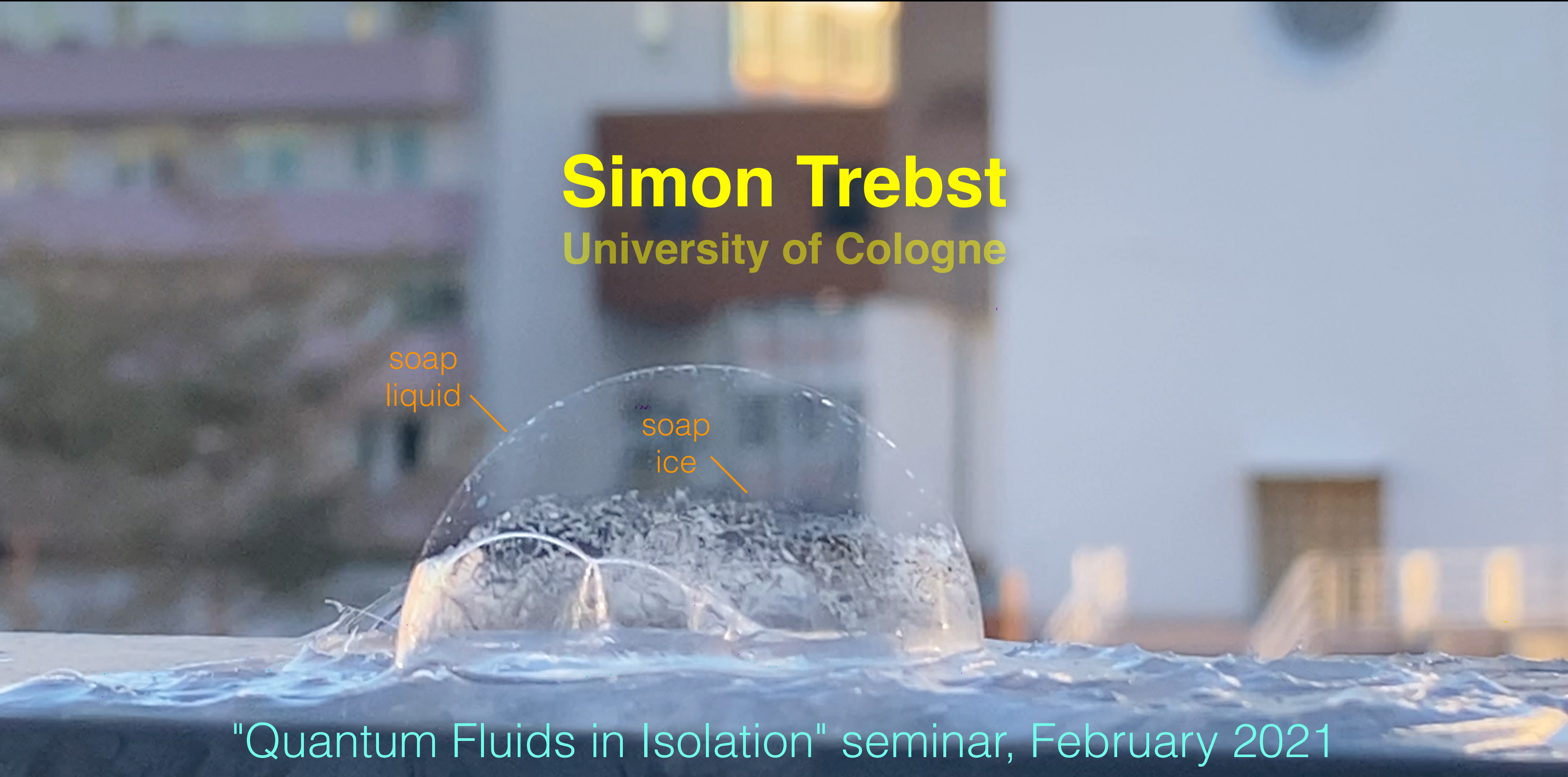
# topology and supersymmetry




**Simon Trebst**
University of Cologne

"Quantum Fluids in Isolation" seminar, February 2021

# collaborators

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Phys. Rev. Research **1**, 032047(R) (2019)
and
Phys. Rev. B **96**, 085145 (2017)
Editors' suggestion.

CRC183
ENTANGLED STATES OF MATTER

Krishanu Roychowdury
Stockholm / Cornell

# topological mechanics

# topological mechanics

C.L. Kane & T.C. Lubensky, Nat. Phys. **10**, 39 (2014)

example #1: “floppy modes” in isostatic lattices

Maxwell relation

$$\nu \equiv N_0 - N_{ss} = d \cdot n_s - n_b$$

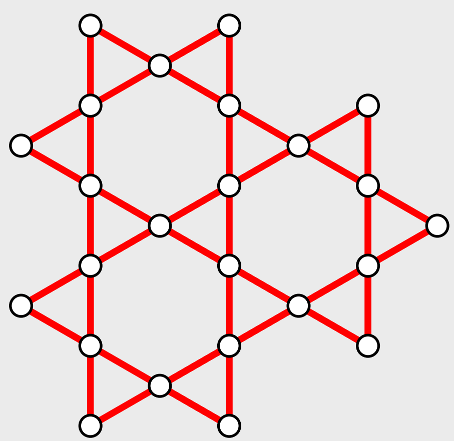
degrees of freedom ($N_0$), states of self stress ($N_{ss}$), sites ($n_s$), bonds ($n_b$)

isostatic lattices

$$\nu = 0$$

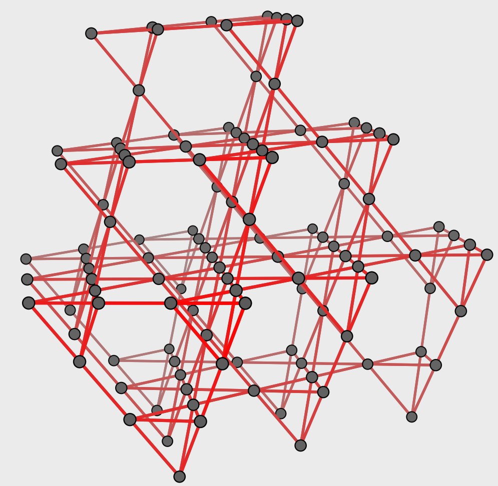
coordination number

$$z = 2 \cdot d$$



kagome lattice

$d = 2 \quad z = 4$



pyrochlore lattice

$d = 3 \quad z = 6$

# topological mechanics

C.L. Kane & T.C. Lubensky, Nat. Phys. **10**, 39 (2014)

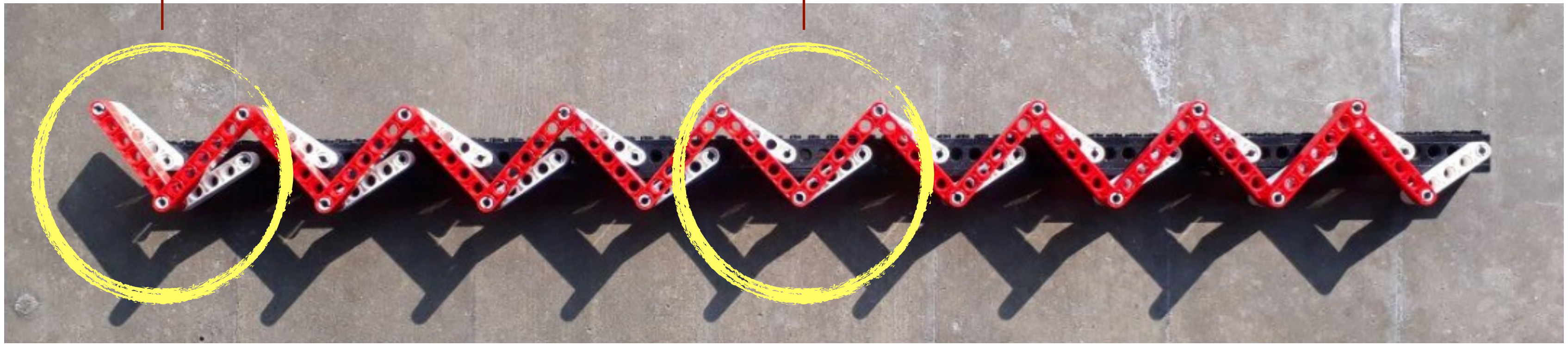
example #1: “floppy modes” in isostatic lattices

Maxwell relation

$$\nu \equiv N_0 - N_{\mathrm{ss}} = d \cdot n_{\mathrm{s}} - n_{\mathrm{b}}$$

$N_0$: degrees of freedom; $N_{\mathrm{ss}}$: states of self stress; $n_{\mathrm{s}}$: sites; $n_{\mathrm{b}}$: bonds

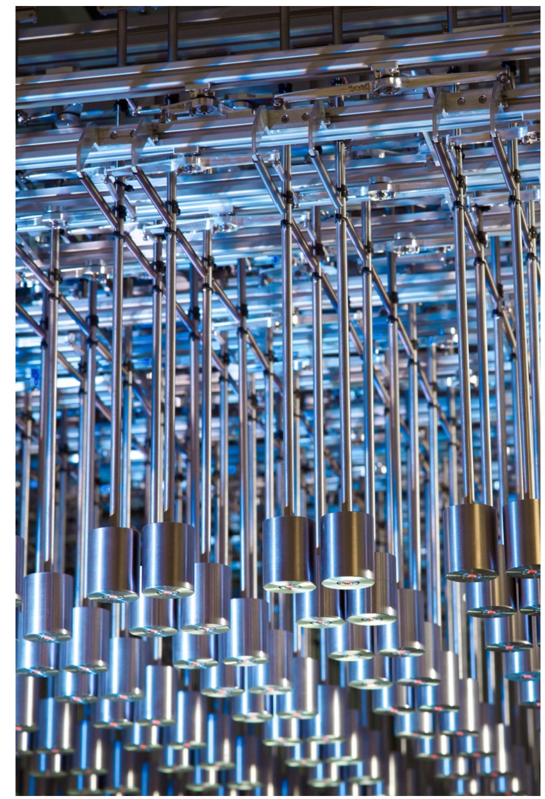
**“floppy mode”** **“rigid mode”**



“mechanical” SSH chain

# topological mechanics

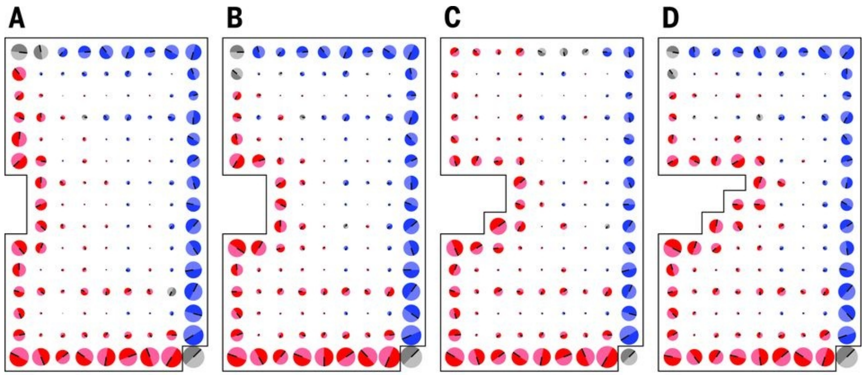
example #2: topological insulator from classical pendula



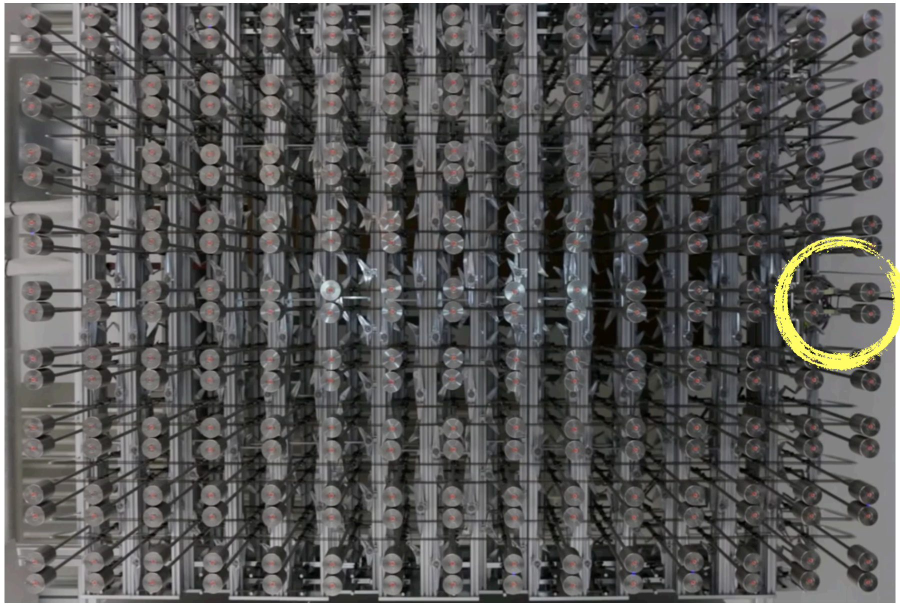
R. Süsstrunk and S. D. Huber, Science **349**, 47 (2015)
S. D. Huber, Nature Phys. **12**, 621 (2016)

