

Local Interactions and Non-Abelian Quantum Loop Gases

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Two-dimensional quantum loop gases are elementary examples of topological ground states with Abelian or non-Abelian anyonic excitations. While Abelian loop gases appear as ground states of local, gapped Hamiltonians such as the toric code, we show that gapped non-Abelian loop gases require nonlocal interactions (or nontrivial inner products). Perturbing a local, gapless Hamiltonian with an anticipated “non-Abelian” ground-state wave function immediately drives the system into the Abelian phase, as can be seen by measuring the Hausdorff dimension of loops. Local quantum critical behavior is found in a loop gas in which all equal-time correlations of local operators decay exponentially.

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Non-Abelian topological phases are the focus of considerable excitement as a result of their universality, their novelty, their beautiful mathematical properties, and their potential application to quantum computing [1]. However, the only concrete physical system in which there is any experimental evidence for a topological phase is the two-dimensional electron gas at high magnetic fields, i.e., in the quantum Hall regime. In order to find such phases elsewhere—in transition metal oxides or in ultracold atomic gases—it is important for theory to serve as a guide by identifying conditions which a system must satisfy in order to support a non-Abelian topological phase.

One simple class of models is associated with *quantum loop gases* (QLG), in which an orthonormal basis of the low-energy Hilbert space can be mapped onto configurations of loops [2]. One remarkable feature of topological phases is that the ground-state wave function encodes many of the quasiparticle properties [2–4], which was exploited as far back as Laughlin’s pioneering work on the fractional quantum Hall effect [5]. Therefore, the ground-state wave function plays a central role in the theory. Many of the properties of a QLG can be deduced by mapping the ground state to a classical statistical mechanical model. However, this mapping must be used with care as we show by example in this Letter and might be an issue for other proposed wave functions being considered in the search for non-Abelian topological phases.

The *toric code* [6] is the classic example of a QLG; the associated classical model is critical percolation. However, the toric code Hamiltonian is in a \mathbb{Z}_2 topological phase which is Abelian; i.e., all of its quasiparticles are Abelian anyons. As we discuss below, non-Abelian topological ground states should be associated with critical $O(n)$ loop models with $n > 1$ [2,7]. (The $n = 1$ case is equivalent to critical percolation.) However, local gapped Hamiltonians with these ground states are not known. In this Letter, we show that such Hamiltonians do not exist. Hamiltonians with the desired ground states have been

constructed [7], but these models are gapless and describe critical points, not stable phases. It was conjectured that by perturbing such a critical model, one could drive the system into a gapped non-Abelian topological phase. In this Letter, we analyze instabilities of such critical models and show that perturbations fundamentally alter the nature of the ground state. For instance, one of the simplest relevant perturbations drives the system into the *Abelian* \mathbb{Z}_2 topological phase. We give a general proof that no local perturbation can give rise to a gapped non-Abelian quantum loop gas. Thus, non-Abelian topological phases require more intricate Hamiltonians.

Loop gas wave function.—QLGs can be realized in lattice models whose low-energy Hilbert space is spanned by states of the form $|\mathcal{L}\rangle$, where the multiloop \mathcal{L} is an arbitrary collection of nonintersecting loops. Here we consider the “ d -isotopy” wave function which generalizes the ground state (GS) of the toric code [7]:

$$|\Psi_0^{(d)}\rangle \propto \sum_{\{\mathcal{L}\}} d^{\ell(\mathcal{L})} |\mathcal{L}\rangle. \quad (1)$$

In this expression, $\ell(\mathcal{L})$ is the number of loops in the multiloop \mathcal{L} . The loops have a “fugacity” d . In the toric code, $d = 1$. This wave function has two key features which hint at its topological nature: On nontrivial surfaces, there is a (degenerate) space of such wave functions corresponding, for instance, to different winding numbers. Second, the wave function amplitude is independent of the lengths of the loops. However, the latter is neither necessary nor sufficient for a gapped topological phase [7].

