Interacting anyons in topological quantum liquids

Program on low dimensional electron systems KITP 2009

Simon Trebst
Microsoft Station Q
UC Santa Barbara

Charlotte Gils

Eddy Ardonne
Adrian Feiguin
Michael Freedman

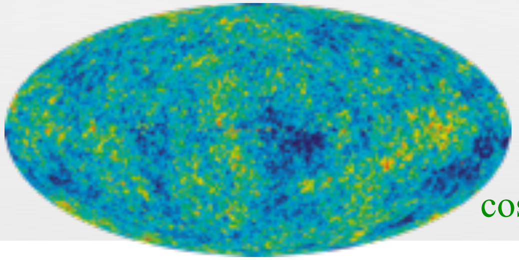
David Huse
Alexei Kitaev
Andreas Ludwig

Didier Poilblanc
Matthias Troyer
Zhenghan Wang

Topological quantum liquids

Spontaneous symmetry breaking

- ground state has **less** symmetry than high-T phase
- Landau-Ginzburg-Wilson theory
- **local** order parameter



cosmic microwave background

Topological order

- ground state has **more** symmetry than high-T phase
- degenerate ground states
- **non-local** order parameter
- quasiparticles have fractional statistics = **anyons**

Anyons and computing

Abelian anyons

$$\psi(x_2, x_1) = e^{i\pi\theta} \cdot \psi(x_1, x_2)$$

fractional phase

Non-Abelian anyons

matrix

$$\psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n)$$

$$\psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n)$$

In general *M* and *N* do not commute!

Topological quantum computing

Degenerate manifold = qubit

Employ **braiding** of non-Abelian anyons to perform computing (unitary transformations).

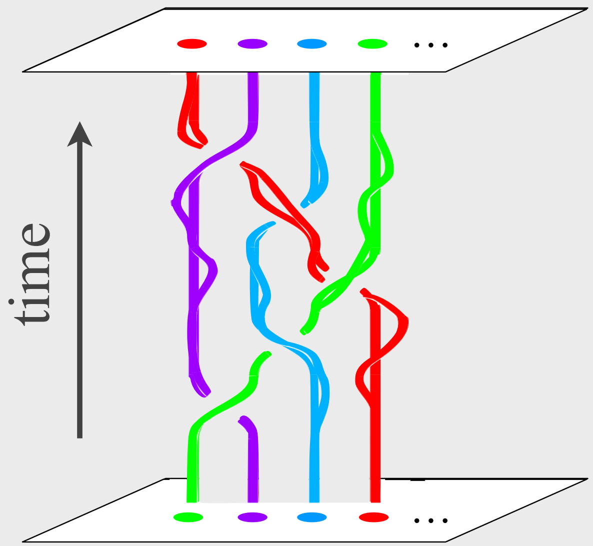


illustration N. Bonesteel

Non-Abelian anyons

Ising anyons = Majorana fermions
p-wave superconductors
Moore-Read state
Kitaev's honeycomb model

$SU(2)_2$

Fibonacci anyons
Read-Rezayi state
Levin-Wen model

$SU(2)_3$

$SU(2)_k$

ordinary spins
quantum magnets

$SU(2)_\infty$

$SU(2)_k$ = 'deformations' of SU(2)

Quantum numbers in $SU(2)_k$

$$0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ \ldots,\ \frac{k}{2}$$

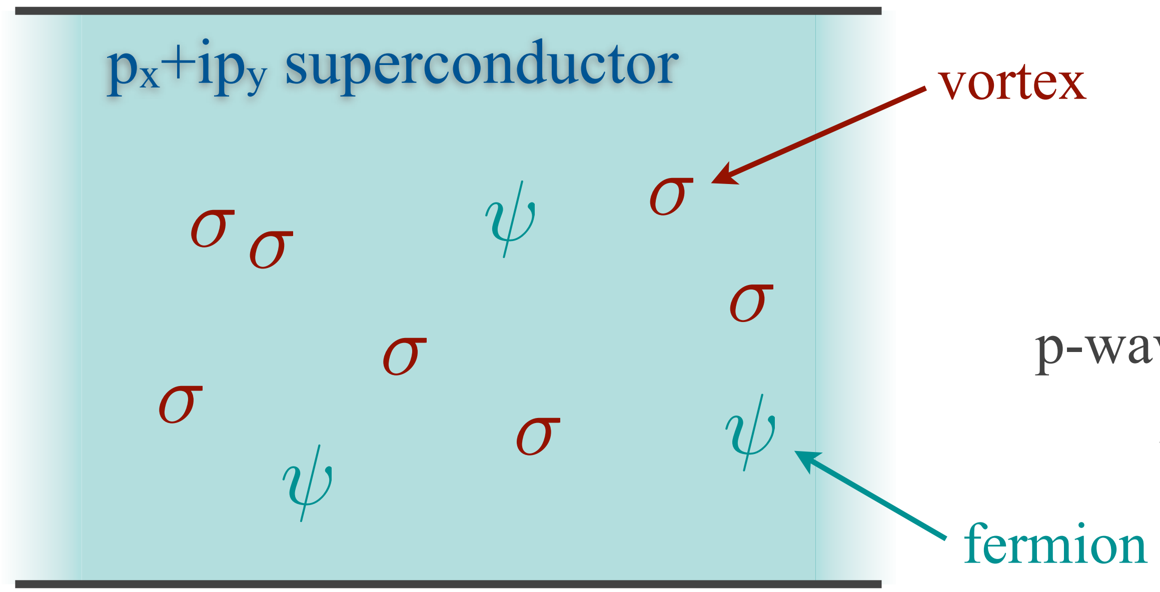
cutoff level k
"quantization"

Fusion rules

$$j_1 \times j_2 = |j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots + \min(j_1 + j_2, k - j_1 - j_2)$$

for all k ≥ 2	for all k ≥ 4
$\frac{1}{2} \times \frac{1}{2} = 0 + 1$	$1 \times 1 = 0 + 1 + 2$

p_x+ip_y superconductors



possible realizations

Sr_2RuO_4

p-wave superfluid of cold atoms

A_1 phase of ^3He films

Topological properties of p_x+ip_y superconductors $SU(2)_2$

Read & Green (2000)

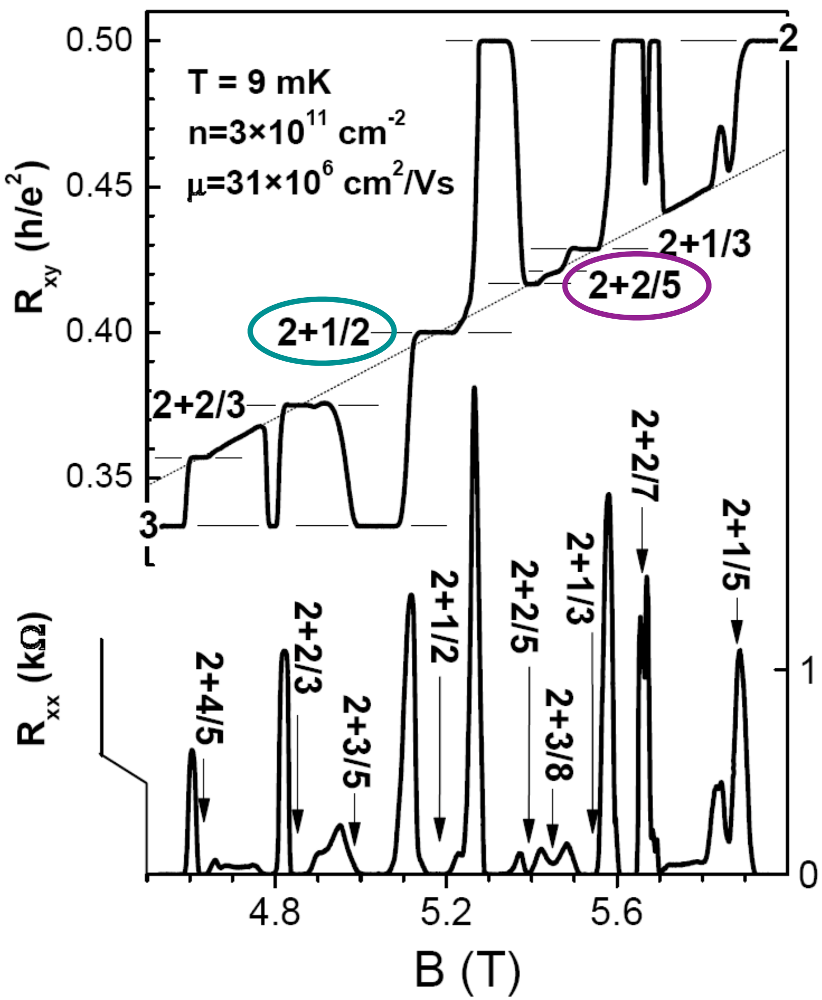
σ-vortices carry "half-flux" $\phi = \frac{hc}{2e}$

characteristic "zero mode"

2N vortices give degeneracy of 2^N.

$$\sigma \times \sigma = 1 + \psi$$

Fractional quantum Hall liquids



J.S. Xia *et al.*, PRL (2004)

“Pfaffian” state

Moore & Read (1994)

Charge e/4 quasiparticles

Ising anyons

Nayak & Wilczek (1996)

$SU(2)_2$

“Parafermion” state

Read & Rezayi (1999)