# topological mechanics

example #2: topological insulator from classical pendula




floppy modes constitute
boundary mode



R. Süsstrunk and S. D. Huber, Science **349**, 47 (2015)
S. D. Huber, Nature Phys. **12**, 621 (2016)

# correspondence principles

**topological mechanics**

electronic system ⟷ mechanical system

(matrix) correspondence

Schrödinger equation

$$i\frac{\partial}{\partial t}\begin{pmatrix}\sqrt{\mathbf{D}}^T x \\ i\dot{x}\end{pmatrix} = \begin{pmatrix}0 & \sqrt{\mathbf{D}}^T \\ \sqrt{\mathbf{D}} & 0\end{pmatrix}\begin{pmatrix}\sqrt{\mathbf{D}}^T x \\ i\dot{x}\end{pmatrix}$$

symmetry class BDI

← square root

Newton's equation

$$\ddot{x} = -\mathbf{D}x$$

dynamical matrix

# correspondence principles

**topological mechanics**

electronic system ⟷ mechanical system

(matrix) correspondence

---

**supersymmetry**

fermions ⟷ bosons

SUSY

supersymmetry

# basic ingredients of SUSY

non-hermitian
**SUSY charge operator**

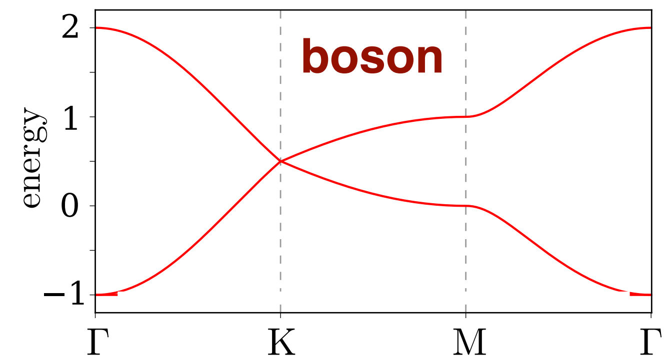
**boson**

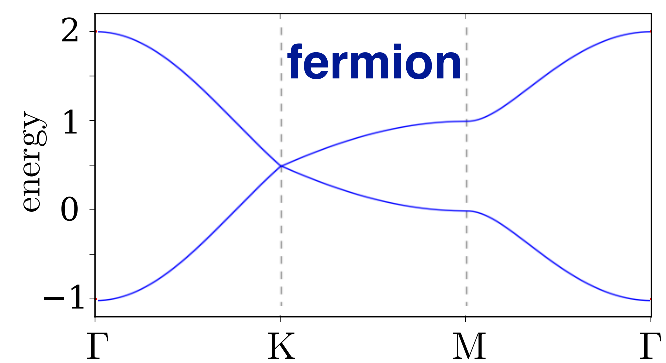
$$\mathcal{Q} = c_i^\dagger \mathbf{R}_{ij} b_j$$

**fermion** arbitrary matrix

supersymmetric **Hamiltonian**

$$\mathcal{H}_{\mathrm{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$




**isospectral**
quadratic
Hamiltonians

# SUSY & topological mechanics

**topological mechanics** – phase space coordinates $(p, q)$
as bosonic degrees of freedom

**real** bosons $[\hat{q}_i, \hat{p}_j] = i\delta_{i,j}$

**real** fermions
= Majorana fermions

natural SUSY partners

SUSY charge
$$\mathcal{Q} = \gamma_i^B \mathbf{1}_{ij} \hat{p}_j + \gamma_i^A \mathbf{A}_{ij} \hat{q}_j$$

real bosons: $\hat{p}_j$, $\hat{q}_j$

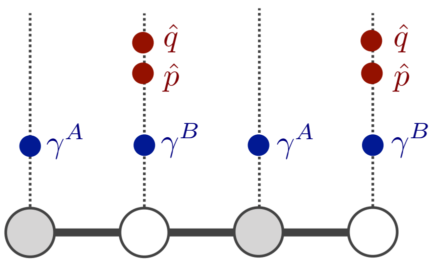
real fermions: $\gamma_i^B$, $\gamma_i^A$

# SUSY & topological mechanics

SUSY charge $\mathcal{Q} = \gamma_i^B \mathbf{1}_{ij} \hat{p}_j + \gamma_i^A \mathbf{A}_{ij} \hat{q}_j$

encodes block-diagonal form

$$\mathbf{R} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}$$




$$H_{\mathrm{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

$$\mathcal{H}_{\mathrm{fermion}} = -i\gamma_j^A \, \mathbf{A}_{jk} \, \gamma_k^B + \mathrm{h.c.}$$

Majoranas hopping on two sublattices AB

$$\mathcal{H}_{\mathrm{boson}} = \hat{p}_i \hat{p}_i + \hat{q}_i (\mathbf{A}^T \mathbf{A})_{ij} \hat{q}_j$$

dynamical matrix

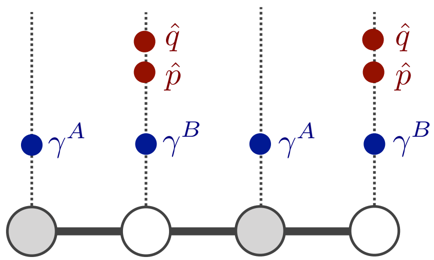
bosons on one sublattice (B)

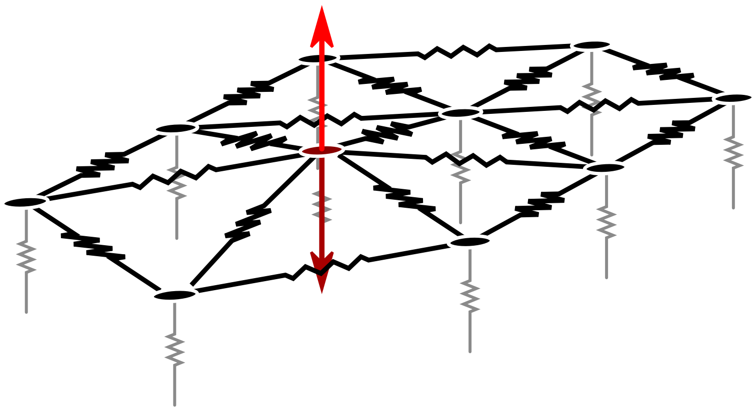
$\mathbf{R}$ is the **rigidity matrix** of the mechanical system.

It allows to directly connect mechanical systems to Majorana analogues, and vice versa.

# SUSY & topological mechanics

From real bosons to classical **balls and springs**.



$$\mathcal{H}_{\text{boson}} = \hat{p}_i \hat{p}_i + \hat{q}_i (\mathbf{A}^T \mathbf{A})_{ij} \hat{q}_j$$

classical limit

$$\mathcal{H} = \sum_i \frac{p_i^2}{2m} + \sum_{ij} \frac{k_{ij}}{2} (q_i - q_j)^2 + \sum_i \frac{\kappa_i}{2} q_i^2$$

$$\sim \sum_i p_i^2 + \sum_{ij} q_i \mathbf{D}_{ij} q_j$$

$$k_{ij} = -2 \sum_{a \in A} \mathbf{A}^T_{ia} \mathbf{A}_{aj} \qquad \kappa_i = 2 \sum_{a \in A} \mathbf{A}^2_{ia} - \sum_{b \in B} k_{ib}$$