In a topological phase, the parameter d also determines the topological properties of excitations, as discussed in Ref. [2]. Excitations can be studied by considering the GS on a surface with punctures: each puncture can be viewed as a localized excitation which is specified by the boundary conditions at the puncture. If loops terminate at the boundary, the excitation is nontrivial (this is sufficient but not necessary). The amplitude to create a pair of such quasi-

particles and annihilate them later is a measure of the number of states of such a quasiparticle called the *quantum dimension*. If there are N quasiparticles with quantum dimension D , there will be $\sim D^N$ degenerate states. For $D > 1$ and not an integer, there will be a large degeneracy which cannot be ascribed locally to the quasiparticles, so they will have non-Abelian braiding statistics. The universal properties of a topological phase are independent of any coordinate system; in particular, space and time can be interchanged. The quantum dimension can thus be determined directly from the GS wave function. For a topological phase with GS (1), the quantum dimension of the fundamental quasiparticle is equivalent to the fugacity d . A loop can be viewed as the projection onto a fixed time slice of a pair creation and annihilation process. For $d = \sqrt{2}$, the fundamental quasiparticle has the same quantum dimension as the σ -field in the Ising topological quantum field theory or the spin-1/2 field in $SU(2)_2$. For arbitrary d , the loop gas GS (1) has no relation to any known topological phase.

The fugacity d also determines the correlation functions of the associated classical statistical model, which is the $O(n)$ loop model with $n = d^2$:

$$Z_{O(n)}(x) \equiv \sum_{\alpha} \left(\frac{x}{n}\right)^{b(\mathcal{L})} n^{\ell(\mathcal{L})}. \quad (2)$$

Here, $b(\mathcal{L})$ is the total length of the multiloop \mathcal{L} . For integer n , the right-hand-side is the expansion in powers of x of the Boltzmann weight $e^{-\beta H} = \prod_{(i,j)} (1 + x \hat{S}_i \cdot \hat{S}_j)$ for a model of classical interacting spins with $O(n)$ symmetry. For $x = n$, $Z_{O(n)}(x) = \langle \Psi_0^{(d)} | \Psi_0^{(d)} \rangle$, so the equal-time ground-state correlations contained in the QLG's $|\Psi_0^{(d)}\rangle$ can be obtained from the known correlations of $Z_{O(n)}(x)$. For $n \leq 2$ and $x \geq x_c = n/\sqrt{2} + \sqrt{2-n}$, this model is in its critical low-temperature phase [8]. It is necessary for the loops to be critical in order for $|\Psi_0^{(d)}\rangle$ to be the ground state of a topological phase. Only then will the endpoints of a broken loop be deconfined. Therefore, for $|\Psi_0^{(d)}\rangle$ to be in a topological phase it is required that $d \leq \sqrt{2}$. It is equally important that correlation functions of local operators decay exponentially in time since a topological phase requires a gap to excited states.

Hamiltonian.—The wave function (1) is the ground state of the following spin-1/2 Hamiltonian, where the spins live on the edges of a honeycomb lattice:

$$\begin{aligned} \mathcal{H}_0^{(d)} = & J \sum_v \left(1 + \prod_{i \in e(v)} \sigma_i^z \right) \\ & + \frac{K}{2} \sum_p \left[\frac{2}{1+d^2} (d\mathbb{1} - F_p) \mathbb{P}_p^0 (d\mathbb{1} - F_p) \right. \\ & \left. + (\mathbb{1} - F_p) \mathbb{P}_p^1 \right]. \end{aligned} \quad (3)$$

The dominant first term ($J \gg K$) enforces the constraint that the low-energy Hilbert space is spanned by configurations with an even number of $\sigma^z = 1$ spins on edges $e(v)$ around all vertices v . The loops are formed by the $\sigma^z = 1$ edges. $F_p \equiv \prod_{i \in p} \sigma_i^x$ flips the six spins around a hexagonal plaquette p and \mathbb{P}_p^m are projectors onto configurations with m loop segments around a given hexagon. (Here we define \mathbb{P}_p^1 so that it annihilates states with a single loop forming the plaquette boundary as in Fig. 1(a); only configurations of the type shown in Fig. 1(b) are not annihilated by \mathbb{P}_p^1 .) Notice that the Hamiltonian (3) includes processes shown in Figs. 1(a) and 1(b), but *not* those in Figs. 1(c) and 1(d). The most salient property of this Hamiltonian is that it is a sum of projectors which simultaneously annihilate the wave function (1) which is, thus, the GS. The low-energy spectrum of the Hamiltonian (3) is gapless for $d \leq \sqrt{2}$: quantum dynamics is inefficient at mixing states with different loop sizes, thereby resulting in gapless modes.