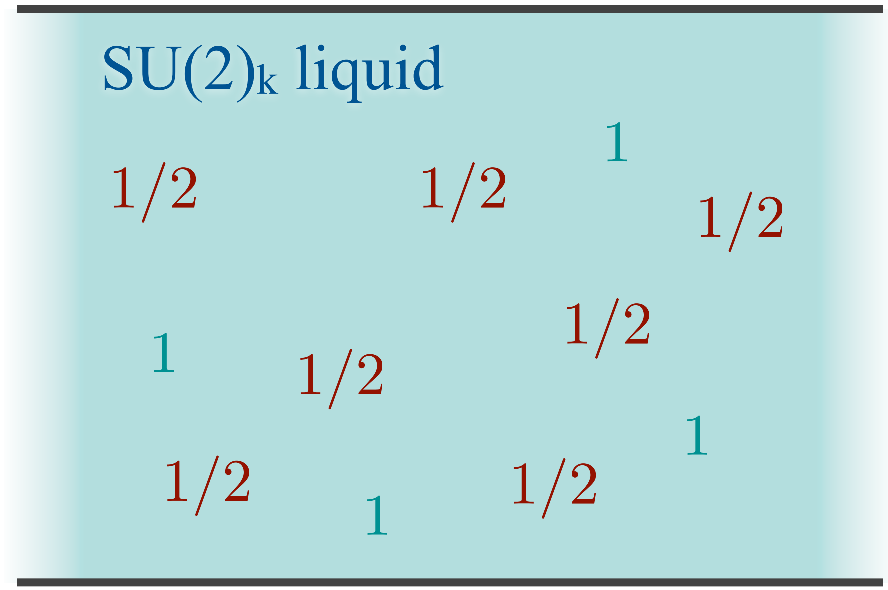
Charge e/5 quasiparticles

Fibonacci anyons

Slingerland & Bais (2001)

$SU(2)_3$

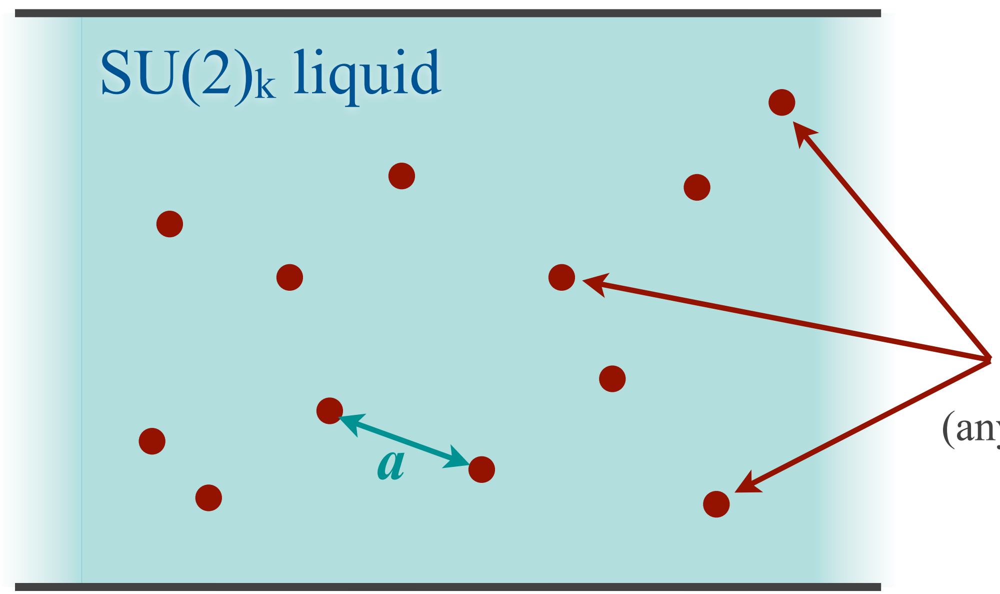
A soup of anyons



What is the **collective state** of
a set of interacting anyons?

Does this collective behavior somehow **affect**
the character of the underlying **parent liquid**?

A soup of anyons

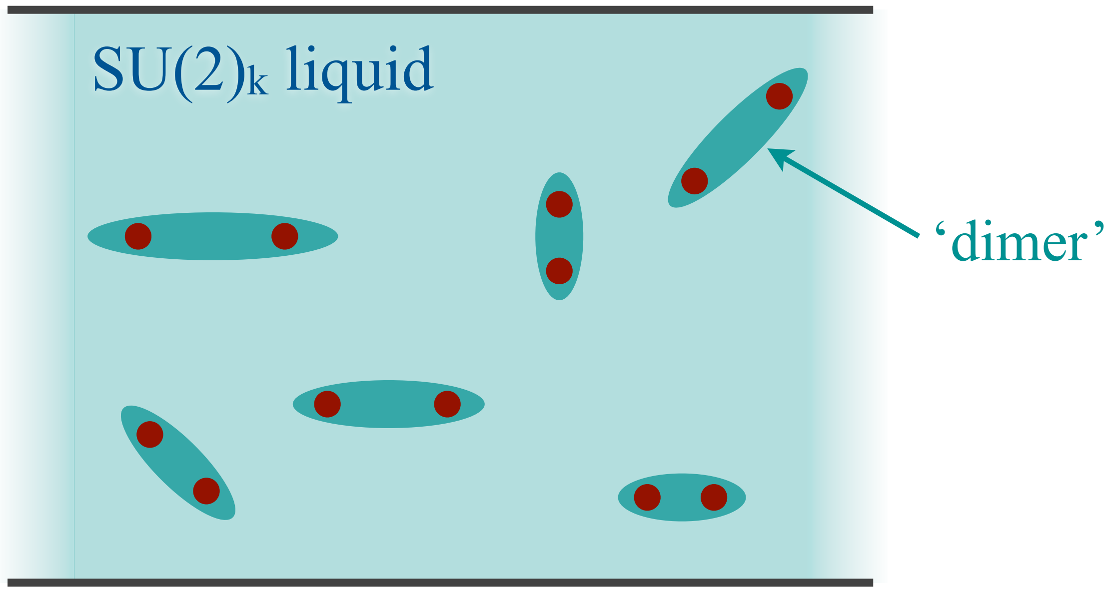


finite density of anyons

(anyons are at fixed positions or '**pinned**')

$a \gg \xi_m$	The ground state has a **macroscopic degeneracy**.
$a \ll \xi_m$	Anyons approach each other and interact. The interactions will **lift the degeneracy**.

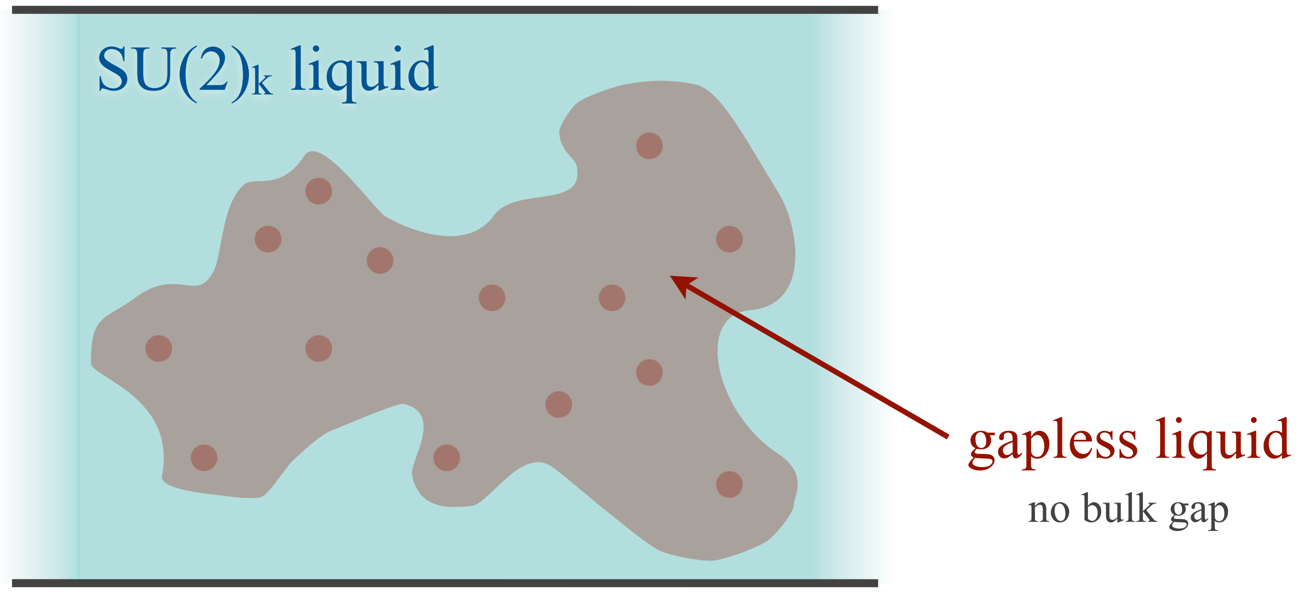
Collective states: possible scenarios



The collective state of anyons is **gapped**.