intersite springs

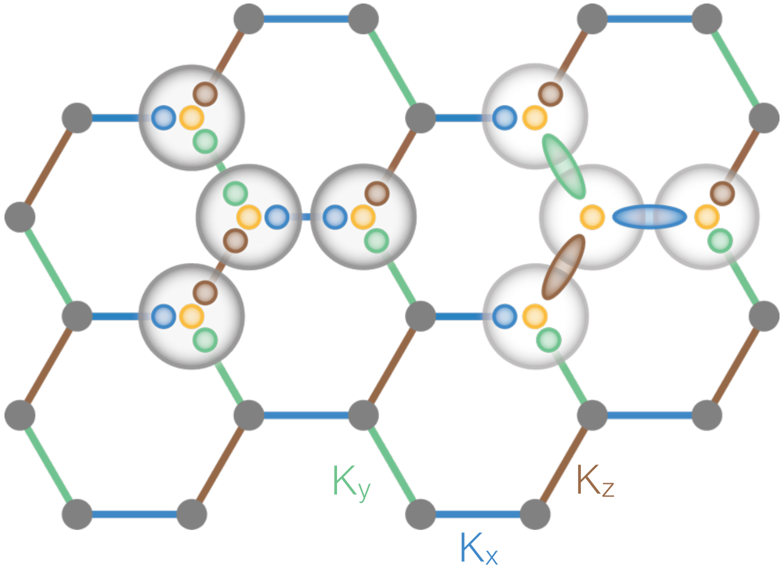
on-site springs

# Kitaev model

mechanical analogue

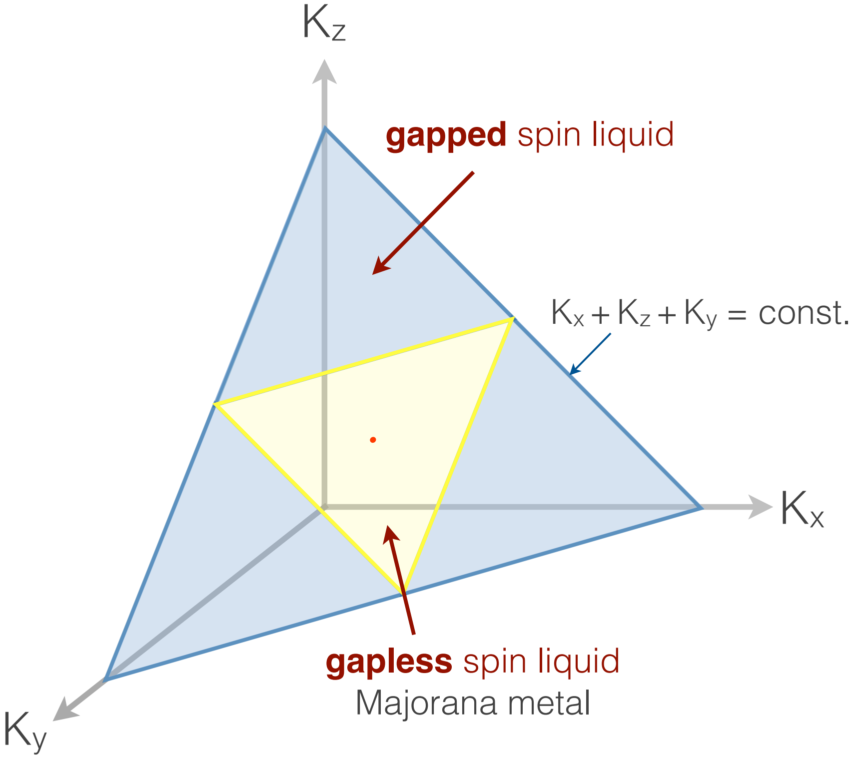
# Kitaev model

$$\mathcal{H} = - \sum_{\gamma-\text{bonds}} K_\gamma \; S_i^\gamma S_j^\gamma$$



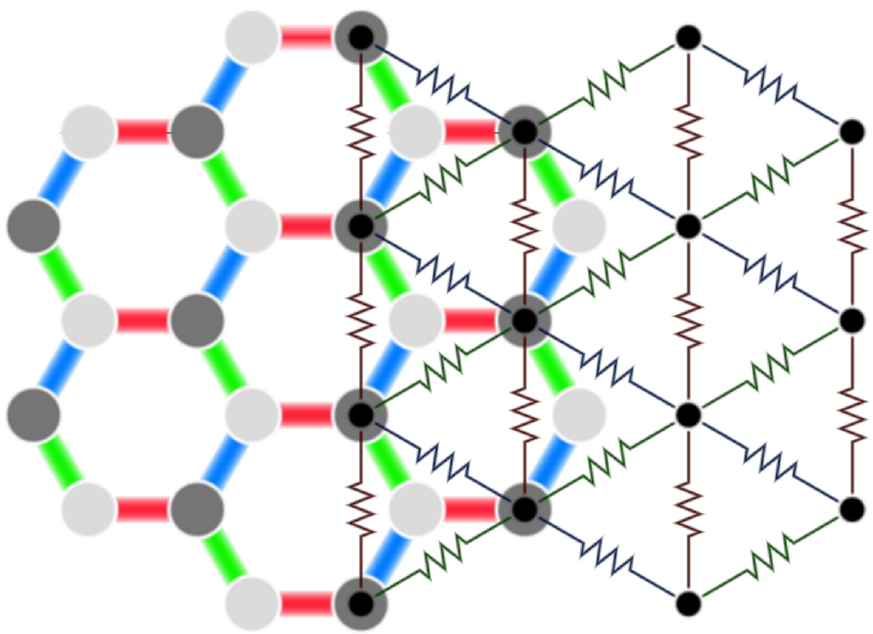

Analytical solution represents spins in terms of four **Majorana fermions**

$$S_i^\gamma = i \; c_i \; c_i^\gamma$$

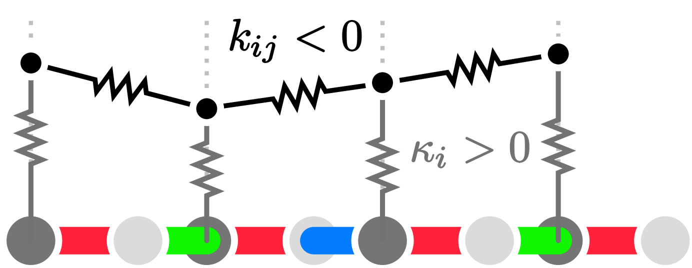


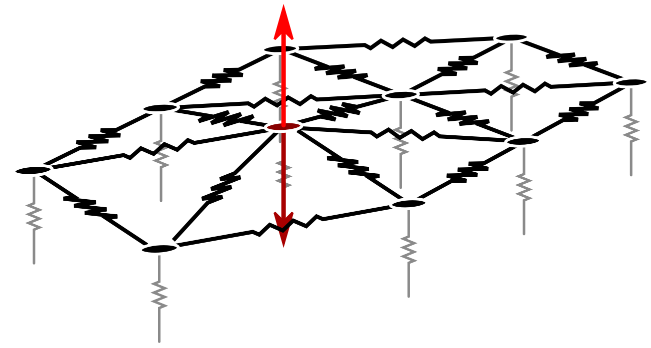
# balls & springs Kitaev model

Majorana fermions
on **honeycomb** lattice

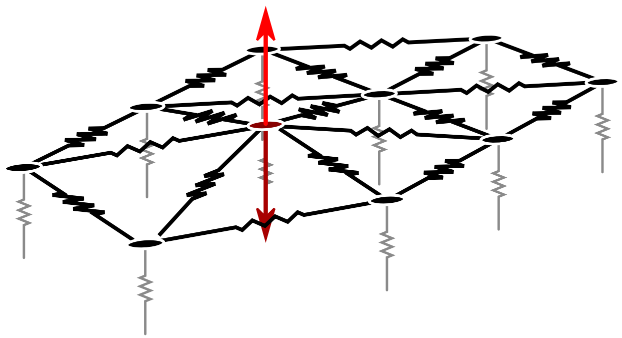


balls & springs
on **triangular** lattice



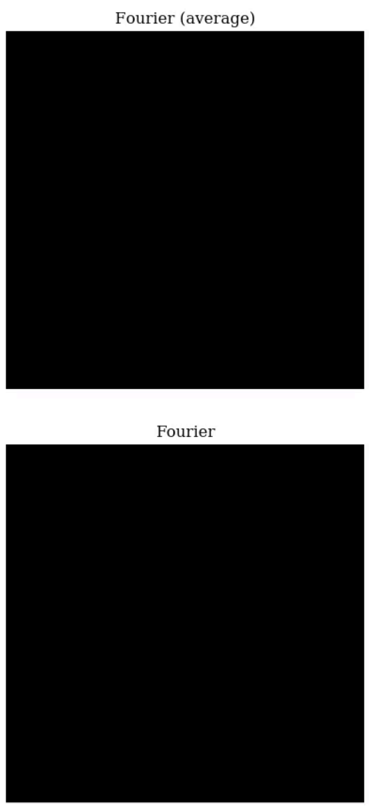



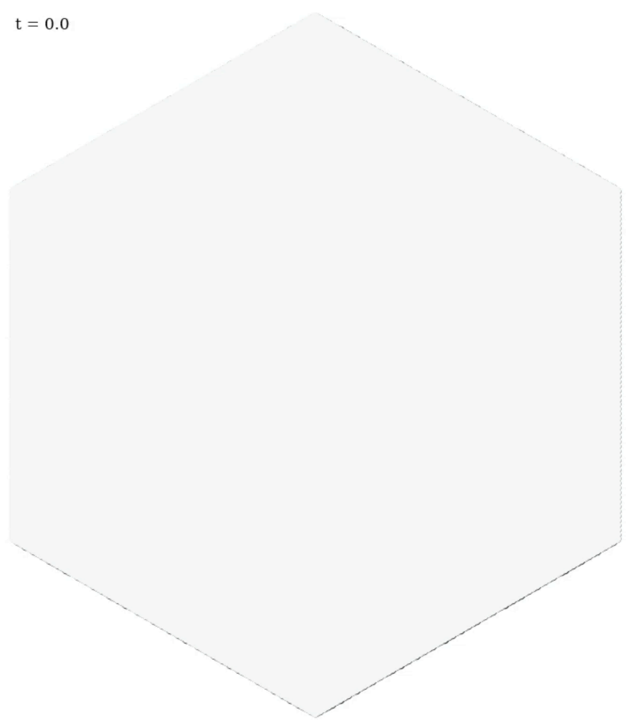
# balls & springs Kitaev model

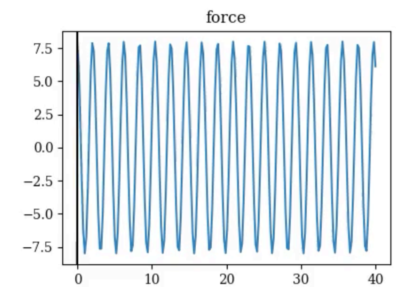


simulate balls & springs model by integrating
**classical equations of motion**

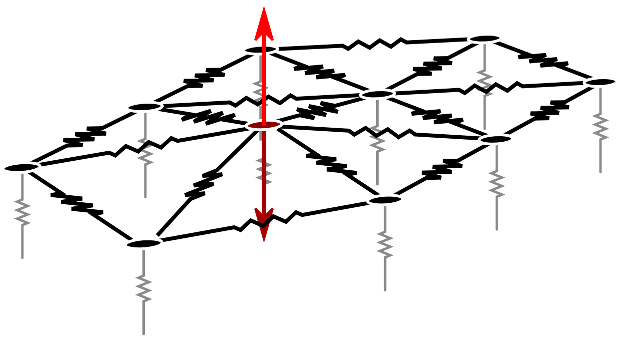
**periodic drive** of a single sites



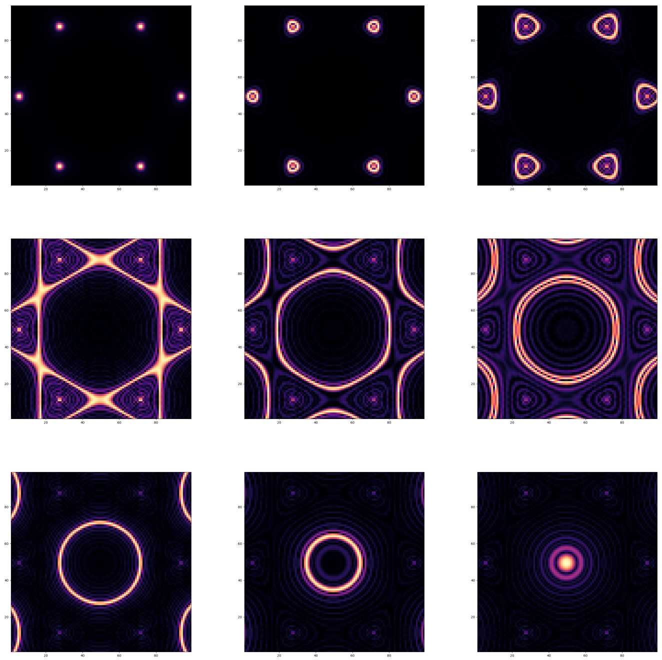



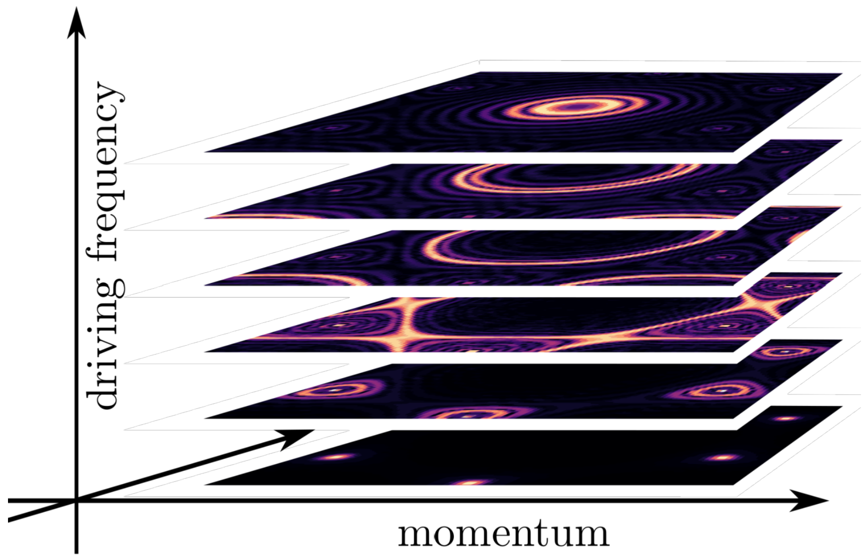
# balls & springs Kitaev model



simulate balls & springs model by integrating **classical equations of motion**

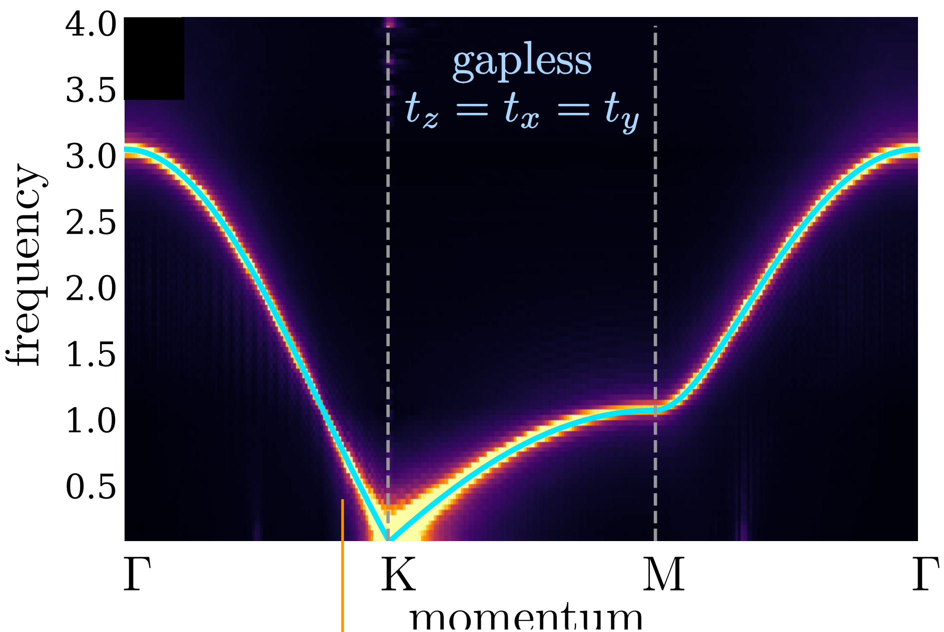
**periodic drive** of a single sites

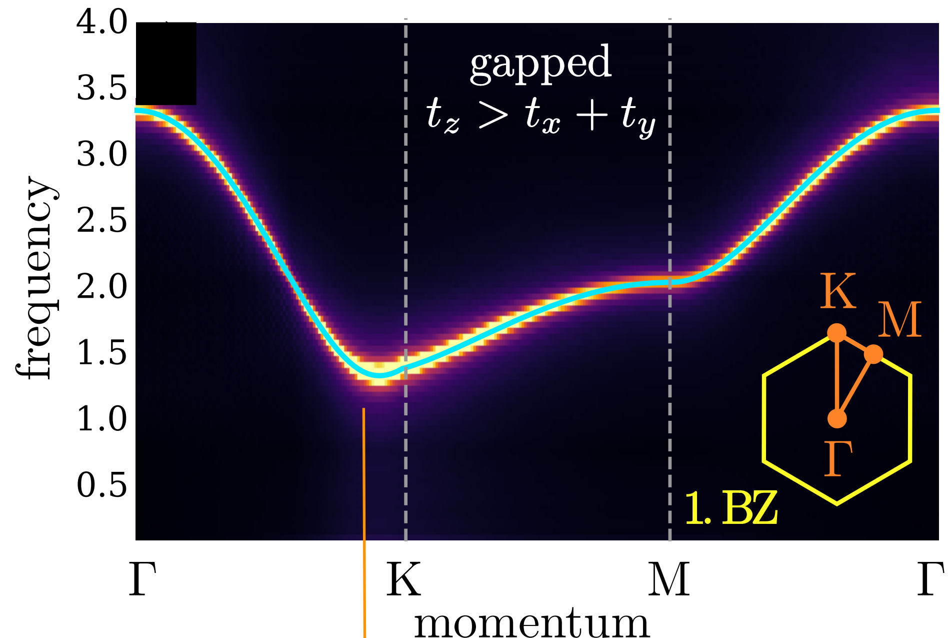




# balls & springs Kitaev model

mechanical energy spectra




Dirac cone with **linear** dispersion
= evidence of many-body physics

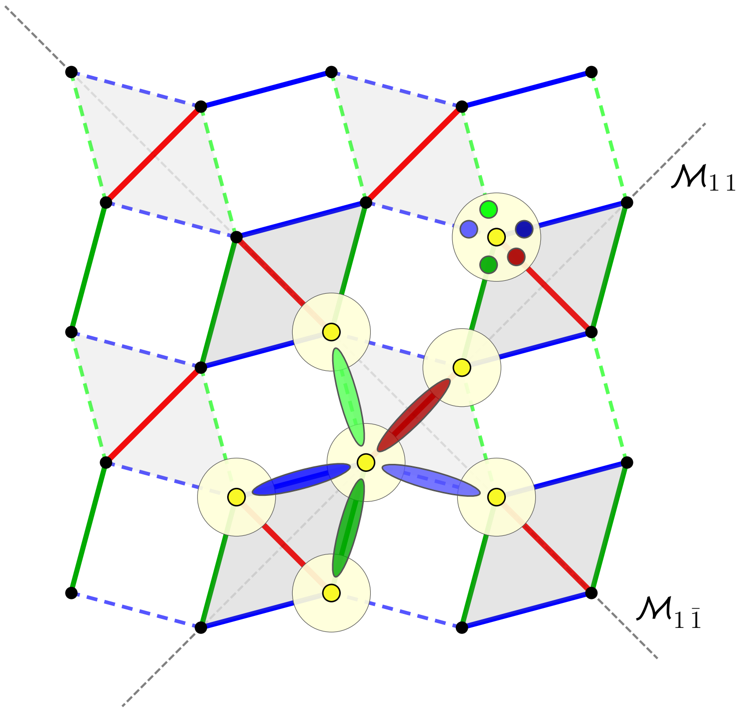
(every single spring has a quadratic dispersion)

energy **gap** = low-frequency rigidity
of a mechanical system (unusual)

(no Goldstone modes here, explicit symmetry breaking)

# mechanical 2nd order TI

## **Quantum spin liquid** with Majorana corner modes



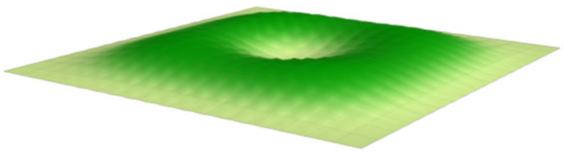

generalized Kitaev model

Shastry-Sutherland lattice

see $SrCu_2(BO_3)_2$

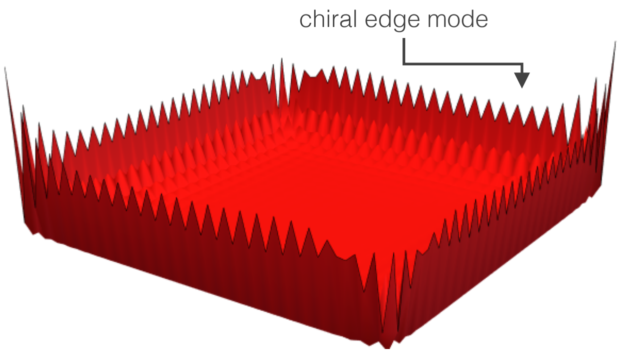
V. Dwivedi, C. Hickey, T. Eschmann & ST, PRB **98**, 054432 (2018)

