Majorana metals

Cargese – June 28th, 2013

Simon Trebst
University of Cologne
Some of the most intriguing phenomena in condensed matter physics arise from the splitting of ‘accidental’ degeneracies.
Motivation

Some of the most intriguing phenomena in condensed matter physics arise from the splitting of ‘accidental’ degeneracies.

- interacting many-body system
- ‘accidental’ degeneracy
- residual effects select ground state

But they are also notoriously difficult to handle analytically, due to

- multiple energy scales
- complex energy landscapes / slow equilibration
- strong coupling
Examples – quantum Hall liquids

interacting many-body system

‘accidental’ degeneracy

residual effects select ground state

Landau level degeneracy

integer quantum Hall

fractional quantum Hall

2\Phi/\Phi_0

orbital states

filled level

incompressible liquid

partially filled level

incompressible liquid

\text{Coulomb repulsion}
Examples – frustrated magnets

- Interacting many-body system
- ‘Accidental’ degeneracy
- Residual effects selecting ground state

$\chi \sim \frac{1}{T - \Theta_{CW}}$

- Long-range order
- Cooperative paramagnet
- High temperature paramagnet

$E$ $\rightarrow$ $0$

$1/\chi$ $\rightarrow$ $\infty$

$T$ $\rightarrow$ $T_c$

$T=0$ residual entropy

Long-range order
Overview

Vortices, quasiholes, anyons, ...

Interactions

Disorder
Vortices
Abelian vs. non-Abelian vortices

Consider a set of ‘pinned’ vortices at fixed positions.

**Abelian**

- **single state**

  Example: Laughlin-wavefunction + quasiholes

**non-Abelian**

- (degenerate) **manifold** of states

  Manifold of states grows **exponentially** with the number of vortices.

  - Ising anyons (Majorana fermions) \( \sqrt{2}^N \)
  - Fibonacci anyons \( \phi^N \)
Abelian vs. non-Abelian vortices

Consider a set of ‘pinned’ vortices at fixed positions.

Abelian

\[ \psi(x_2, x_1) = e^{i\pi\theta} \cdot \psi(x_1, x_2) \]

\textbf{single state}

non-Abelian

\[ \psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n) \]
\[ \psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n) \]

\textbf{(degenerate) manifold of states}

\textbf{matrix}

In general \( M \) and \( N \) do not commute!
Topological quantum computing

Degenerate manifold = qubit

Employ **braiding** of non-Abelian vortices to perform computing (unitary transformations).

Matrix depends only on the topology of the braid swept out by quasiparticle world lines!

Robust quantum computation? *(Kitaev '97; Freedman, Larsen and Wang '01)*

In general $M$ and $N$ do not commute!

$\psi(x_1 \leftrightarrow x_3) = M \cdot \psi(x_1, \ldots, x_n)$

$\psi(x_2 \leftrightarrow x_3) = N \cdot \psi(x_1, \ldots, x_n)$
Vortex-vortex interactions

\[ a \gg \xi_m \]

\[ a \approx \xi_m \]

Exponential decay \( \propto \exp(-a/\xi_m) \)

RKKY-like oscillation \( \propto k_F \)

Vortex separation

\[ \delta E \]
Vortex-vortex interactions

\[ a \gg \xi_m \]

Exponential decay \( \propto \exp\left(-a/\xi_m\right) \)

RKKY-like oscillation \( \propto k_F \)

pair splitting [mK]

\[ \delta E \]

\[ a \approx \xi_m \]

Vortex separation

middle of plateau
detuning \( \Delta B / B_0 \)
edge of plateau
Vortex-vortex interactions

**Vortex quantum numbers**

$SU(2)_k = \text{‘deformation’ of } SU(2)$

with finite set of representations

$0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \frac{k}{2}$

**Fusion rules**

$j_1 \times j_2 =

|j_1 - j_2| + (|j_1 - j_2| + 1) + \ldots +

\text{min}(j_1 + j_2, k - j_1 - j_2)$

**Example**

$k = 2$

$1/2 \times 1/2 = 0 + 1$

**Vortex pair**

$1/2 \times 1/2 = 0 + 1$

**Energetics for many vortices**

$H = J \sum \prod_{i,j}^0 \langle ij \rangle_{ij}$

“Heisenberg Hamiltonian” for vortices
Vortex-vortex interactions

**Microscopics**

Which channel is favored is not universal, but microscopic detail.