The parent liquid remains **unchanged.**

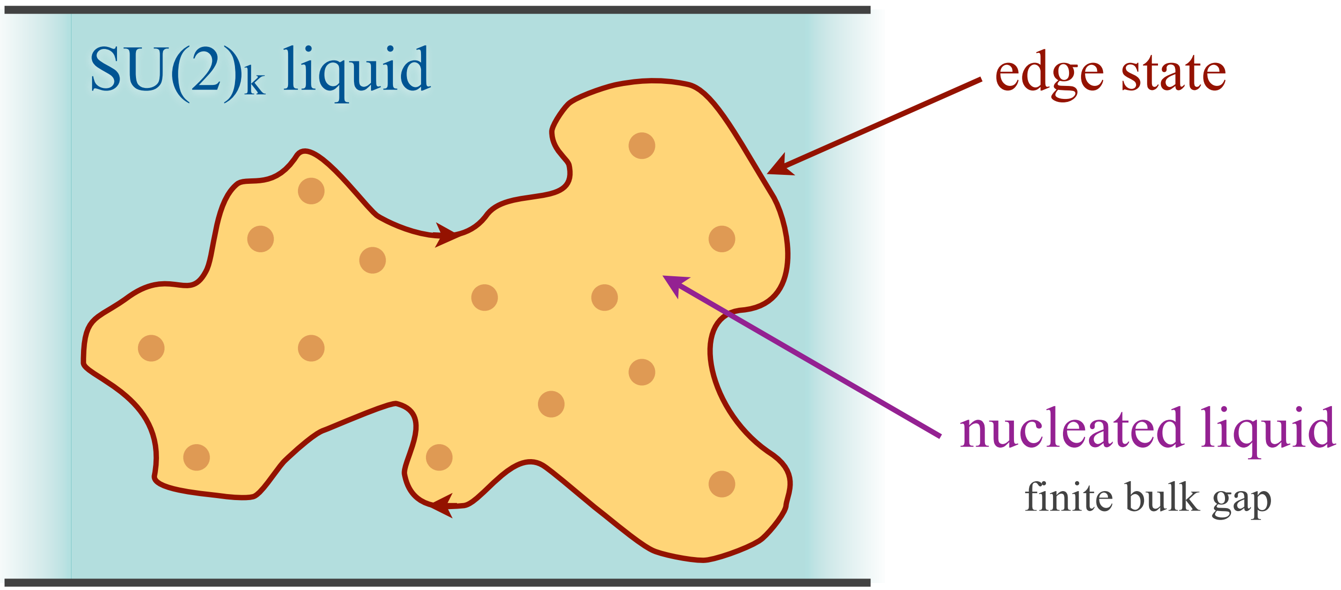
Collective states: possible scenarios



The collective state of anyons is a **gapless quantum liquid**.

A **gapless phase nucleates** within the parent liquid.

Collective states: possible scenarios

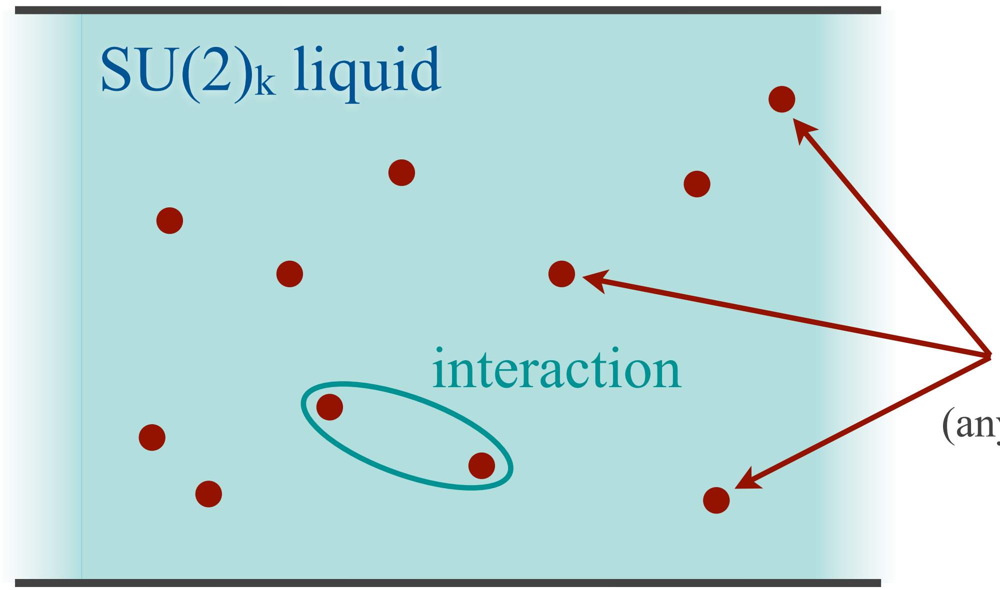


The collective state of anyons is a **gapped quantum liquid**.

A **novel liquid is nucleated** within the parent liquid.

A soup of anyons

Phys. Rev. Lett. **98**, 160409 (2007).



finite density of anyons
(anyons are at fixed positions or '**pinned**')

SU(2)$_k$ fusion rules

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

energetically split
multiple fusion outcomes

"Heisenberg" Hamiltonian

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

Anyonic Heisenberg model

$SU(2)_k$ fusion rules

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

"Heisenberg" Hamiltonian

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

energetically split
multiple fusion outcomes

Which fusion channel is favored? – Non-universal

p-wave superconductor
M. Cheng *et al.*, arXiv:0905.0035

$1/2 \times 1/2 \rightarrow \mathbf{0}$
short distances, then oscillates

Moore-Read state
M. Baraban *et al.*, arXiv:0901.3502

$1/2 \times 1/2 \rightarrow \mathbf{1}$
short distances, then oscillates

Kitaev's honeycomb model
V. Lathinen *et al.*, Ann. Phys. **323**, 2286 (2008)

$1/2 \times 1/2 \rightarrow \mathbf{0}$

Connection to topological charge tunneling: P. Bonderson, arXiv:0905.2726

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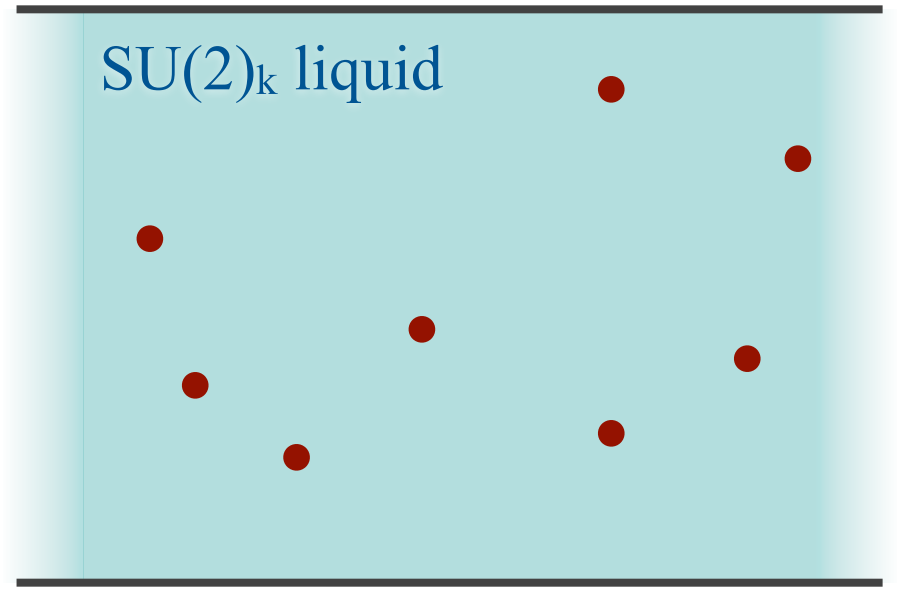

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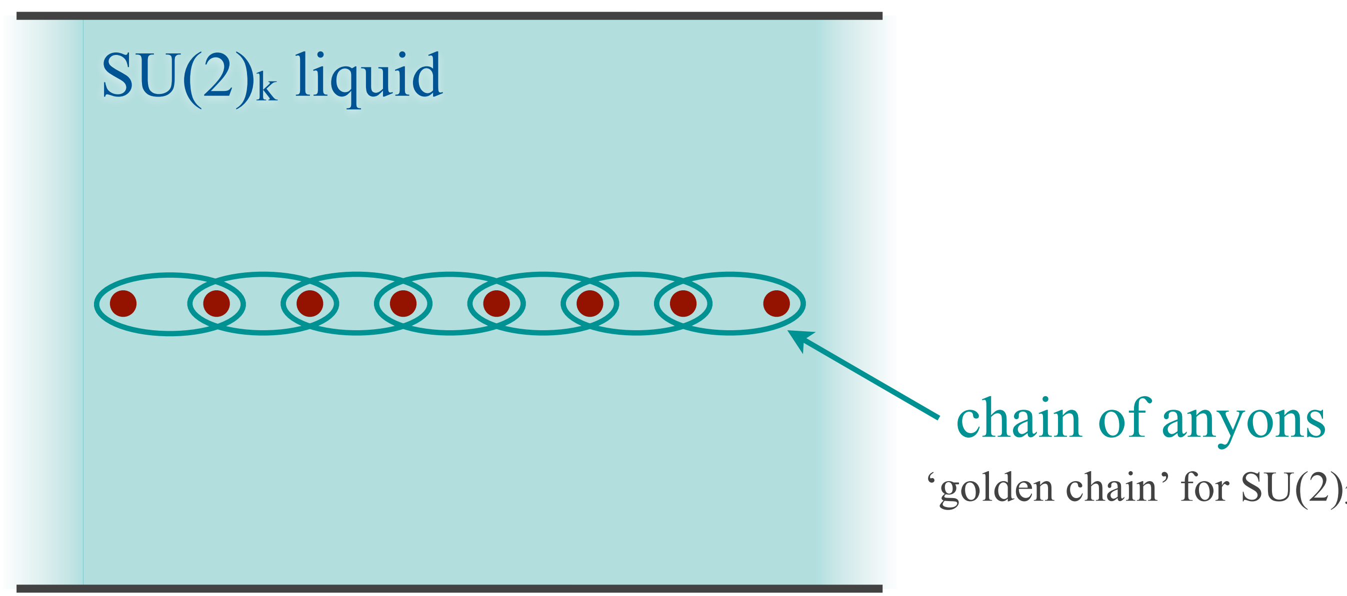

Anyonic Heisenberg model

Prog. Theor. Phys. Suppl. **176**, 384 (2008).