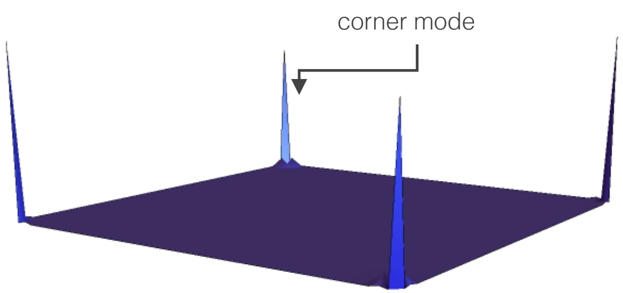
T. Eschmann, V. Dwivedi, H. F. Legg, C. Hickey & ST, PRR **2**, 043159 (2020)



**"trivial" spin liquid**
= band insulator

**chiral spin liquid**
= Chern insulator

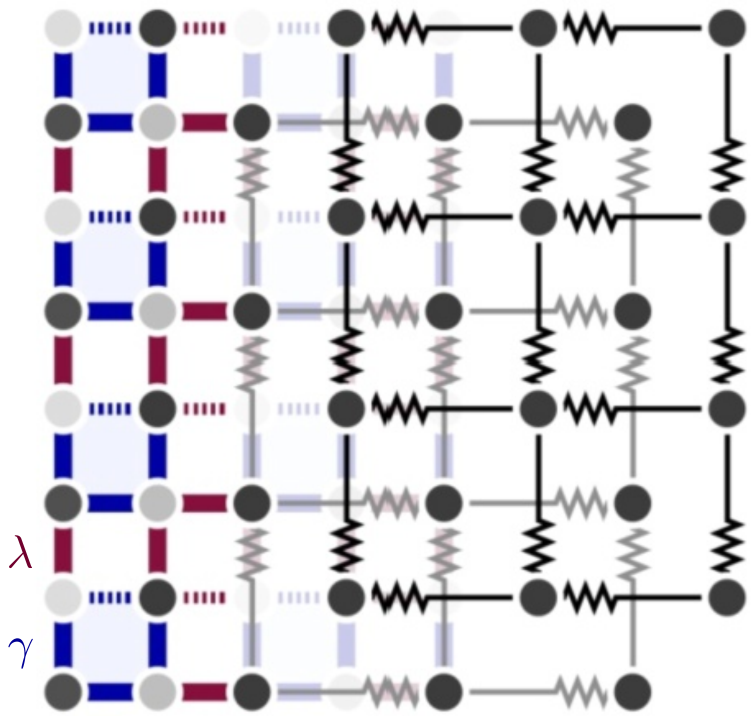


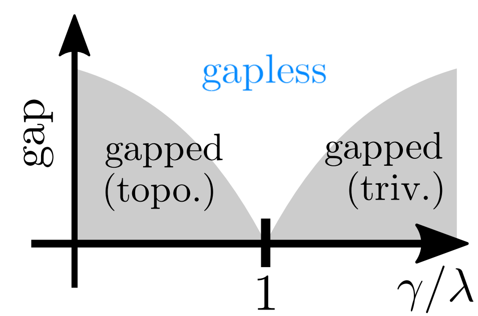



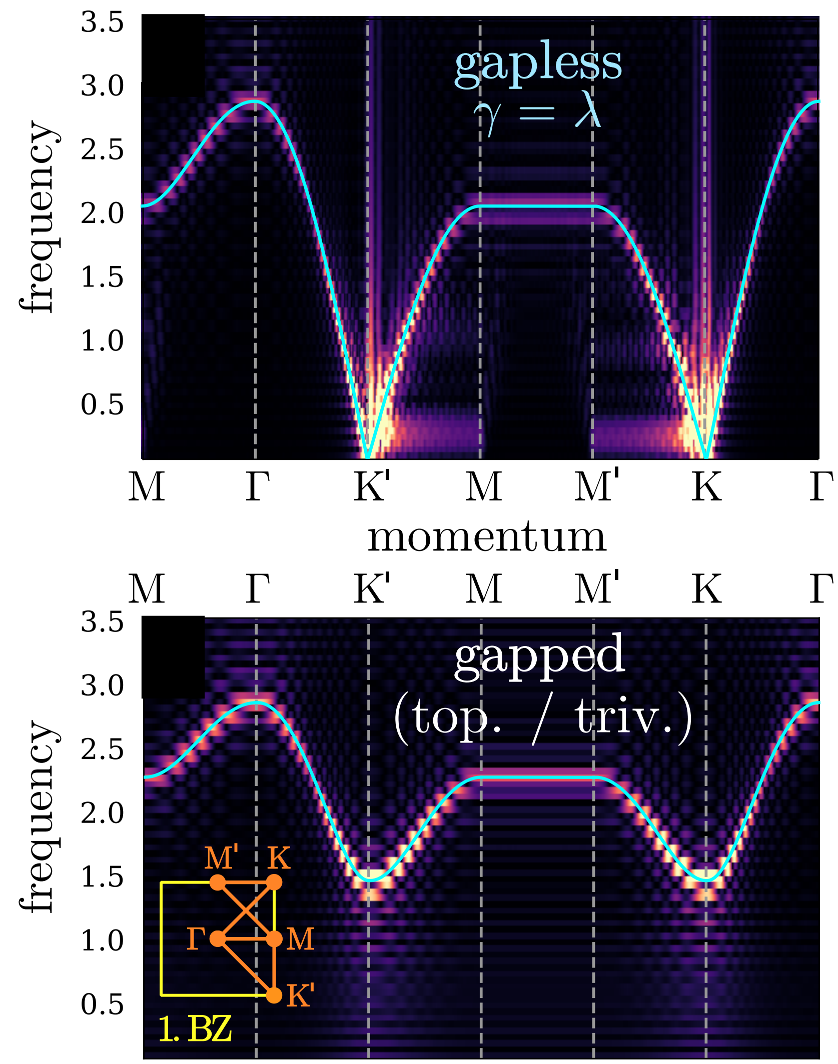

**2nd order spin liquid**
= 2nd order topological insulator

# mechanical 2nd order TI

**Majorana spin liquid** V. Dwivedi, C. Hickey, T. Eschmann & ST, PRB **98**, 054432 (2018)




**phonon insulator** M. Serra-Garcia, …, S. D. Huber, Nature **555**, 342 (2018)

# topological invariants

# topological invariants

This SUSY construction allows to explore
**topological properties of bosonic systems**
by connecting the symplectic bosonic eigenfunctions
with a **fermionic Berry phase** of its SUSY partner.

**fermionic** Berry curvature

$$\mathcal{A} = \langle u_m(\mathbf{k}) | i \nabla_k | u_n(\mathbf{k}) \rangle$$

fermionic eigenstates

via **rigidity matrix**

$$|u_m(\mathbf{k})\rangle = \frac{\mathbf{R}(\mathbf{k})}{\sqrt{|\omega_m(\mathbf{k})|}} |v_m(\mathbf{k})\rangle \equiv \tilde{\mathbf{R}}(\mathbf{k}) |v_m(\mathbf{k})\rangle$$

bosonic eigenstates

**bosonic** Berry curvature

$$\begin{aligned} \mathcal{A}_{\mathrm{SUSY}} &= \langle v_m(\mathbf{k}) | i \tilde{\mathbf{R}}^\dagger \nabla_k \left( \tilde{\mathbf{R}} | v_n(\mathbf{k}) \rangle \right) \\ &= \langle v_m(\mathbf{k}) | i \sigma_2 \left( \nabla_k + \underbrace{\sigma_2 \tilde{\mathbf{R}}^\dagger \nabla_k \tilde{\mathbf{R}}}_{\text{additional covariant derivative}} \right) | v_n(\mathbf{k}) \rangle \end{aligned}$$

# route to classify bosonic systems

This SUSY construction allows to explore
**topological properties of bosonic systems**
by connecting the symplectic bosonic eigenfunctions
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**SUSY** Berry curvature

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**Bosonic systems** that are trivial with regard to conventional definition of Berry phase
can be non-trivial with regard to SUSY Berry phase!

# route to classify bosonic systems

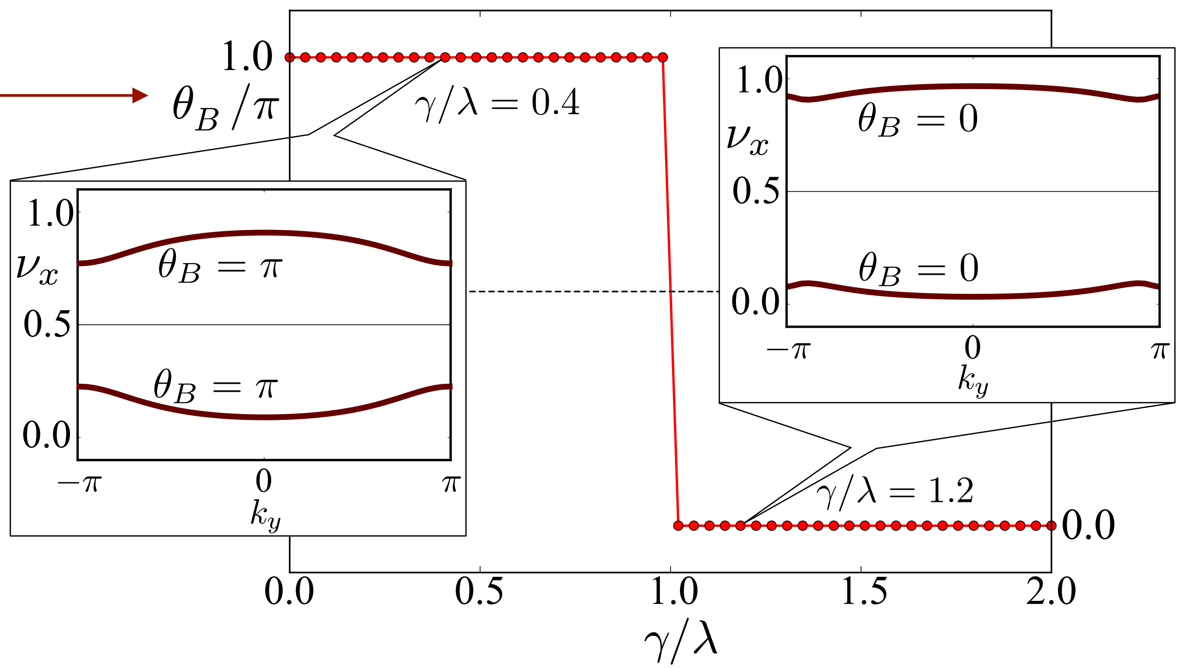
**SUSY** Berry curvature

$$\mathcal{A}_{\mathrm{SUSY}} = \langle v_m(\mathbf{k})| i\tilde{\mathbf{R}}^\dagger \nabla_k \left(\tilde{\mathbf{R}}|v_n(\mathbf{k})\rangle\right)$$

$$= \langle v_m(\mathbf{k})| i\sigma_2 \left(\nabla_k + \sigma_2 \tilde{\mathbf{R}}^\dagger \nabla_k \tilde{\mathbf{R}}\right) |v_n(\mathbf{k})\rangle$$

additional covariant derivative

bosonic Berry curvature




**Bosonic systems** that are trivial with regard to conventional definition of Berry phase can be non-trivial with regard to SUSY Berry phase!

# spin spirals

Dirac magnons

spin liquids

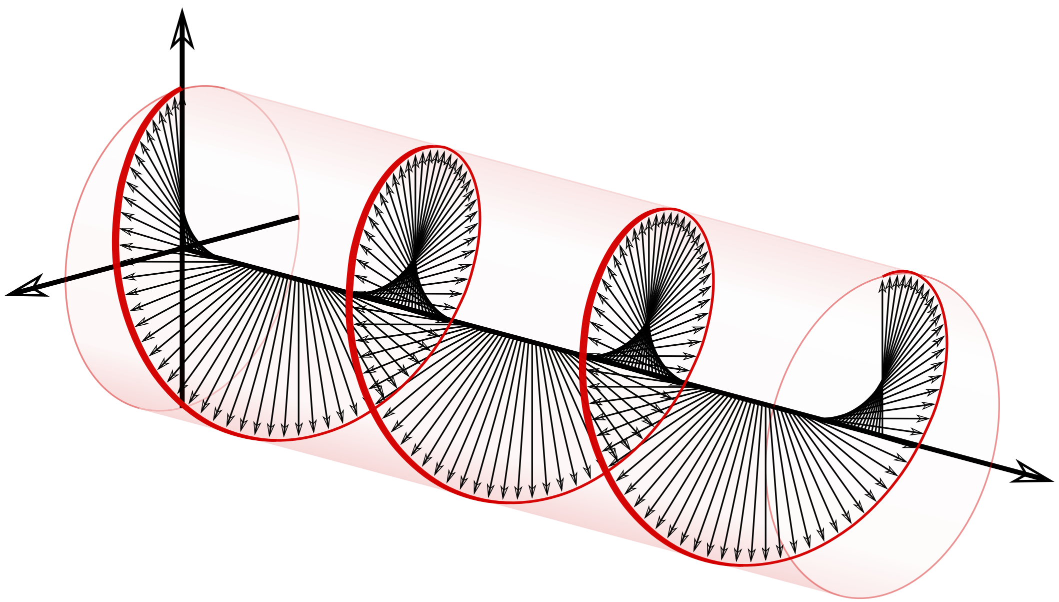
...