For $d = 1$ a gap to all excited states can be opened *without* changing the GS $|\Psi_0^{(1)}\rangle$ by augmenting (3) with the “loop surgery” terms in Figs. 1(c) and 1(d)

$$\mathcal{H}_{\text{TC}} = \mathcal{H}_0^{(d=1)} + \frac{K}{2} \sum_p [(\mathbb{1} - F_p)(\mathbb{P}_p^2 + \mathbb{P}_p^3)]. \quad (4)$$

This is the honeycomb lattice version of the toric code [6]; hence, its GS is in a \mathbb{Z}_2 topological phase. Clearly, the existence of an energy gap depends on the Hamiltonian.

Augmenting the Hamiltonian (3) by the surgery terms of Figs. 1(c) and 1(d) for $d \neq 1$ is not straightforward, as these terms generically do not conserve the number of loops and hence skew the loop amplitudes in Eq. (1). To preserve the correct amplitudes, one may propose

$$\mathcal{H}_1 = \mathcal{H}_0^{(d)} + \frac{K}{2} \sum_p [(\mathbb{1} - d^{\Delta l} F_p)(\mathbb{P}_p^2 + \mathbb{P}_p^3)], \quad (5)$$

where Δl is the change in the number of loops when plaquette p is flipped. Although our Monte Carlo (MC) simulations show that this opens a gap $\Delta \approx 2K$, the problem is that Δl is a *nonlocal* operator: its eigenvalue depends on how loop segments are connected elsewhere.

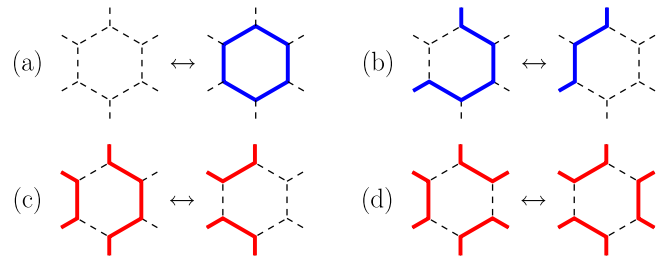


FIG. 1 (color online). Four types of moves enacted by the plaquette flip term F_p . Term (a) enforces the loop fugacity d , (b) enforces isotopy invariance. Collectively they are known as “ d -isotopy” moves. Terms (c) and (d) are the surgery terms.

Local perturbations.—The question that we now want to address is, Can the non-Abelian state $|\Psi_0^{(d)}\rangle$ with $d \neq 1$ be the GS of a *local*, gapped Hamiltonian or is this specific to the Abelian case $d = 1$? One proposal is adding a *local* $(k + 1)$ -strand surgery term (not shown) given by a lattice version of the Jones-Wenzl projector [2,9] which also annihilates the GS $|\Psi_0^{(d)}\rangle$ if $d = 2 \cos \pi / (k + 2)$. However, most of the states in this sequence $d = 1, \sqrt{2}, (1 + \sqrt{5})/2, \dots$ ($k = 1, 2, 3, \dots$) occur at values of d for which the loops are not critical [8]. The case $d = 1$ ($k = 1$) is the surgery term discussed above, and therefore we focus on $d = \sqrt{2}$ ($k = 2$). It can be argued that this term will *not* open a gap, because there is a vanishing probability for three *long* strands to meet [10].

However, it is possible for two strands to meet, so we consider the effect of the *local* 2-strand surgery term:

$$\mathcal{H}_2 = \mathcal{H}_0^{(d=\sqrt{2})} + \frac{\epsilon}{2} \sum_p [(1 - F_p)(\mathbb{P}_p^2 + \mathbb{P}_p^3)]. \quad (6)$$

The state $|\Psi_0^{(\sqrt{2})}\rangle$ is no longer the GS of Hamiltonian (6), because the second term is the surgery term for $d = 1$. Nevertheless, one may hope that for $\epsilon \ll K$, a weakly-perturbed, *gapped* GS will emerge which might still be in the desired non-Abelian topological phase. We simulate the model (6) on a torus of $L \times L$ plaquettes with tilt angle 60° using a variant of the path-integral ground-state (PIGS) algorithm [11]. We measure ground-state expectation values $\langle 0|A|0\rangle$ by sampling the continuous-time path-integral representation of

$$\langle 0|A|0\rangle = \lim_{\beta \rightarrow \infty} \frac{\langle \Psi_0^{(\sqrt{2})} | e^{-\beta H/2} A e^{-\beta H/2} | \Psi_0^{(\sqrt{2})} \rangle}{\langle \Psi_0^{(\sqrt{2})} | e^{-\beta H} | \Psi_0^{(\sqrt{2})} \rangle} \quad (7)$$