$SU(2)_k$ fusion rules

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

energetically split
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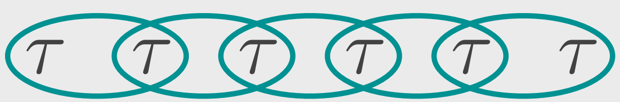
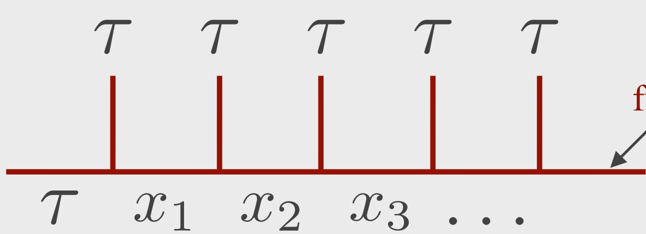
"Heisenberg" Hamiltonian

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

Example: chains of anyons



$(\tau = 1/2)$



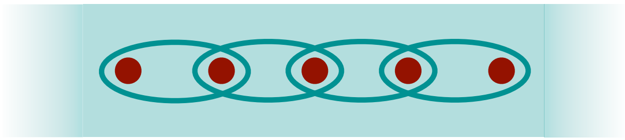
Hilbert space

$$|x_1, x_2, x_3, \ldots \rangle$$

Hamiltonian

$$H = \sum_i F_i \, \Pi_i^0 \, F_i$$

F-matrix = 6j-symbol

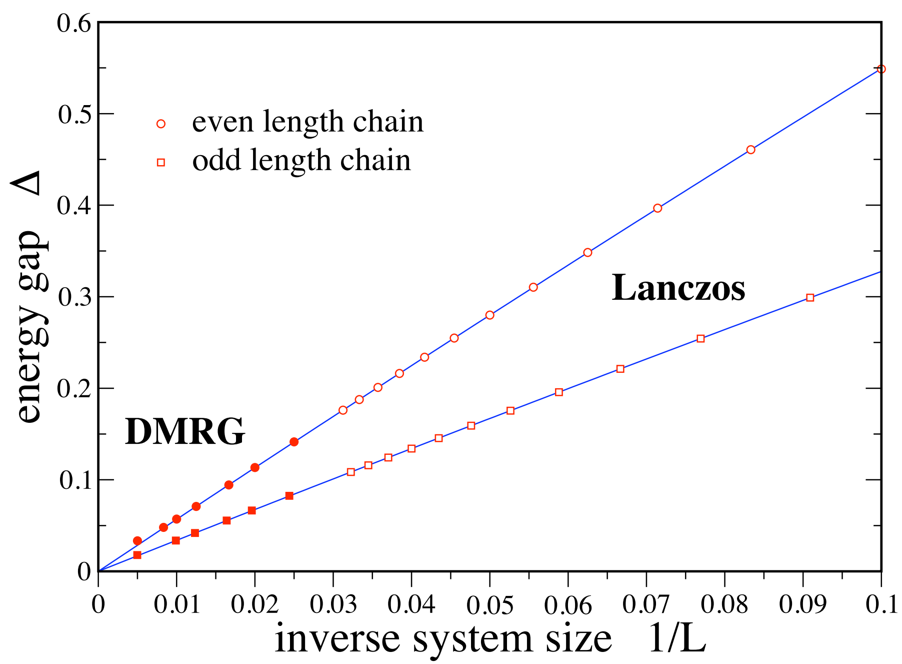


Critical ground state

Finite-size gap

$$\Delta(L) \propto (1/L)^{z=1}$$

conformal field theory description

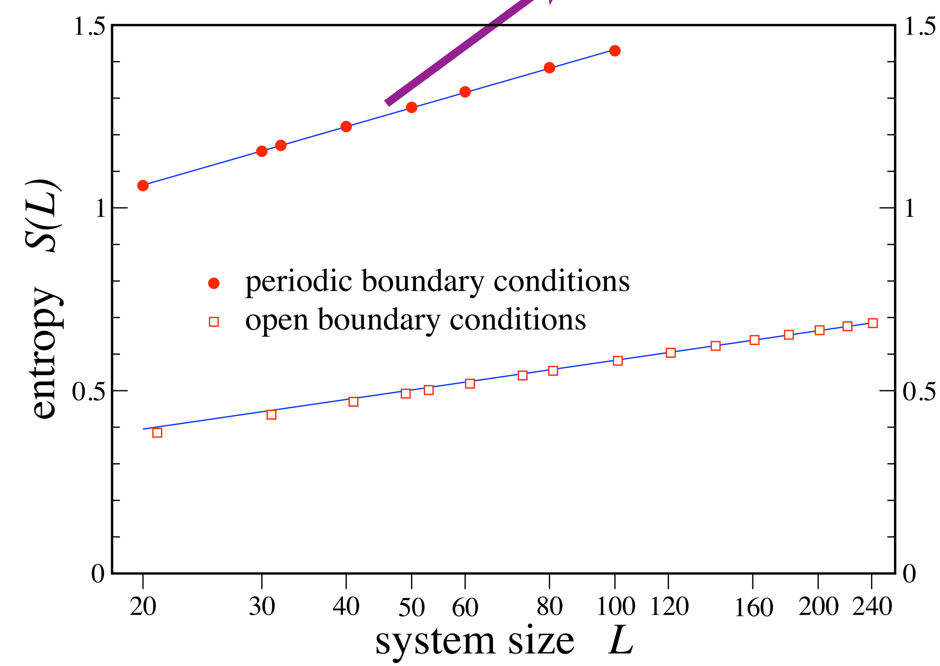


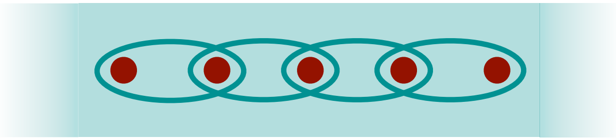
Entanglement entropy

$$S_{\mathrm{PBC}}(L) \propto \frac{c}{3} \log L$$

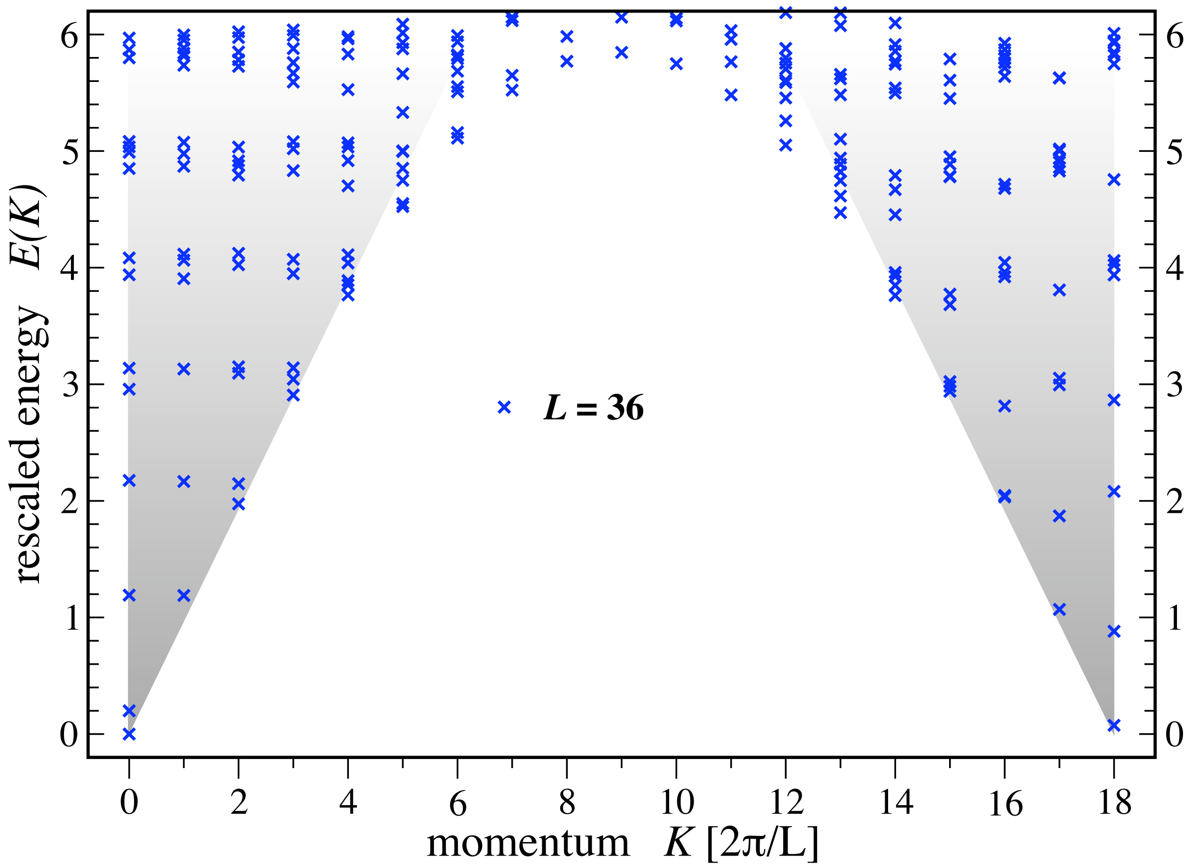
central charge

$c = 7/10$





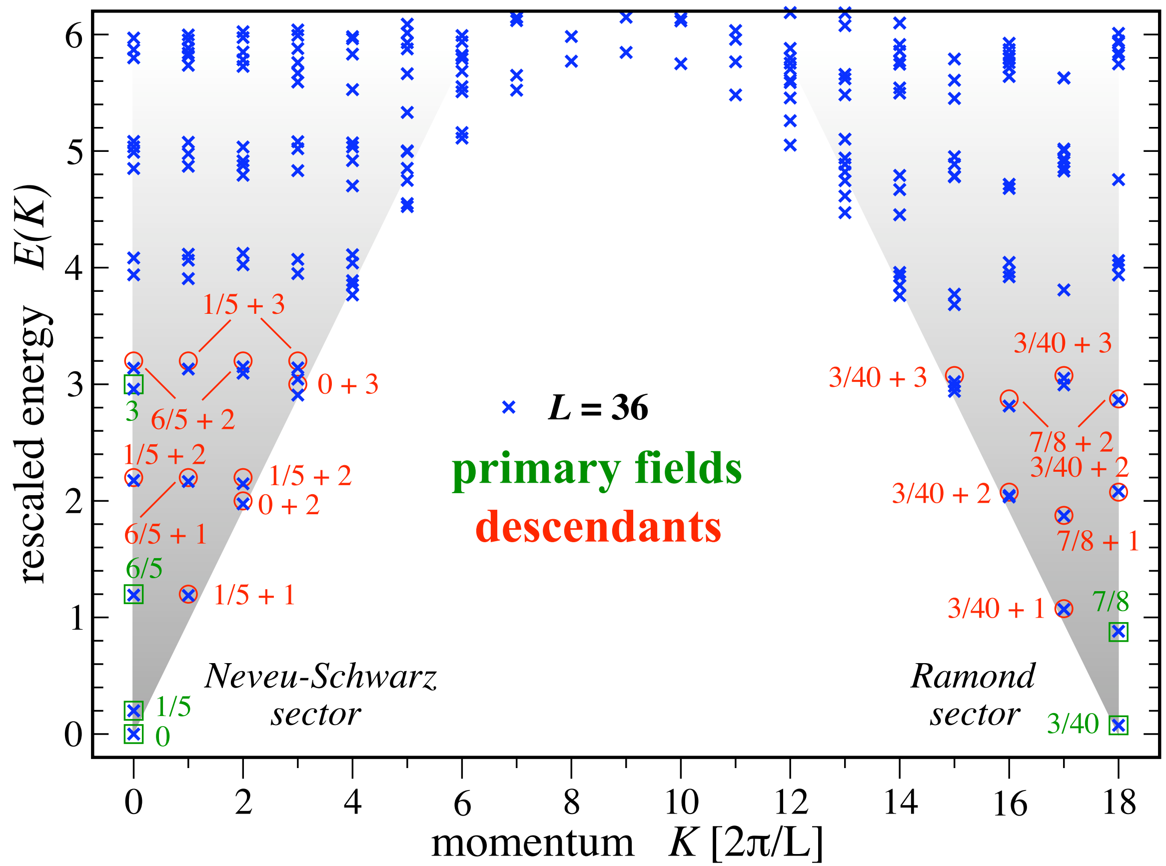
Conformal energy spectra



Z_2 sublattice symmetry

$$E = E_1 L + \frac{2\pi v}{L}\left(-\frac{c}{12} + \underbrace{h_L + h_R}_{\text{scaling dimension}}\right)$$

Conformal energy spectra



Z_2 sublattice symmetry