# spin spirals

**Coplanar spirals** typically arise in the presence of **competing interactions**

Elementary ingredient for

- multiferroics
- spin textures/multi-q states
  - skyrmion lattices
  - $Z_2$ vortex lattices
- spiral spin liquids



Description in terms of a single wavevector

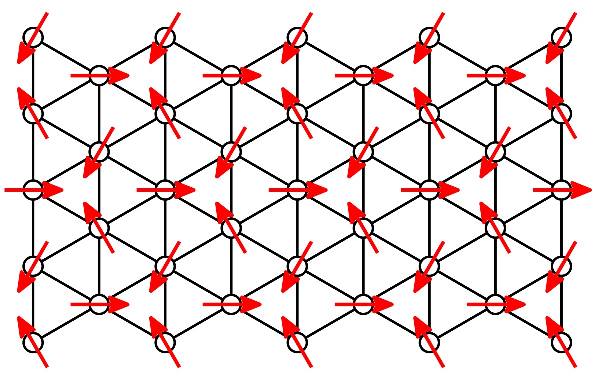
$$\vec{S}(\vec{r}) = \mathrm{Re}\left(\left(\vec{S}_1 + i\vec{S}_2\right) e^{i\vec{q}\vec{r}}\right)$$

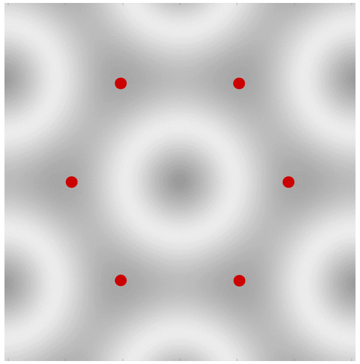
# spin spirals

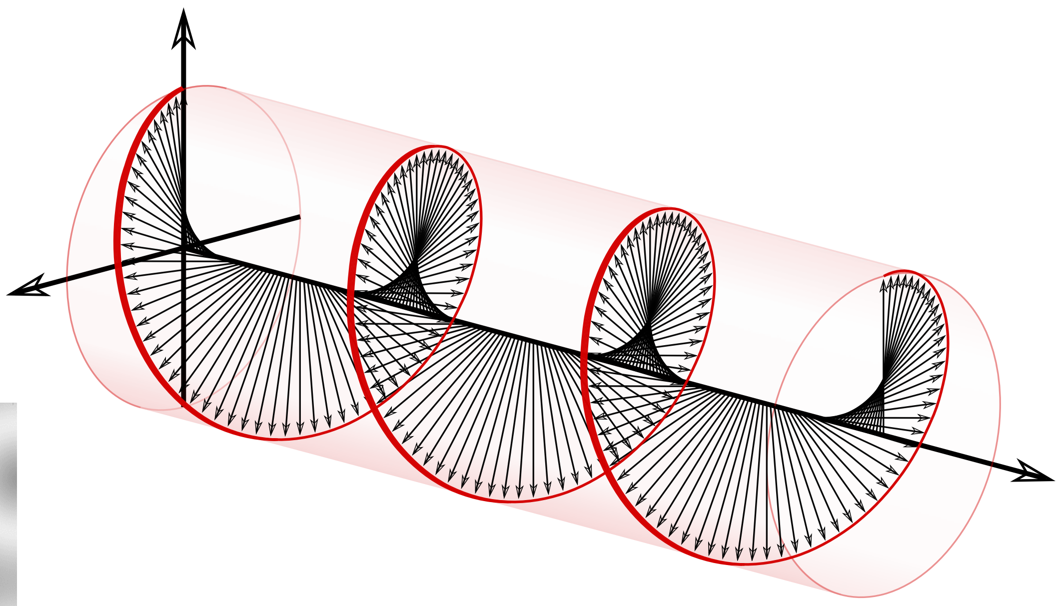
**Coplanar spirals** typically arise in the presence of **competing interactions**

Familiar example

- **120º order** of Heisenberg AFM on triangular lattice





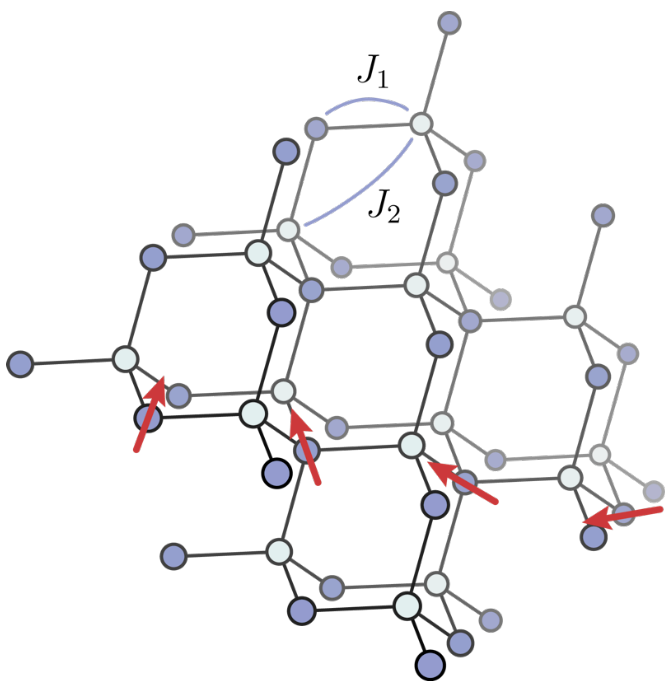


$$\vec{S}(\vec{r}) = \mathrm{Re}\left( \left( \vec{S}_1 + i\vec{S}_2 \right) e^{i\vec{q}\vec{r}} \right)$$

$$\vec{q} = \left( \pm\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$$

# spin spiral materials

## Frustrated diamond lattice antiferromagnets

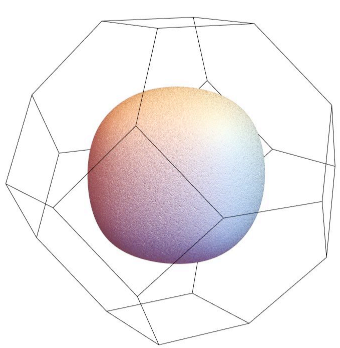



**A-site spinels**

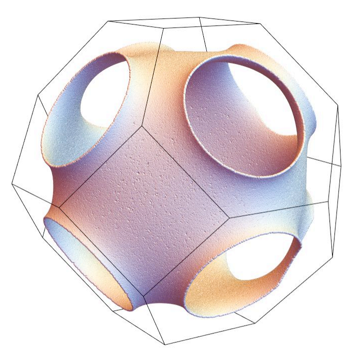
| | |
|---|---|
| **$MnSc_2S_4$** | S=5/2 |
| $FeSc_2S_4$ | S=2 |
| $CoAl_2O_4$ | S=3/2 |
| $NiRh_2O_4$ | S=1 |

$$\mathcal{H} = J_1 \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j + J_2 \sum_{\langle\langle i,j \rangle\rangle} \vec{S}_i \vec{S}_j$$

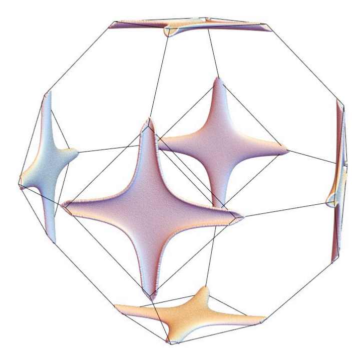
degenerate coplanar spirals form
**spin spiral surfaces** in *k*-space



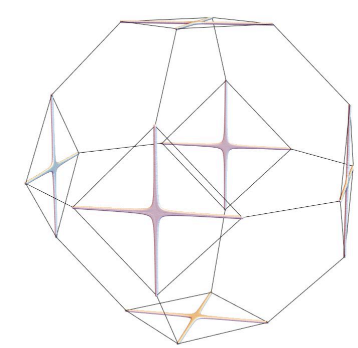
$J_2/J_1 = 0.2$



$J_2/J_1 = 0.4$



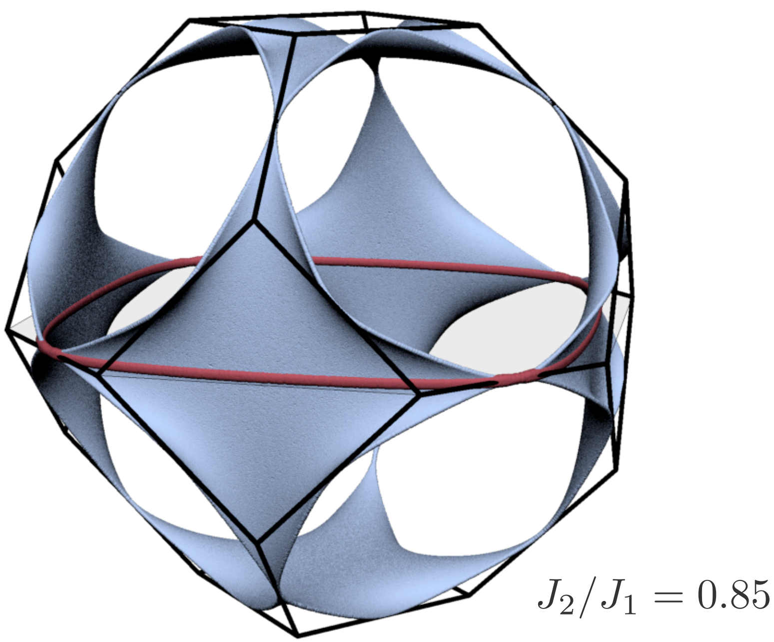
$J_2/J_1 = 3$



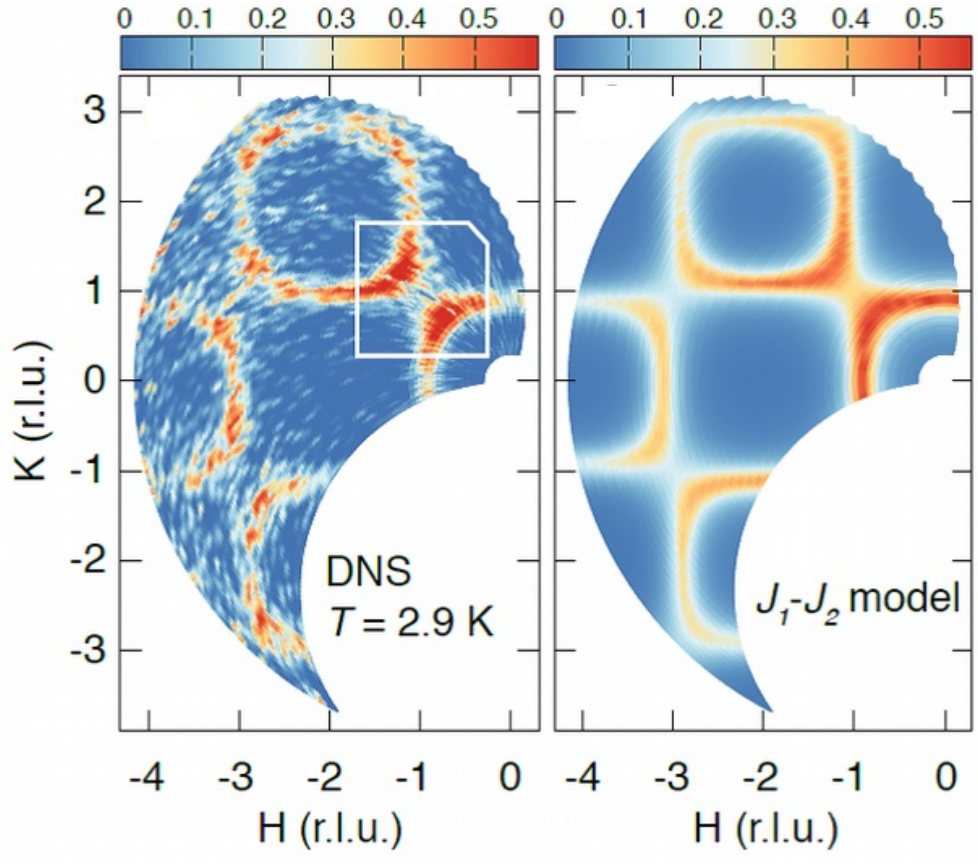
$J_2/J_1 = 100$

# spin spiral materials

Experimental observation of spin spiral surface in inelastic neutron scattering of $MnSc_2S_4$.




Nature Phys. **3**, 487 (2007)

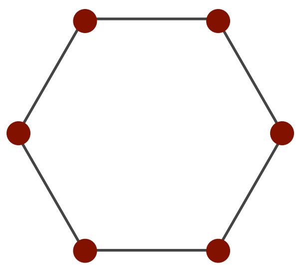



Nature Phys. **13**, 157 (2017)

# spin spiral manifolds

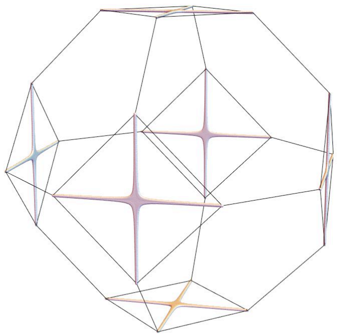
**Spiral manifolds** are extremely reminiscent of **Fermi surfaces**

triangular lattice



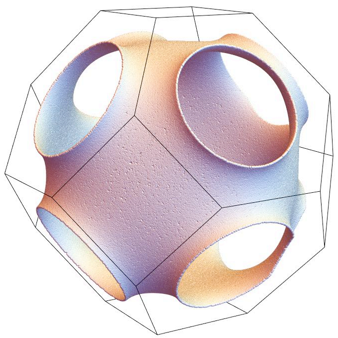
Dirac points

FCC lattice



nodal lines

diamond lattice



Fermi surface

But:

Spiral manifolds describe ground state of classical spin system, while Fermi surfaces are features in the middle of the energy spectrum of an electronic quantum system.