using local updates. We choose the projection time β large enough to project out all the states but the ground state. We find that this perturbation immediately opens a gap, since the time dependent local correlation functions $C(\tau) = \langle 0 | \sigma_i^z e^{-\tau(H-E_0)} \sigma_i^z | 0 \rangle$, decrease exponentially as $C(\tau) \sim A(\tau) \exp(-\tau\Delta)$, where the prefactor $A(\tau) = (1 - \exp(-\tau a k^2)) / a \tau$ is determined by a quadratic dispersion $E(k) = \Delta + a k^2$ above the gap Δ . The two-strand surgery term is expected to be a relevant perturbation, and an infinitesimal ϵ should be sufficient to open a gap. Shown in Fig. 2, a substantial gap exists already for $\epsilon = 0.01$.

We now need to determine whether this gapped ground state is in an Abelian or non-Abelian phase. While these two phases can be distinguished by the fugacity of large loops, this is hard to measure. A simpler, more direct measurement differentiating the two phases is the Hausdorff dimension d_H of the longest loop. The Coulomb gas solution [8] of the $O(n)$ loop model allows to calculate d_H exactly [12] with $d_H = 7/4$ for $|\Psi_0^{(1)}\rangle$ and $d_H = 3/2$ for $|\Psi_0^{(\sqrt{2})}\rangle$. We first measure these Hausdorff dimensions by sampling multiloops \mathcal{L} in a MC simulation

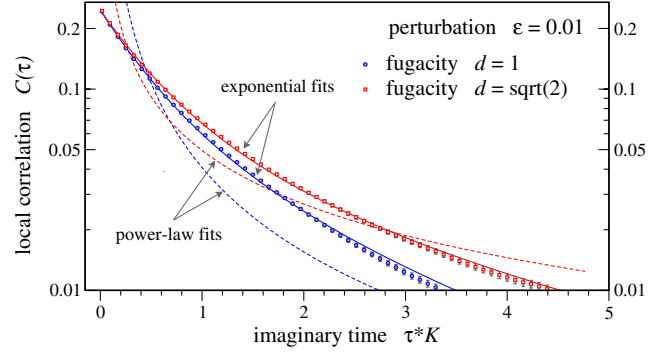


FIG. 2 (color online). The local imaginary time correlation function $C(\tau)$ for the loop gas with fugacity d and surgery term $\epsilon = 0.01$. Data shown are for system size $L = 16$ and inverse temperature $\beta = 16$.

with weights $d^{2\ell(L)}$. The length distributions of the longest loops is shown in Fig. 3 for fugacities $d = 1$ and $d = \sqrt{2}$. Rescaling the data for various system sizes by the expected Hausdorff dimension L^{d_H} we find excellent data collapses. The characteristic shape of the distributions originates from loops with different winding numbers.

We have explicitly checked that the Hausdorff dimension is universal and constant for the full extent of a gapped phase by sampling the loop configurations created by domain walls of a ferromagnetic Ising model in its high temperature phase. At infinite temperature the Ising model on the triangular lattice is just percolation at the critical point, and the boundaries of percolation clusters form a loop gas $|\Psi_0^{(1)}\rangle$. Although at any finite temperature above

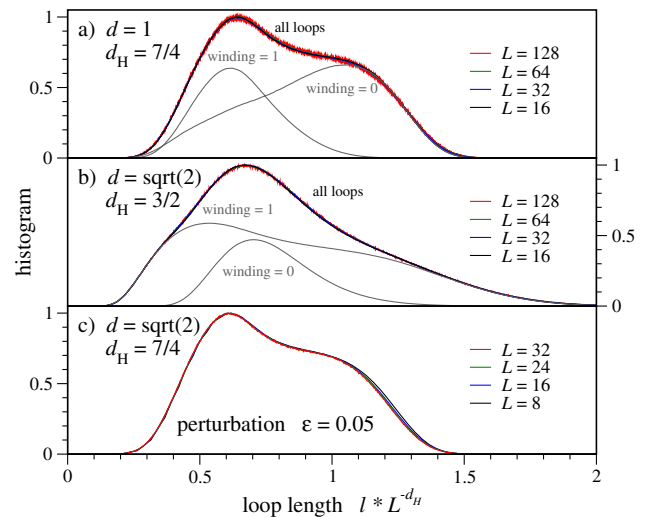


FIG. 3 (color online). Length distribution of the longest loop in a quantum loop gas with fugacity d ; data for various system sizes are rescaled by L^{d_H} with Hausdorff dimension d_H . (a) Abelian QLG with $d = 1$, $d_H = 7/4$. (b) Non-Abelian QLG with $d = \sqrt{2}$, $d_H = 3/2$. (c) Data collapse for $d = \sqrt{2}$, $d_H = 7/4$ with surgery term $\epsilon = 0.05$ at inverse temperature $\beta = 16$.