central charge $c = 7/10$

primary fields	I	ϵ	ϵ'	ϵ''	σ	σ'
scaling dimensions	0	1/5	6/5	3	3/40	7/8

thermal operators $K = 0$ (for I, ϵ, ϵ', ϵ'') · spin op. $K = \pi$ (for σ, σ')

Mapping & exact solution

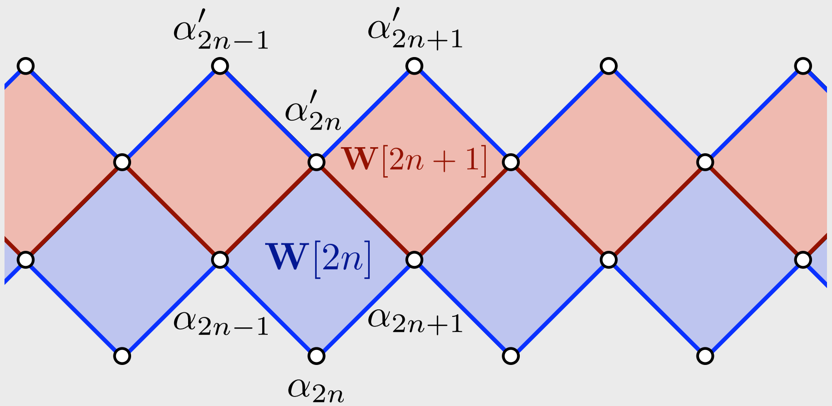
The operators $X_i = -d\, H_i$ form a representation of the **Temperley-Lieb algebra**

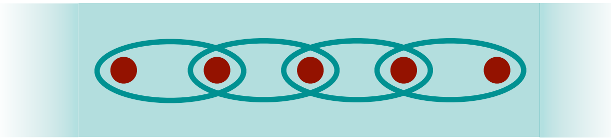
$$(X_i)^2 = d \cdot X_i \qquad X_i X_{i\pm1} X_i = X_i \qquad [X_i, X_j] = 0$$

for $|i-j| \geq 2$

$$d = 2\cos\left(\frac{\pi}{k+2}\right)$$

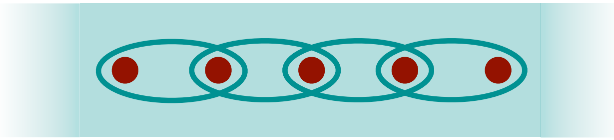
The transfer matrix
is an integrable representation
of the RSOS model.



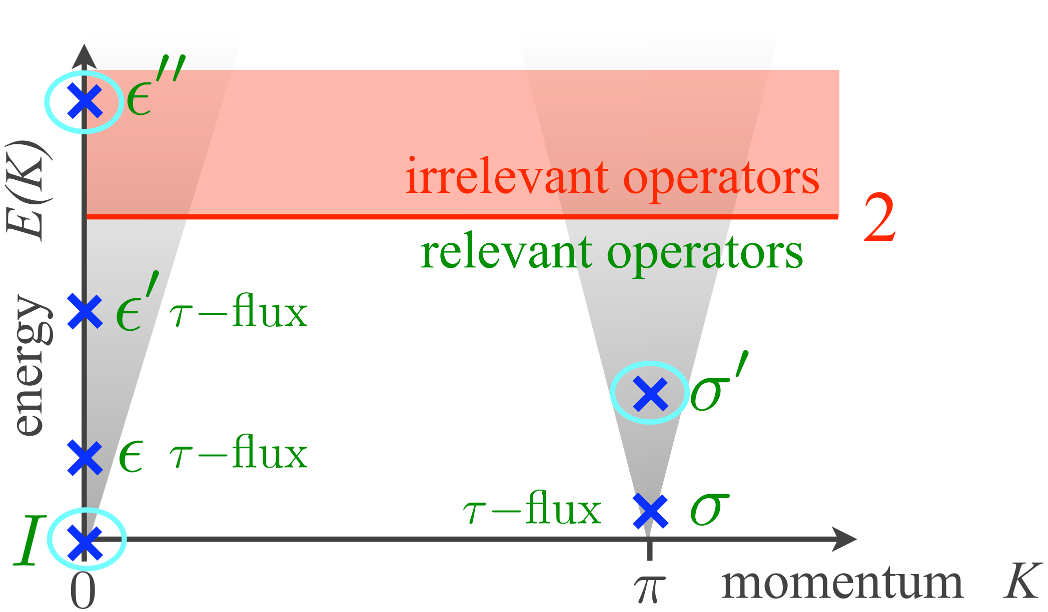


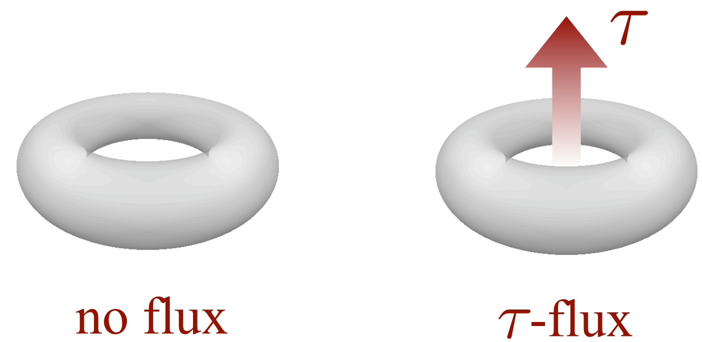
Deformed spin-1/2 chains

level k	$1/2 \times 1/2 \rightarrow \mathbf{0}$ ‘antiferromagnetic’	$1/2 \times 1/2 \rightarrow \mathbf{1}$ ‘ferromagnetic’
2	Ising c = 1/2	Ising c = 1/2
3	tricritical Ising c = 7/10	3-state Potts c = 4/5
4	$\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$	$\frac{SU(2)_k}{U(1)}$
5		
k	**k-critical Ising** c = 1-6/(k+1)(k+2)	**Z_k-parafermions** c = 2(k-1)/(k+2)
∞	Heisenberg AFM c = 1	Heisenberg FM c = 2