# spin spiral manifolds

Luttinger-Tisza

## **spin spirals** in a nutshell

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij}\, \vec{S}_i \vec{S}_j$$

Fourier transform of spin model

$$= \sum_{\vec{k}} \sum_{A,B} S^A_{\vec{k}}\, \mathbf{M}_{A,B}(\vec{k})\, S^B_{-\vec{k}}$$

diagonalize matrix

$$\mathbf{M}_{A,B}(\vec{k}) = \sum_{\vec{r}^A_{B,j}} J_{\vec{r}_j}\, e^{-i\vec{k}\cdot\vec{r}_j}$$

find **minimal** eigenvalues

$$\lambda_j(\vec{k})$$

## **free fermions** in a nutshell

$$\mathcal{H} = \sum_{\langle i,j \rangle} t_{ij}\, c_i^\dagger c_j$$

Fourier transform of spin model

$$= \sum_{\vec{k}} \sum_{A,B} c^\dagger_{A,\vec{k}}\, \mathbf{H}_{A,B}(\vec{k})\, c_{B,\vec{k}}$$

diagonalize matrix

$$\mathbf{H}_{A,B}(\vec{k}) = \sum_{\vec{r}^A_{B,j}} t_{\vec{r}_j}\, e^{-i\vec{k}\cdot\vec{r}_j}$$

find **zero** eigenvalues

$$\epsilon_j(\vec{k})$$

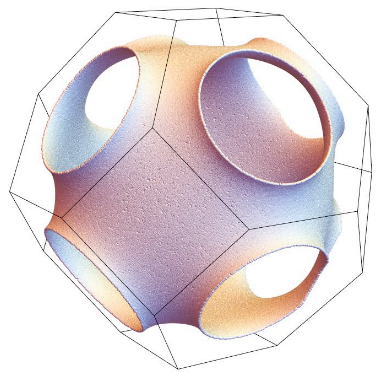
# spin spiral manifolds

## **spin spirals** in a nutshell

$\mathbf{M}_{A,B}(\vec{k})$

with **minimal** eigenvalues

$\lambda_j(\vec{k})$



make ansatz

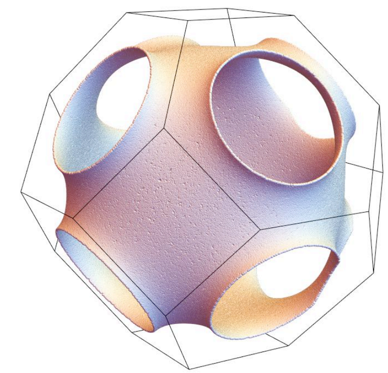
$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

## **free fermions** in a nutshell

$\mathbf{H}_{A,B}(\vec{k})$

with **zero** eigenvalues

$\epsilon_j(\vec{k})$



$\mathbf{H}(\vec{k})^2$ has eigenvalues $\epsilon_j(\vec{k})^2$

zero eigenvalues of $\mathbf{H}(\vec{k})$ are minimal eigenvalues of $\mathbf{H}(\vec{k})^2$

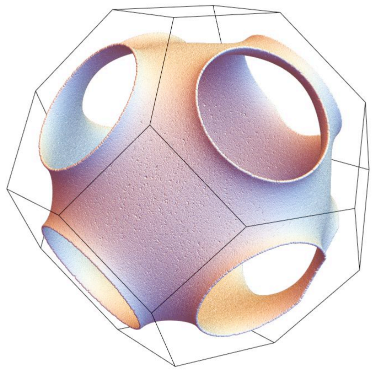
# matrix correspondence

**spin spirals** in a nutshell

$\mathbf{M}_{A,B}(\vec{k})$

with **minimal** eigenvalues

$\lambda_j(\vec{k})$



$\mathbf{H}(\vec{k})^2$

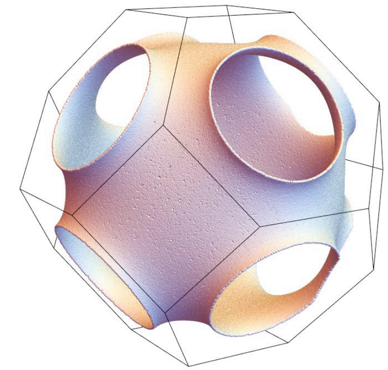
$\sqrt{\mathbf{M}(\vec{k})}$

**free fermions** in a nutshell

$\mathbf{H}_{A,B}(\vec{k})$

with **zero** eigenvalues

$\epsilon_j(\vec{k})$



$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

mapping of a classical to quantum system
(of same spatial dimensionality)
via a 1:1 matrix correspondence

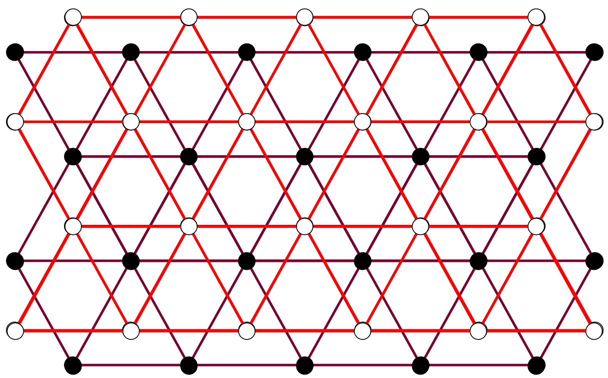
→ reminiscent of "topological mechanics"

# lattice construction

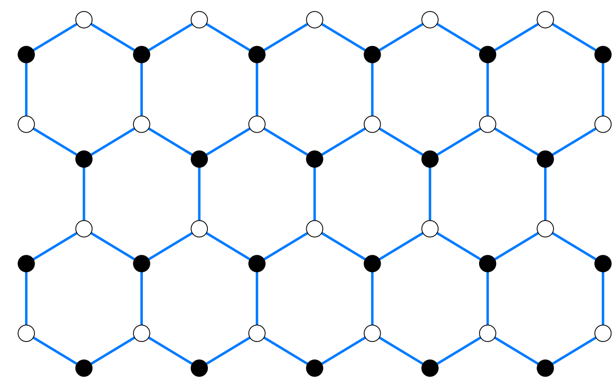
$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

What does **“squaring”** of quantum system mean?

Explicit **lattice construction**.



$$\mathbf{H}(\vec{k})^2$$



coplanar spirals on triangular lattice

$$\vec{q} = \left( \pm\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$$

free fermions on honeycomb lattice

$$\vec{q} = \left( \pm\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}} \right)$$

# lattice construction

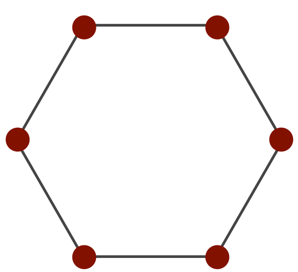
$$\mathbf{M}(\vec{k}) = \mathbf{H}(\vec{k})^2 - E_0 \cdot \mathbf{1}$$

What does **"squaring"** of quantum system mean?

Explicit **lattice construction**.

spin spirals
triangular lattice

120º order

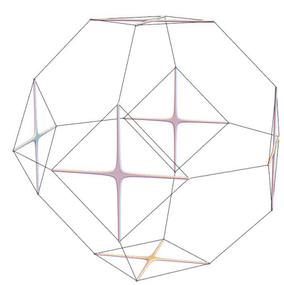


Dirac points

free fermions
honeycomb lattice

spin spirals
FCC lattice

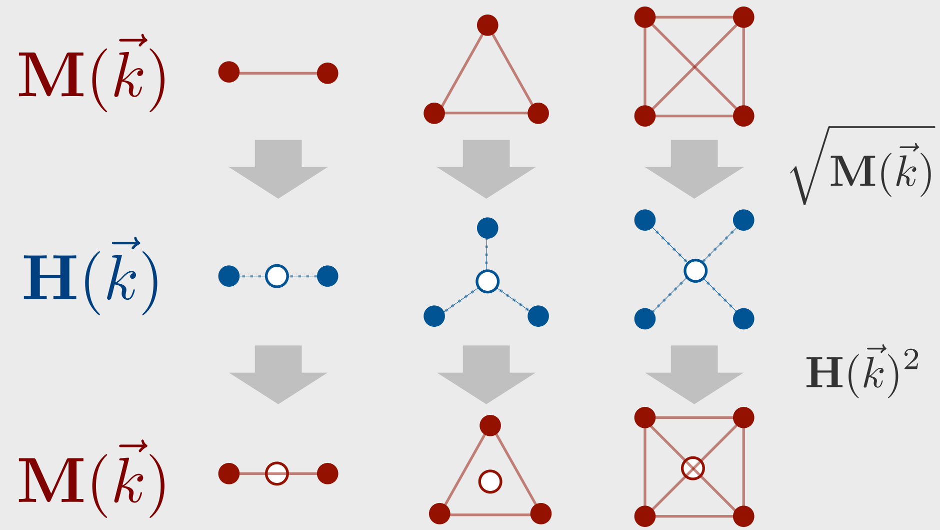
degenerate spirals



nodal lines

free fermions
diamond lattice

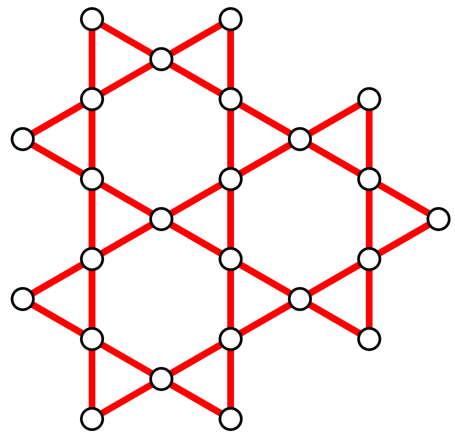
general lattice construction

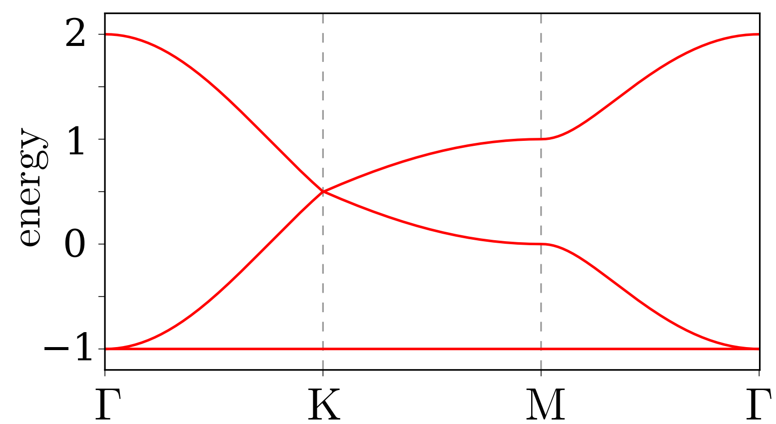


# lattice construction – examples

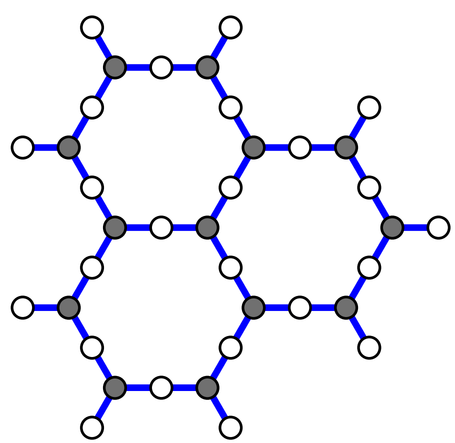
Spectra of the **kagome** and **extended honeycomb** lattice.

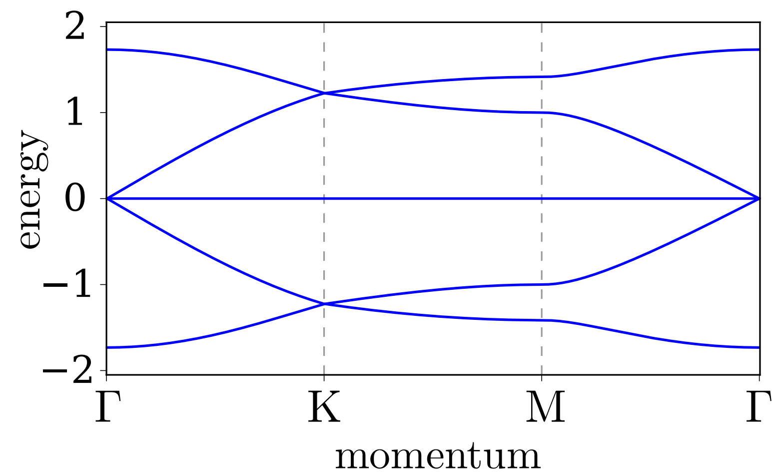
spins






fermions

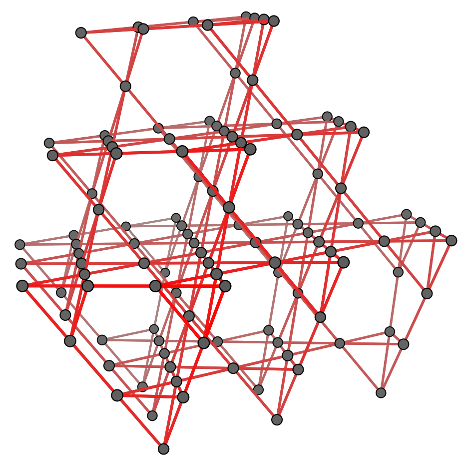


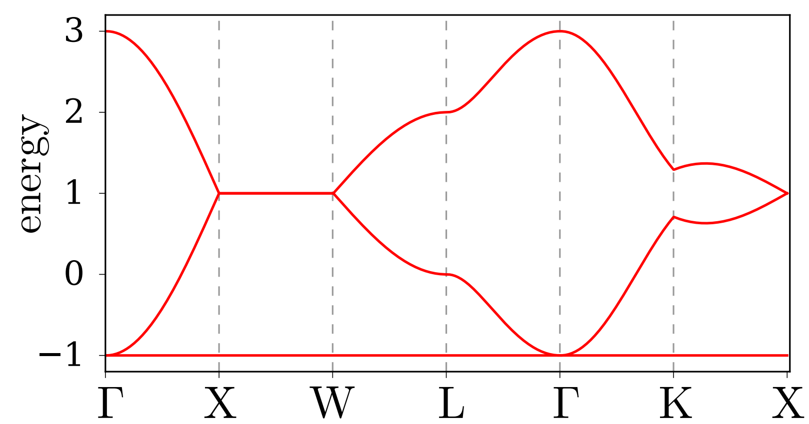


# lattice construction – examples

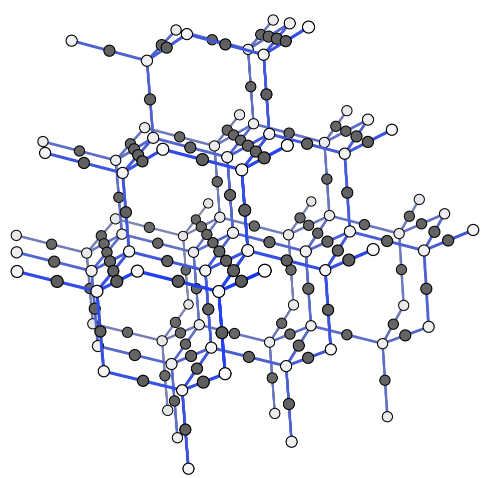
Spectra of the **pyrochlore** and **extended diamond** lattice.

