
SU(2)$_2$ liquid

Grosfeld & Stern PRB (2006)
strong pairing SC
weak pairing SC

Kitaev unpublished (2006)
Levin & Halperin PRB (2009)

2D anyon systems
All of these previous results fit into our more general framework.
A powerful correspondence

SU(2)\_k liquid

finite density interactions

SU(2)\_k liquid

collective states of anyonic spin chains

edge states of topological liquids

nucleation of novel topological liquids

arXiv:0810.2277
Anyonic spin-1 chains

$\text{SU}(2)_\infty$

$J_{S=2} = -\cos \theta$ \hspace{1cm} $J_{S=1} = \sin \theta$

$\text{SU}(2)_k$

$J_{S=2} = -\cos \theta$ \hspace{1cm} $J_{S=1} = \sin \theta$

\[ \text{critical SU}(3) \]

\[ \text{nematic} \quad c = 2 \]

\[ \text{Haldane} \]

\[ \text{ferromagnet} \quad c = 2 \]

\[ \text{dimerized} \]

\[ \text{gapped SU}(3) \]

\[ \text{SU}(2)_2 \quad c = 3/2 \]

\[ \bullet \text{AKLT} \]

\[ \text{super CFT} \quad (N = 1) \]

\[ \frac{\text{su}(2)_{k-4} \times \text{su}(2)_4}{\text{su}(2)_k} \]

\[ \frac{\text{su}(2)_k}{\text{u}(1)_k} \]

\[ \frac{\text{su}(2)_{k-1} \times \text{su}(2)_1}{\text{su}(2)_k} \]
Conclusions

• Interacting non-Abelian anyons can support a wide variety of collective states:
  
  **stable** gapless states, gapped states, quasiparticles, ...

• In a topological liquid a **finite density** of interacting anyons nucleates a new topological liquid
  
  gapless states = edge states between top. liquids

arXiv:0810.2277