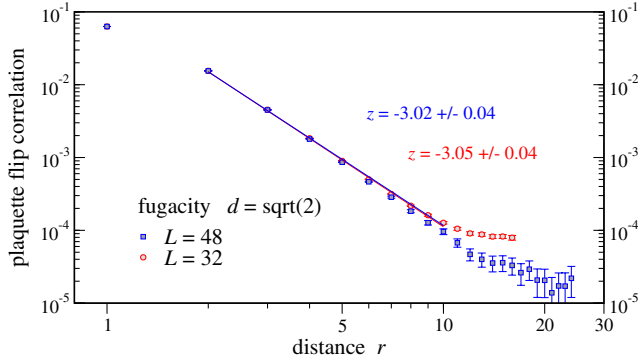


FIG. 4 (color online). Algebraic decay of the plaquette flip correlations between plaquettes at distance r in the QLG $|\Psi_0^{(\sqrt{2})}\rangle$.

the critical point, corresponding to different GS wave functions within the same topological phase, the fugacity for small loops is changed, we find that the Hausdorff dimension of the longest loop stays $d_H = 7/4$.

Returning to model (6), we now analyze the Hausdorff dimension of the ground state in the gapped phase for small ϵ . As shown in Fig. 3(c) the perturbation changes the Hausdorff dimension from $d_H = 3/2$ to $d_H = 7/4$ and the characteristic loop distribution to that of the *Abelian* phase. This can be understood in a renormalization-group sense by noting that $\mathcal{H}_0^{(d)}$ enforces the loop fugacity d only on microscopic loops, while the perturbation (which is the surgery term of the $d = 1$ loop gas) acts on large loops, and is thus a *relevant* perturbation, driving the system to the Abelian $d = 1$ fixed point.

One might still hope that a different local perturbation to the Hamiltonian might open a gap but leave the system in the non-Abelian phase. We will now show that such hope is futile by showing that they are power-law correlation between *local* operators in the GS $|\Psi_0^{(d)}\rangle$ for $1 < d \leq \sqrt{2}$. While the correlations of most local operators are short-ranged, those of the plaquette flip operator F_p decay algebraically as r^{-z} with $z \approx 3$ for $d = \sqrt{2}$ as shown in Fig. 4 and smaller values of z for $d < \sqrt{2}$. A theorem by Hastings [13] then proves that because of this algebraic (and not exponential) decay of the correlation function between two local operators, the $d \neq 1$ loop gas wave function cannot be the GS of a gapped local Hamiltonian.

The origin of the algebraic decay is the fractal nature of the loop gas: plaquette flips performing surgery between two segments of the same loop are correlated, since the change in loop number Δl is not just the sum of the changes of the individual flips. Since the flip matrix element is $d^{\Delta l}$ this results in an algebraically decaying correlation function for $d \neq 1$. This argument applies not only to the wave function (1) but to any other wave function with long critical loops and a fugacity $d \neq 1$. Since the

surgery term requires at least two loops to pass through the same plaquette, e.g., four loop segments emanating from the same plaquette, the $\langle F_p(r)F_p(0) \rangle$ correlation function should not decay faster than the $O(n)$ model exponent associated with four-line vertices which gives $z \leq 4$ [8].

We can thus prove that there exists no local, gapped Hamiltonian whose GS wave function is a loop gas with non-Abelian excitations. Our approach maps out a general strategy to analyze proposed wave functions for non-Abelian phases. To construct local Hamiltonians with non-Abelian excitations one needs to consider more intricate Hamiltonians, such as string-net models [14,15]. They can be constructed systematically from loop gases with modified inner products [16].

For $d = 1$ the Hamiltonian (3) is critical, but exhibits short-ranged equal-time correlations of *all* local operators. However, by direct calculation we find that on-site operators have power-law decay in time, e.g., $\langle \sigma^z(0)\sigma^z(\tau) \rangle \sim 1/\tau$. This is the first microscopic model exhibiting local quantum criticality [17] without dissipative baths.

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