Topological symmetry





Relevant perturbations

~~$\sigma_L \sigma_R$~~ ~~$\sigma'_L \sigma'_R$~~ prohibited by **translational symmetry**

~~$\epsilon_L \epsilon_R$~~ ~~$\epsilon'_L \epsilon'_R$~~ prohibited by **topological symmetry**

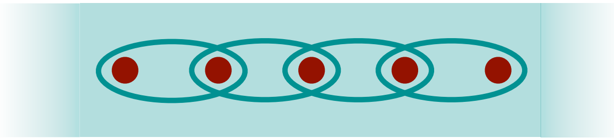
Symmetry operator

$$\langle x'_1, \ldots, x'_L | Y | x_1, \ldots, x_L \rangle = \prod_{i=1}^{L} \left(F^{x'_{i+1}}_{\tau x_i \tau} \right)^{x'_i}_{x_{i+1}}$$

with eigenvalues

$$S_{\tau-\text{flux}} = \phi \quad S_{\text{no flux}} = -\phi^{-1}$$

$$[H, Y] = 0$$

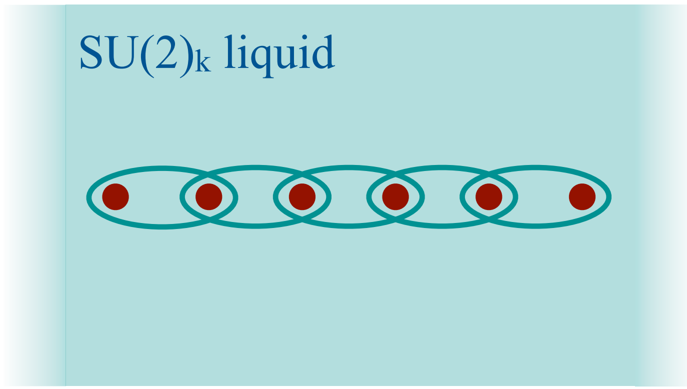


Topological protection

level k	$1/2 \times 1/2 \to \mathbf{0}$ ‘antiferromagnetic’		$1/2 \times 1/2 \to \mathbf{1}$ ‘ferromagnetic’	
2	Ising c = 1/2	✓	Ising c = 1/2	✓
3	tricritical Ising c = 7/10	✓	3-state Potts c = 4/5	✓
4	tetracritical Ising c = 4/5	✓	c = 1	✓
5	pentacritical Ising c = 6/7	✓	c = 8/7	✓
k	k-critical Ising c = 1-6/(k+1)(k+2)	✓	Z_k-parafermions c = 2(k-1)/(k+2)	✓
∞	Heisenberg AFM c = 1	✗	Heisenberg FM c = 2	✗

Gapless modes & edge states

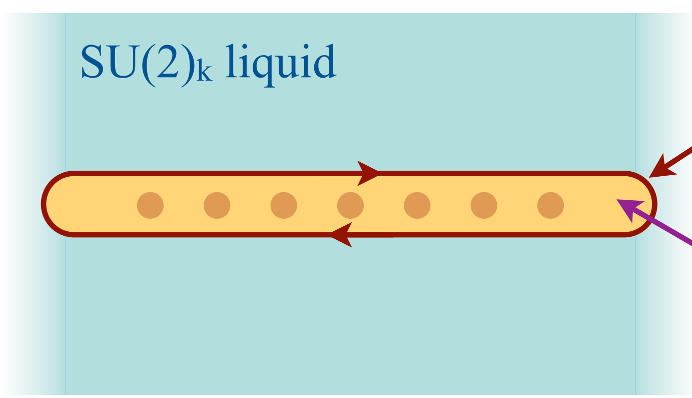
arXiv:0810.2277



critical theory
(AFM couplings)

$$\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$$

finite density
interactions



gapless modes = edge states

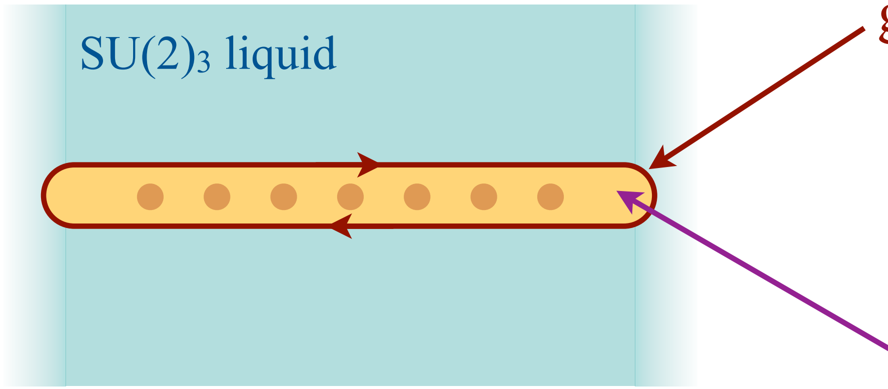
$$\frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k}$$

nucleated liquid

$$SU(2)_{k-1} \times SU(2)_1$$

Example: Ising meets Fibonacci

arXiv:0810.2277

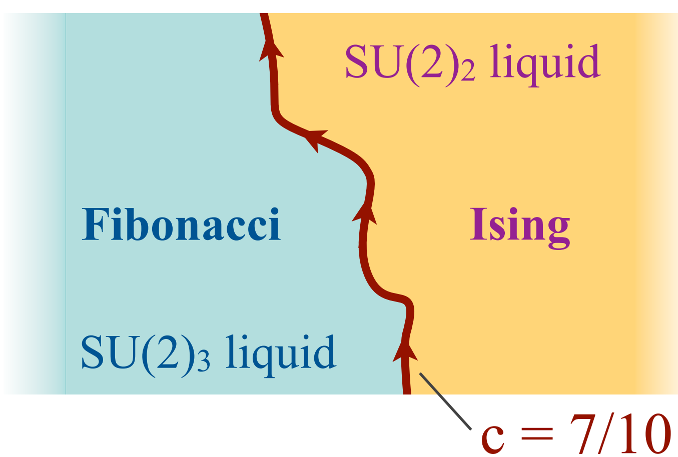


gapless modes = edge states

$$\frac{SU(2)_2 \times SU(2)_1}{SU(2)_3} \qquad c = 7/10$$

nucleated liquid

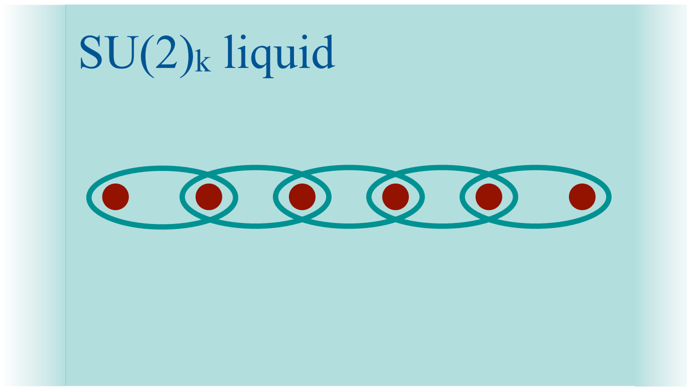
$$SU(2)_2 \times SU(2)_1$$



When Ising meets Fibonacci:
a tricritical Ising edge (c = 7/10)

Gapless modes & edge states

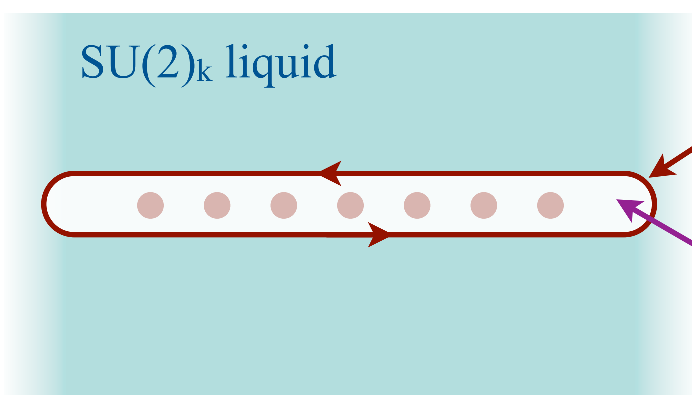
arXiv:0810.2277



critical theory
(FM couplings)