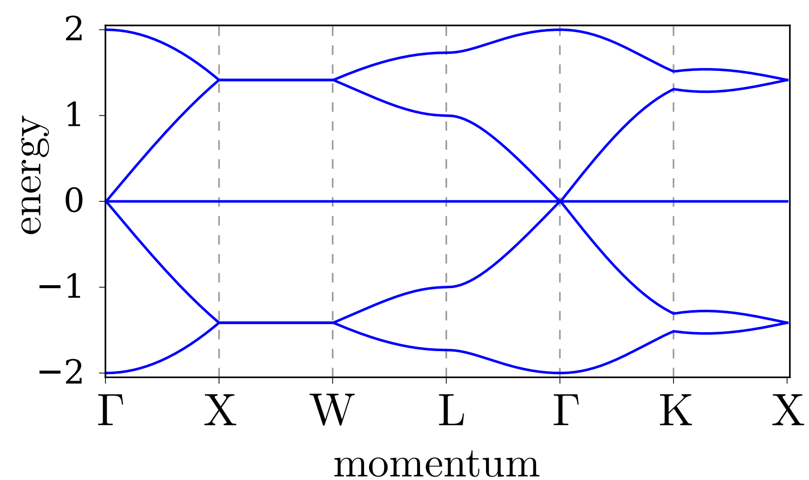
spins



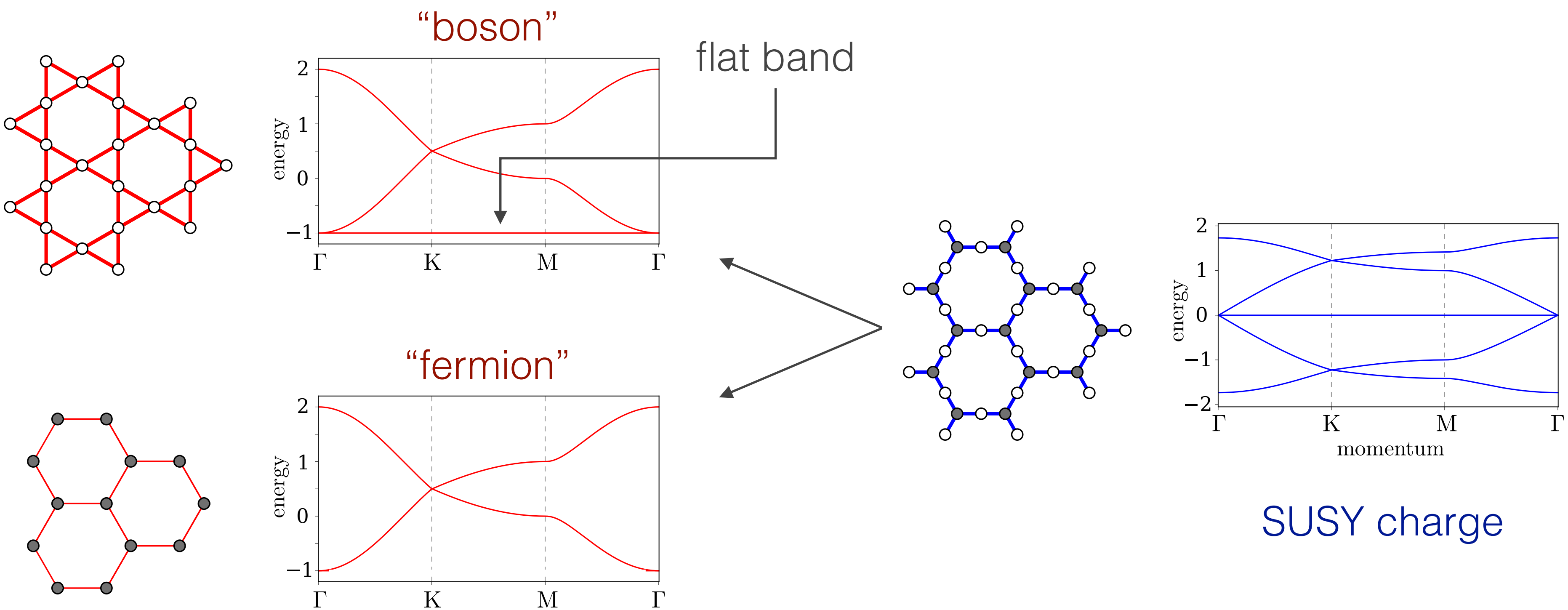



fermions





# SUSY formulation




$$\mathcal{H}^2 = \begin{pmatrix} \mathbf{Q}^\dagger \mathbf{Q} & 0 \\ 0 & \mathbf{Q}\mathbf{Q}^\dagger \end{pmatrix}$$

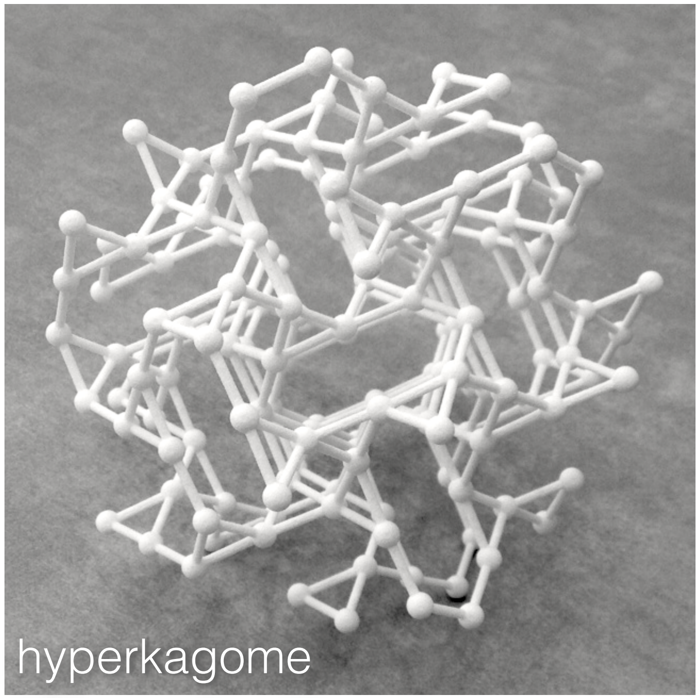
square root

$$\mathcal{H} = \begin{pmatrix} 0 & \mathbf{Q}^\dagger \\ \mathbf{Q} & 0 \end{pmatrix}$$

# quantum spin liquids

## parton constructions

# quantum spin liquids




PRL 99, 137207 (2007) PHYSICAL REVIEW LETTERS week ending 28 SEPTEMBER 2007

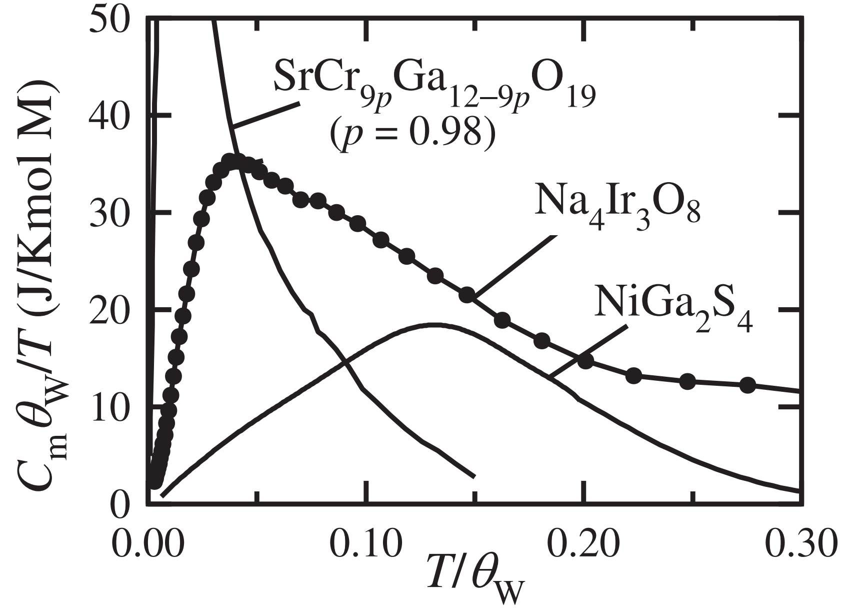
**Spin-Liquid State in the $S = 1/2$ Hyperkagome Antiferromagnet $Na_4Ir_3O_8$**

Yoshihiko Okamoto,[1,*] Minoru Nohara,[2] Hiroko Aruga-Katori,[1] and Hidenori Takagi[1,2]
[1]*RIKEN (The Institute of Physical and Chemical Research), 2-1 Hirosawa, Wako, Saitama 351-0198, Japan*
[2]*Department of Advanced Materials, University of Tokyo and CREST-JST, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8561, Japan*
(Received 19 May 2007; revised manuscript received 24 July 2007; published 27 September 2007)

PRL 101, 197202 (2008) PHYSICAL REVIEW LETTERS week ending 7 NOVEMBER 2008

**Gapless Spin Liquids on the Three-Dimensional Hyperkagome Lattice of $Na_4Ir_3O_8$**

Michael J. Lawler,[1] Arun Paramekanti,[1] Yong Baek Kim,[1] and Leon Balents[2]
[1]*Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada*
[2]*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*
(Received 30 June 2008; published 3 November 2008)

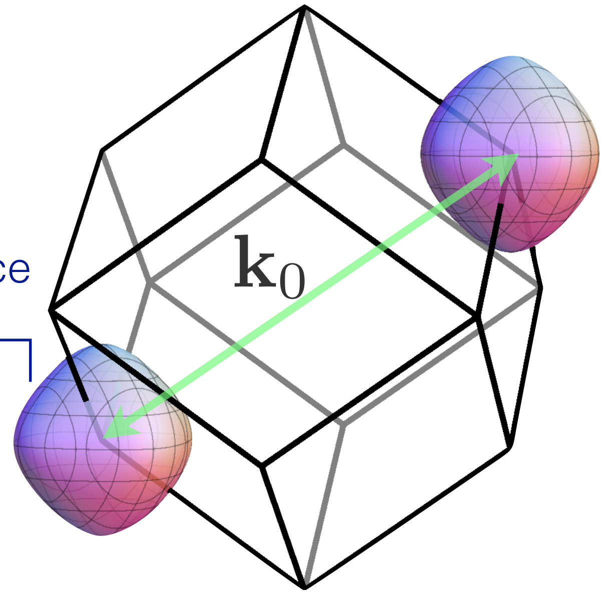


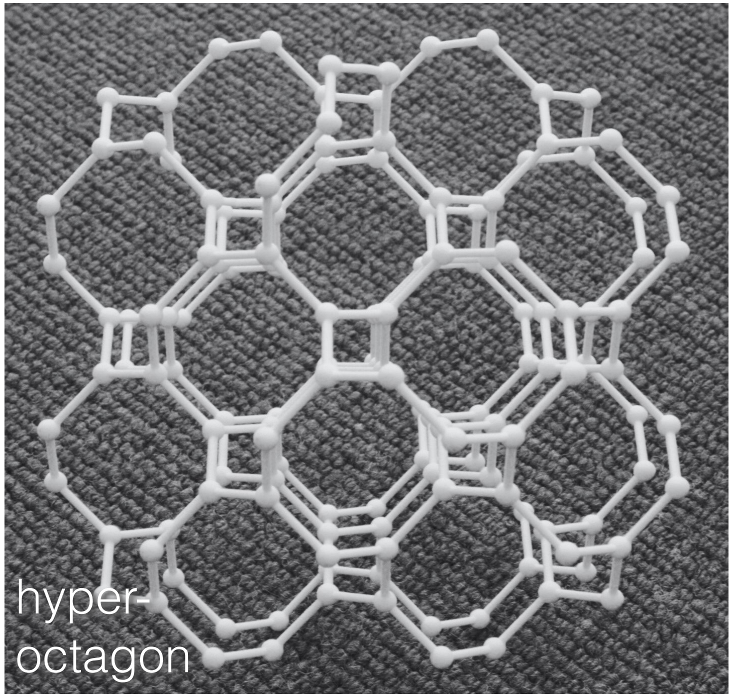

Gapless quantum spin liquid with a **spinon Fermi surface**.

Parton construction with complex fermions coupled to U(1) gauge field

$$C(T) \propto T \ln(1/T)$$

# quantum spin liquids




Gapless quantum spin liquid with a **Majorana Fermi surface**.