- $J$: p-wave SC, Kitaev model
- $1/2 \times 1/2 \rightarrow 0$
- $1/2 \times 1/2 \rightarrow 1$
- Moore-Read state

**Vortex pair**

$$1/2 \times 1/2 = 0 + 1$$

**Energetics for many vortices**

$$H = J \sum_{ij} \prod_{j}^{0}$$

"Heisenberg Hamiltonian" for vortices
The many-vortex problem

quantum liquid

\[ a \gg \xi_m \]

E

vortex-vortex interactions

macroscopic degeneracy
The many-vortex problem

\[ a \gg \xi_m \]

**macroscopic degeneracy**

\[ E \]

vortex-vortex interactions

\[ a \approx \xi_m \]

**unique ground state**

\[ \Delta \]
The collective state

Quantum liquid

Bulk gap

Quantum liquid

Anyonic Heisenberg chains

\( (\tau = 1/2) \)

Hilbert space

\( |x_1, x_2, x_3, \ldots \rangle \)

Hamiltonian

\[ H = \sum_i F_i \Pi_i^0 F_i \]

F-matrix = 6j-symbol
### Finite-size gap

\[ \Delta(L) \propto \left( \frac{1}{L} \right)^{z=1} \]

**conformal field theory description**

### Entanglement entropy

\[ S(L) \propto \frac{c}{3} \log L \]

- **central charge**
  - \( c = 7/10 \)
  - **DMRG**
  - **Lanczos**

---

**Graphs:**
- **Finite-size gap**
  - \( \Delta(L) \)
  - **DMRG**
  - **Lanczos**
  - **Inverse system size** \( 1/L \)

- **Entropy**
  - \( S(L) \)
  - **System size** \( L \)
  - **Entropy** \( S(L) \) vs. **System size** \( L \)
<table>
<thead>
<tr>
<th>level $k$</th>
<th>$1/2 \times 1/2 \rightarrow 0$</th>
<th>\text{‘antiferromagnetic’}</th>
<th>$1/2 \times 1/2 \rightarrow 1$</th>
<th>\text{‘ferromagnetic’}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Ising</td>
<td>$c = 1/2$</td>
<td>Ising</td>
<td>$c = 1/2$</td>
</tr>
<tr>
<td>3</td>
<td>tricritical Ising</td>
<td>$c = 7/10$</td>
<td>3-state Potts</td>
<td>$c = 4/5$</td>
</tr>
<tr>
<td>4</td>
<td>\begin{align*} SU(2)_{k-1} \times SU(2)_1 \ SU(2)_k \end{align*}</td>
<td>\null</td>
<td>\begin{align*} SU(2)_k \ U(1) \end{align*}</td>
<td>\null</td>
</tr>
<tr>
<td>5</td>
<td>k-critical Ising</td>
<td>$c = 1-6/(k+1)(k+2)$</td>
<td>Z$_k$-parafermions</td>
<td>$c = 2(k-1)/(k+2)$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>Heisenberg AFM</td>
<td>$c = 1$</td>
<td>Heisenberg FM</td>
<td>$c = 2$</td>
</tr>
</tbody>
</table>
Gapless modes & edge states

SU(2)_k liquid

critical theory
1/2 × 1/2 → 0
\[ \frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k} \]

interactions

gapless modes = edge states
\[ \frac{SU(2)_{k-1} \times SU(2)_1}{SU(2)_k} \]

nucleated liquid
\[ SU(2)_{k-1} \times SU(2)_1 \]
Interactions + disorder
Disorder induced phase transition

\[ a \gg \xi_m \]

\[ a \approx \xi_m \]

**quantum liquid**

- Disorder + vortex-vortex interactions
- Macroscopic degeneracy

**thermal metal**

- Degeneracy is split
Interactions and disorder

\[ H = \sum \left( J_{jk} \Pi_{jk} \right) \]

\[ \langle jk \rangle \]