$$\frac{SU(2)_k}{U(1)}$$

finite density
interactions

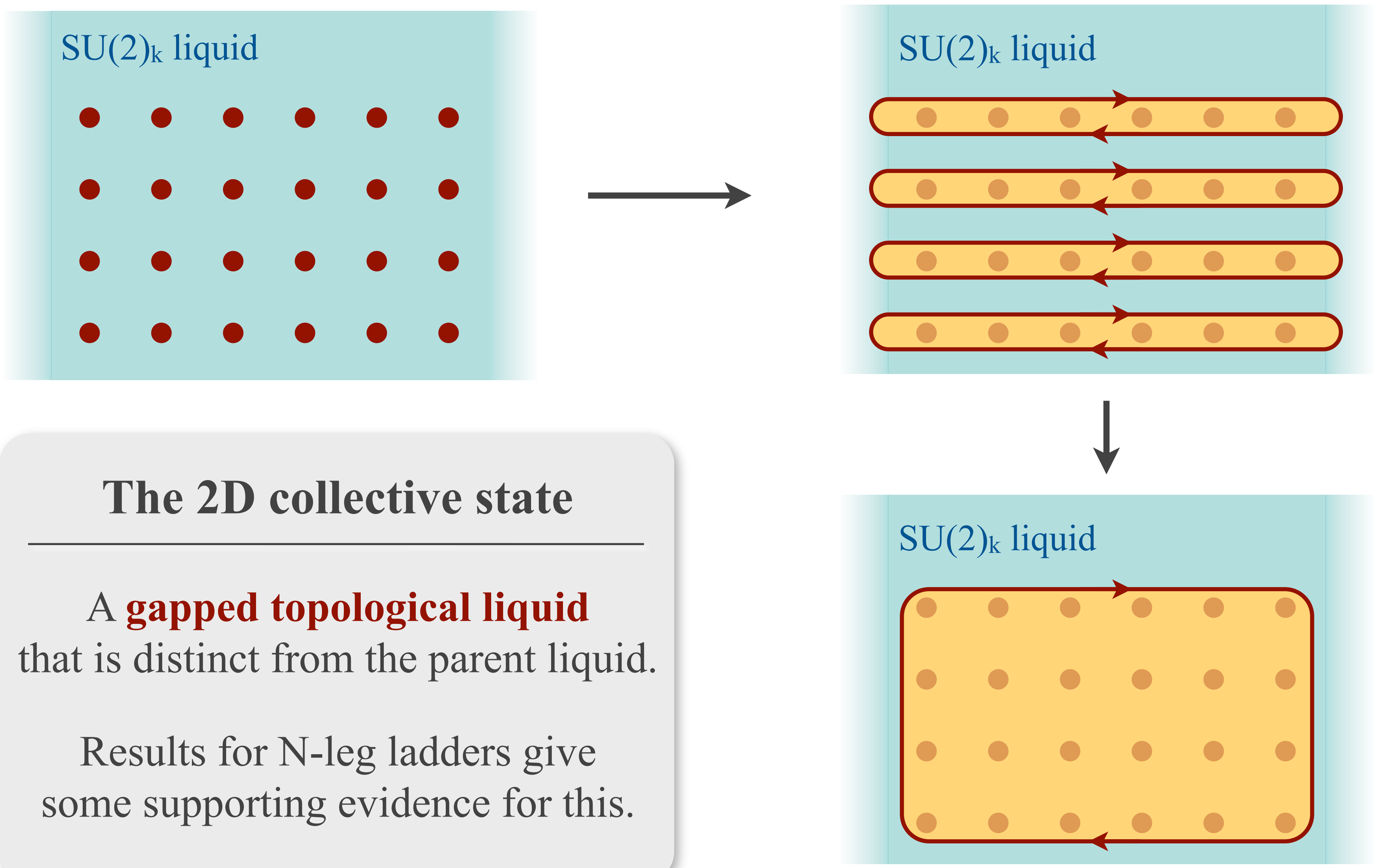


gapless modes = edge states

$$\frac{SU(2)_k}{U(1)}$$

nucleated liquid $U(1)$
(Abelian)

Approaching two dimensions

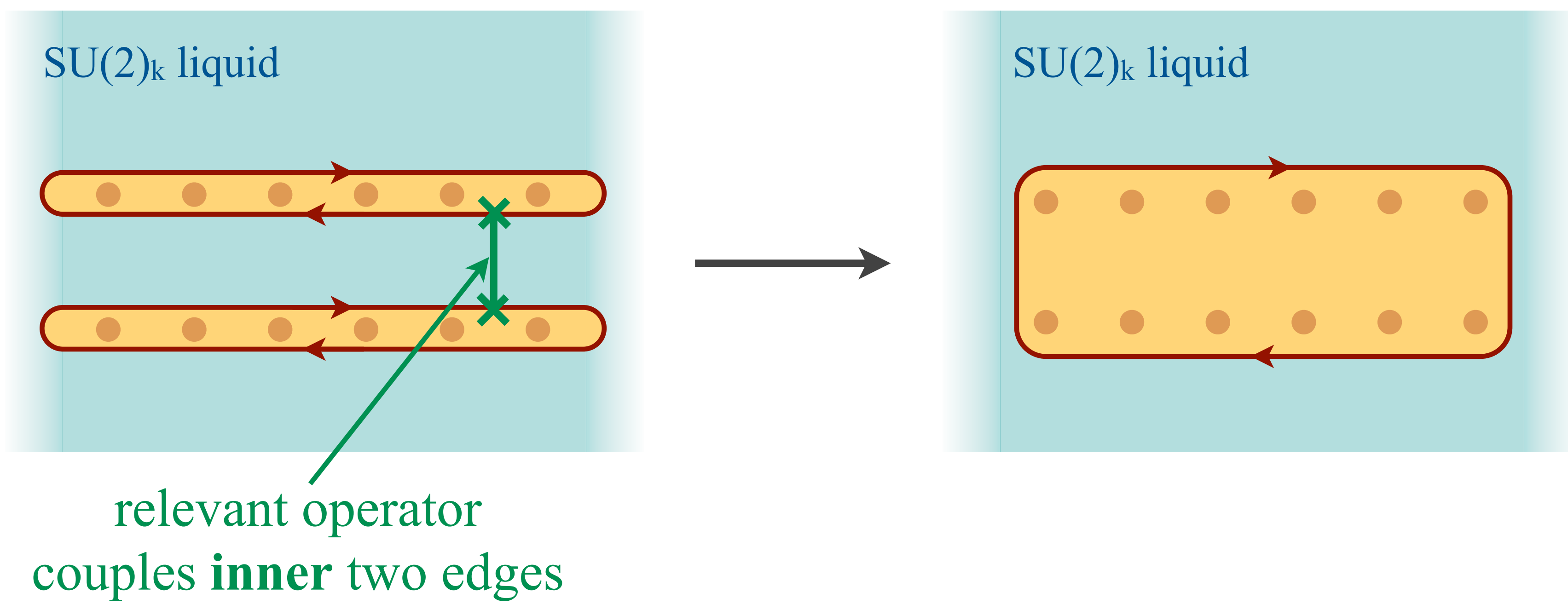


The 2D collective state

A **gapped topological liquid** that is distinct from the parent liquid.

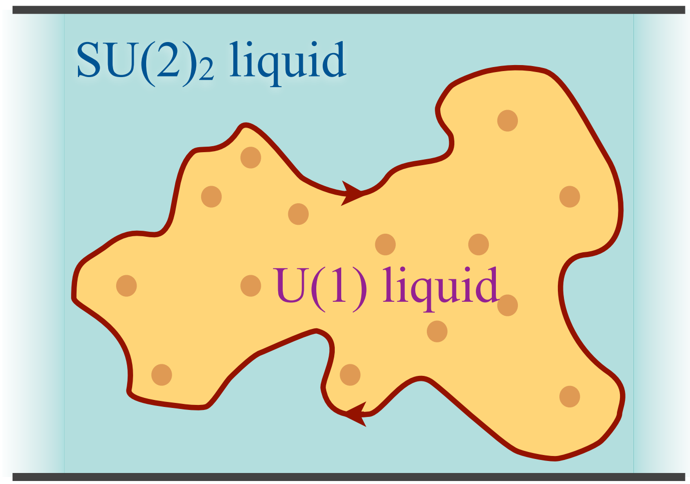
Results for N-leg ladders give some supporting evidence for this.

Coupling two chains

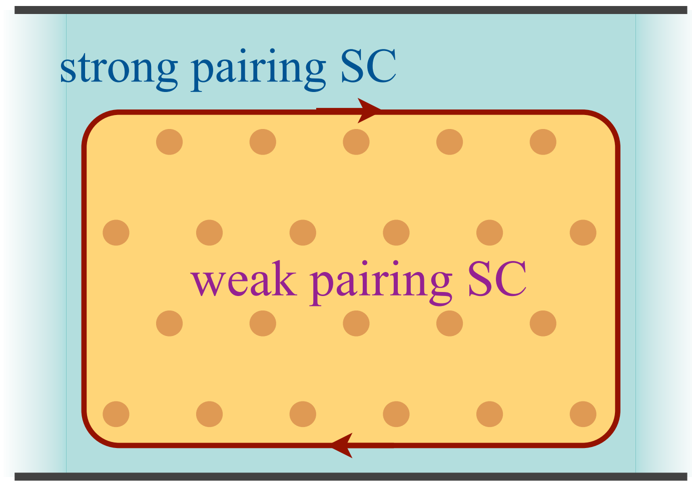


Earlier work for Majorana fermions

Read & Ludwig PRB (2000)



Grosfeld & Stern PRB (2006)



Grosfeld & Schoutens arXiv:0810.1955

Kitaev unpublished (2006)
Levin & Halperin PRB (2009)

2D anyon systems

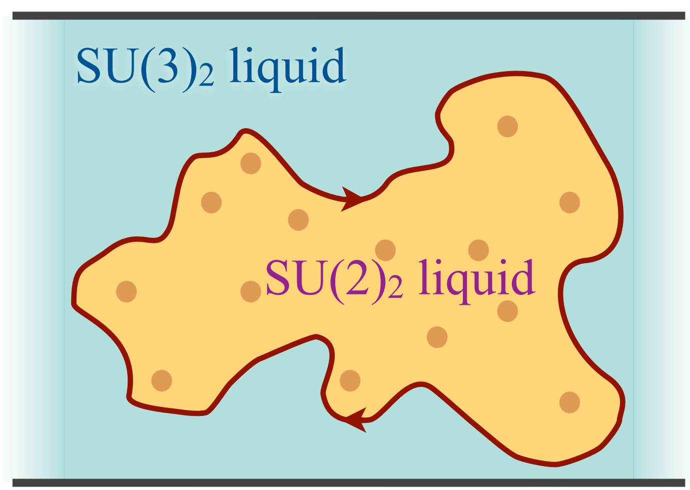
All of these previous results fit into our more general framework.