Parton construction with Majorana fermions coupled to $Z_2$ gauge field

$$C(T) \propto T$$

PHYSICAL REVIEW B **93**, 085101 (2016)

**Classification of gapless $\mathbb{Z}_2$ spin liquids in three-dimensional Kitaev models**

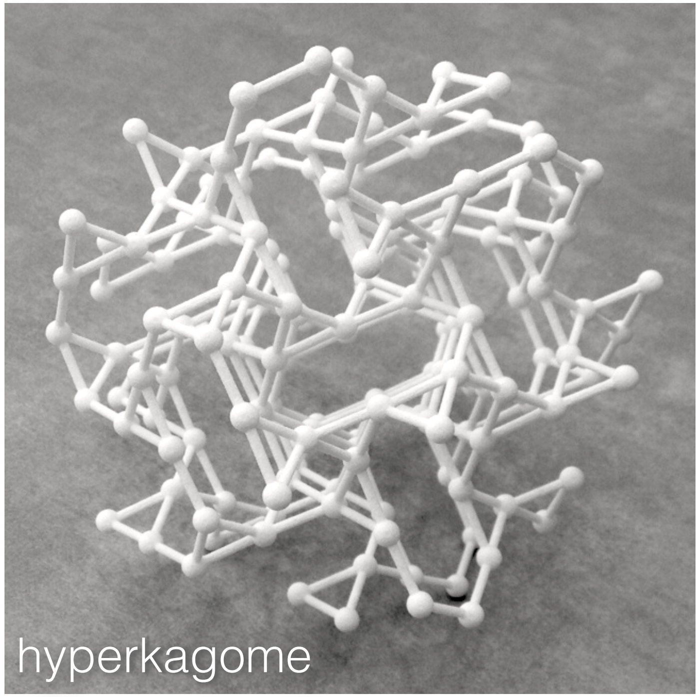
Kevin O'Brien, Maria Hermanns, and Simon Trebst
*Institute for Theoretical Physics, University of Cologne, 50937 Cologne, Germany*
(Received 17 November 2015; published 1 February 2016)

PHYSICAL REVIEW B **89**, 235102 (2014)

**Quantum spin liquid with a Majorana Fermi surface on the three-dimensional hyperoctagon lattice**

M. Hermanns and S. Trebst
*Institute for Theoretical Physics, University of Cologne, 50937 Cologne, Germany*
(Received 10 April 2014; revised manuscript received 15 May 2014; published 2 June 2014)

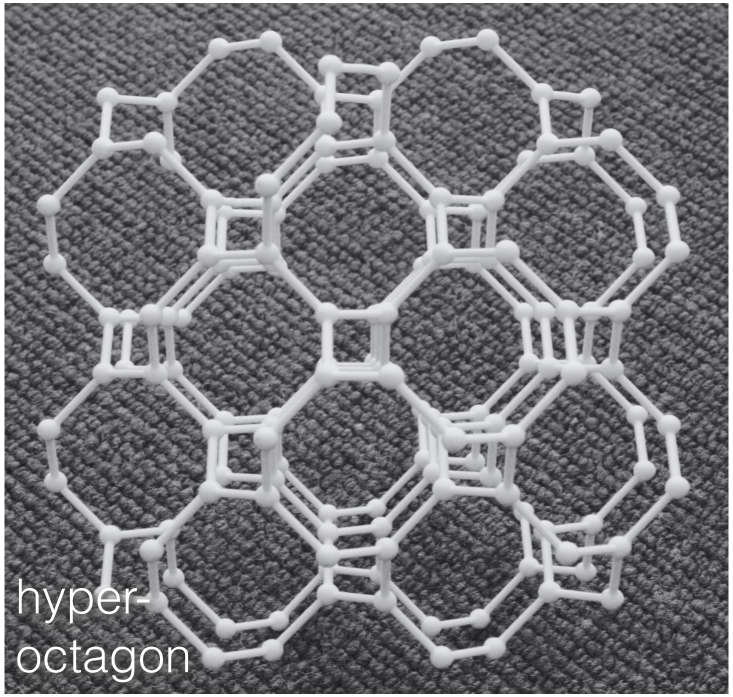
# quantum spin liquids




medial

lattice

premedial

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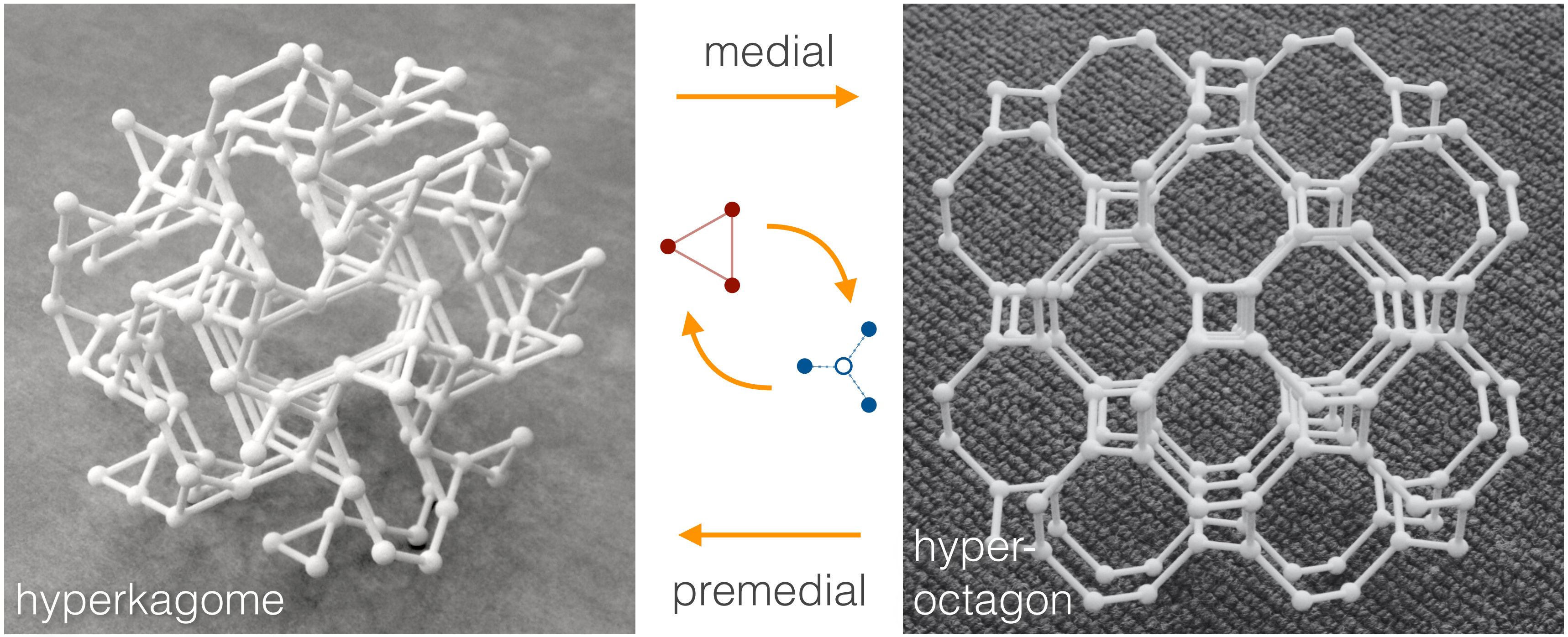
# quantum spin liquids

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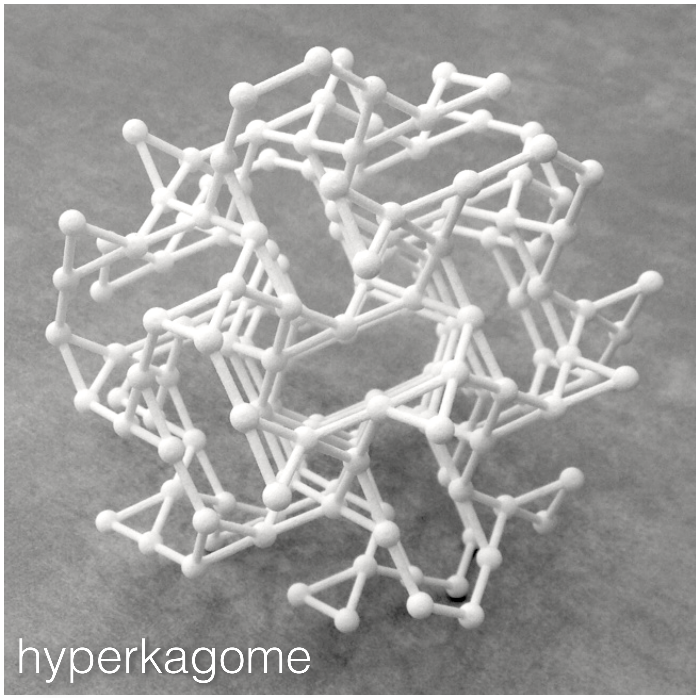
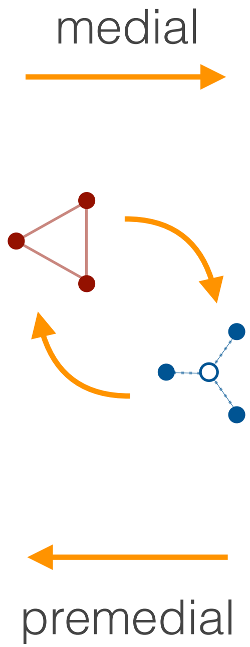
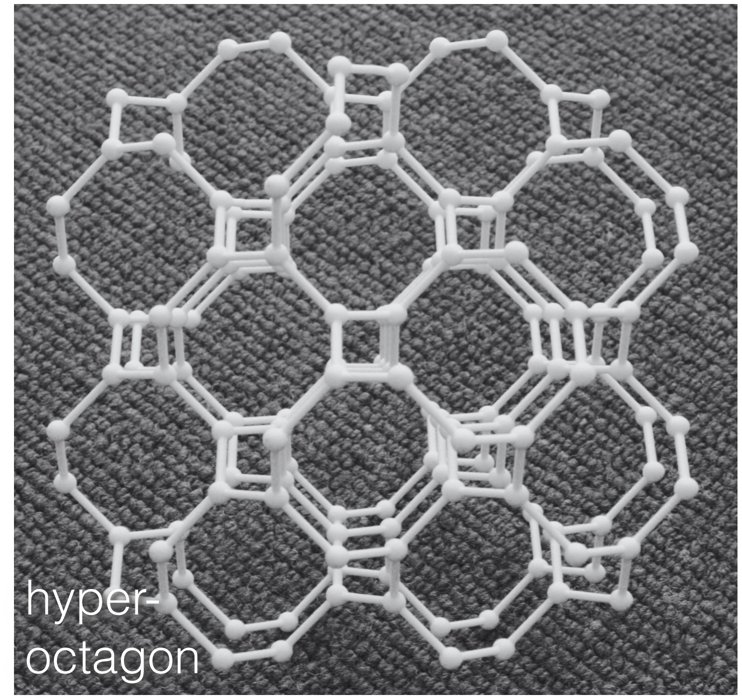
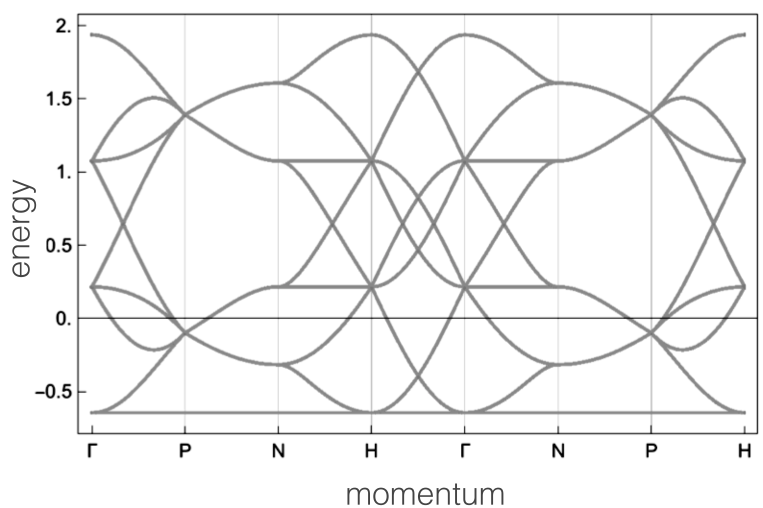
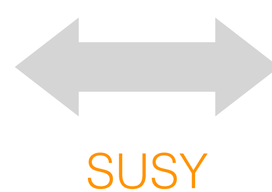
# quantum spin liquids

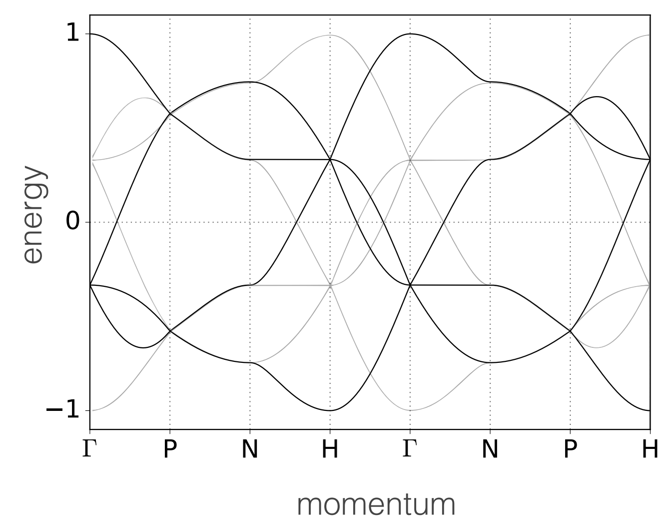


# summary

# Summary

## topological mechanics from supersymmetry

$$H_{\text{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$$

$$\mathcal{H}_{\text{fermion}} = -i\gamma_j^A \, \mathbf{A}_{jk} \, \gamma_k^B + \text{h.c.}$$

**Majorana** system (AB lattice)

$$\mathcal{H}_{\text{boson}} = \hat{p}_i\hat{p}_i + \hat{q}_i(\mathbf{A}^T\mathbf{A})_{ij}\hat{q}_j$$

**mechanical** system (B sublattice)

## novel topological invariant for boson systems

$$\mathcal{A}_{\text{SUSY}} = \langle v_m(\mathbf{k})| i\sigma_2 \left(\nabla_k + \underbrace{\sigma_2\tilde{\mathbf{R}}^\dagger\nabla_k\tilde{\mathbf{R}}}_{\text{additional covariant derivative}}\right)|v_n(\mathbf{k})\rangle$$

**many other SUSY pairs –** spin spirals & fermions, spin liquids, …

Phys. Rev. Research **1**, 032047(R) (2019) and Phys. Rev. B **96**, 085145 (2017), Editors' suggestion.

# Thanks!

@SimonTrebst