Natural analytical tool: strong-randomness RG

Unfortunately, this does not work. The system flows away from strong disorder under the RG. No infinite randomness fixed point.
From Ising anyons to Majorana fermions

$$H = \sum \langle jk \rangle J_{jk} \Pi_{jk}$$

**interacting** Ising anyons

“anyonic Heisenberg model”

$$\mathcal{H} = - \sum \langle jk \rangle iJ_{jk} \gamma_j \gamma_k$$

**free** Majorana fermion hopping model
From Ising anyons to Majorana fermions

Majorana operators
\[ \{ \gamma_i, \gamma_j \} = \delta_{ij} \]
\[ \gamma_i^\dagger = \gamma_i \]
\[ \gamma_{i1} = (c_i^\dagger + c_i) / 2 \]
\[ \gamma_{i2} = (c_i^\dagger - c_i) / 2i \]

\[ \mathcal{H} = - \sum_{\langle jk \rangle} i J_{jk} \gamma_j \gamma_k \]

free Majorana fermion hopping model

particle-hole symmetry ✓  

time-reversal symmetry ✗  

Symmetry class D
A disorder-driven metal-insulator transition

Density of states indicates phase transition.

Griffith physics

sign disorder \( p \)

\[ J = \begin{cases} 
  +1, & p \\
  -1, & 1-p
\end{cases} \]

Density of states indicates phase transition.
The thermal metal

Density of states **diverges** logarithmically at zero energy.
Taking a closer look

Oscillations in the DOS fit the prediction from random matrix theory for symmetry class D

\[ \rho(E) = \alpha + \frac{\sin(2\pi \alpha EL^2)}{2\pi EL^2} \]
The thermal metal

Inverse participation ratios (moments of the GS wavefunction) indicate multifractal structure characteristic of a metallic state.

\[ I_q = \int d^2r |\psi(r)|^{2q} \sim \frac{1}{L^{\tau_q}} \]
Disorder induced phase transition

Quantum liquid

Thermal metal

\[ a \gg \xi_m \]

\[ a \approx \xi_m \]

Disorder + vortex-vortex interactions

Macroscopic degeneracy

Degeneracy is split
Heat transport

Caltech thermopower experiment

Heat transport along the sample edges changes quantitatively

Bulk heat transport diverges logarithmically as $T \to 0$.

$$\kappa_{xx}/T \propto \log T$$

middle of plateau

electrical transport remains unchanged
Other places to look for Majorana metals ...
Spin-orbit assisted Mott physics in Iridates

(Na,Li)$_2$IrO$_3$

$\Theta_{CW} \approx -33$ K

$\Theta_{CW} \approx -125$ K

$T_N \approx 15$ K

Spin-orbit assisted Mott physics in Iridates

\[ (\text{Na}, \text{Li})_2\text{IrO}_3 \]

Ir\(^{4+}\) (5d\(^5\))

d-orbitals

octahedral crystal field

Ir\(_6\) cage

\[ \begin{align*}
\text{e}_g & \sim 3\text{eV} \\
\text{t}_{2g} &
\end{align*} \]

\[ L \cdot \vec{\sigma} \]

spin-orbit coupling

\[ j = 1/2 \]

\[ j = 3/2 \]

\[ H_{\text{Kitaev}} = \sum_{\gamma-\text{links}} J_{\gamma} \sigma^\gamma_i \sigma^\gamma_j \]

Rare combination of a model of fundamental conceptual importance (harboring topological phases) and an exact analytical solution.
The Kitaev model

\[ H_{\text{Kitaev}} = \sum_{\gamma-\text{links}} J_\gamma \sigma_i^\gamma \sigma_j^\gamma \]

Rare combination of a model of **fundamental conceptual importance** (harboring topological phases) and an **exact analytical solution**.

- **Gapped spin liquid** with \( Z_2 \) topological order
- **Gapless spin liquid** with Majorana fermion excitations
- 2 Dirac cones

The Kitaev model + field

\[ H_{\text{Kitaev}+h} = \sum_{\gamma-\text{links}} J_{\gamma} \sigma_i^\gamma \sigma_j^\gamma - \sum_i \vec{h} \cdot \vec{\sigma}_i \]

\[ \vec{h} = h (1, 1, 1) \]

vortices are Ising anyons with Majorana fermion zero modes

gapped spin liquid

Z\(_2\) topological order

non-Abelian topological order
Kitaev model + field + disorder
Topological excitations + interactions + disorder can give rise to a plethora of collective phenomena.

- Topological **liquid nucleation**
- **Thermal metal**
- Distinct experimental **bulk observable** (heat transport) in search for Majorana fermions.