Recent work for Fibonacci anyons

Read & Ludwig PRB (2000)

Grosfeld & Stern PRB (2006)

Grosfeld & Schoutens arXiv:0810.1955



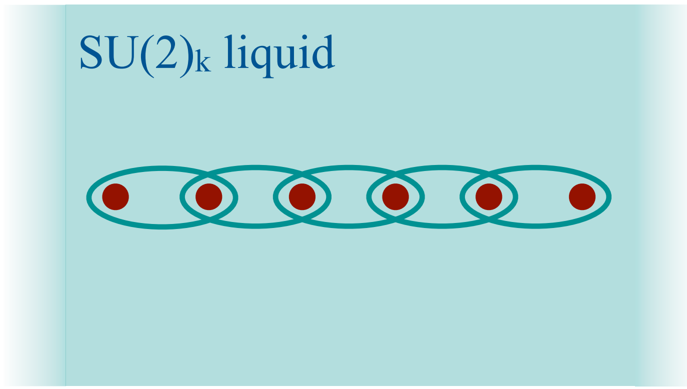
Kitaev unpublished (2006)
Levin & Halperin PRB (2009)

2D anyon systems

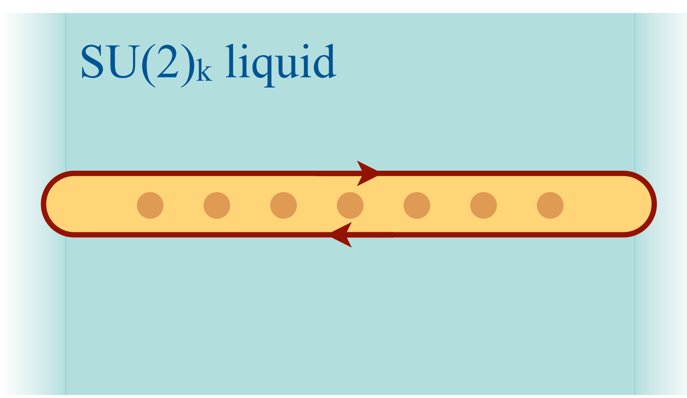
All of these previous results
fit into our more general framework.

A powerful correspondence

arXiv:0810.2277



finite density
interactions



collective states
of anyonic spin chains

↓

edge states
of topological liquids

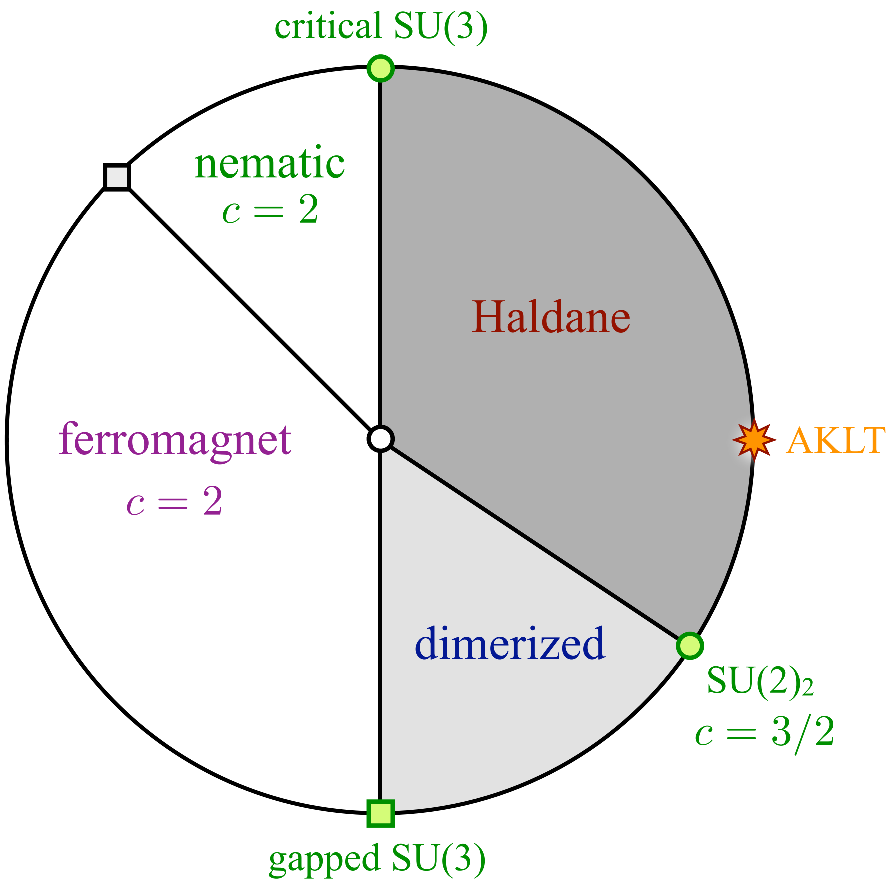
↓

nucleation of novel
topological liquids

Anyonic spin-1 chains

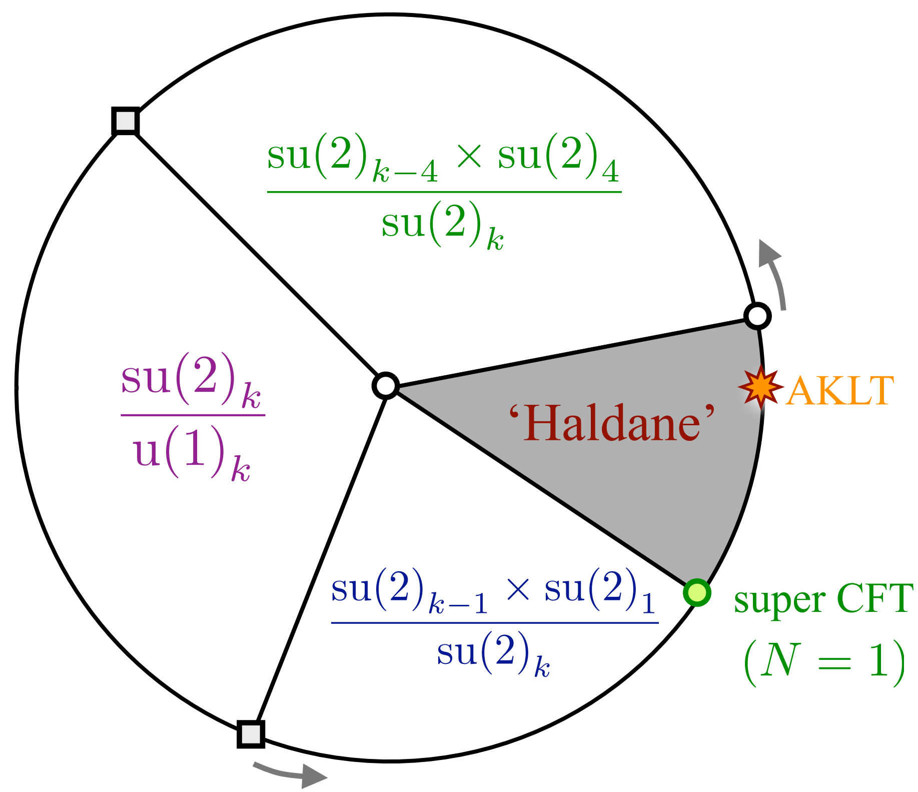
$\mathrm{SU(2)}_\infty$

$J_{S=2} = -\cos\theta \quad J_{S=1} = \sin\theta$



$\mathrm{SU(2)}_k$

$J_{S=2} = -\cos\theta \quad J_{S=1} = \sin\theta$



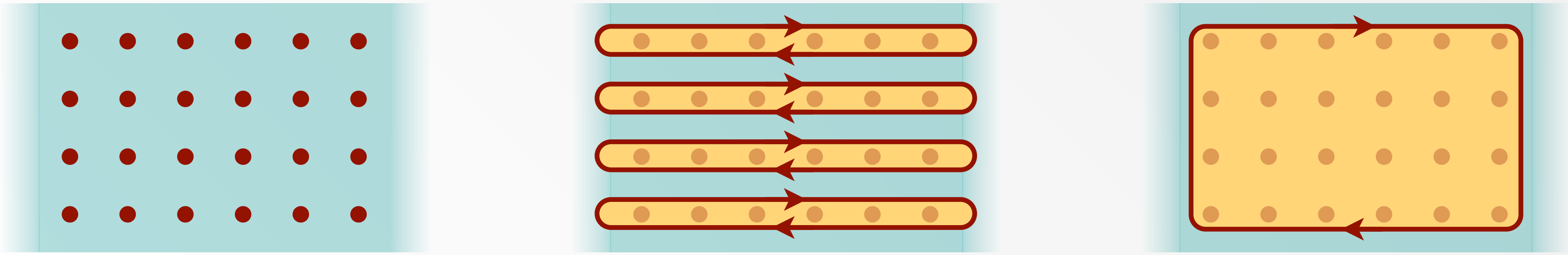
Conclusions

- Interacting non-Abelian anyons can support a wide variety of collective states:

    stable gapless states, gapped states, quasiparticles, ...

- In a topological liquid a **finite density** of interacting anyons nucleates a new topological liquid

    gapless states = edge states between top. liquids



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Phys. Rev. Lett. **101**, 050401 (2008).

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