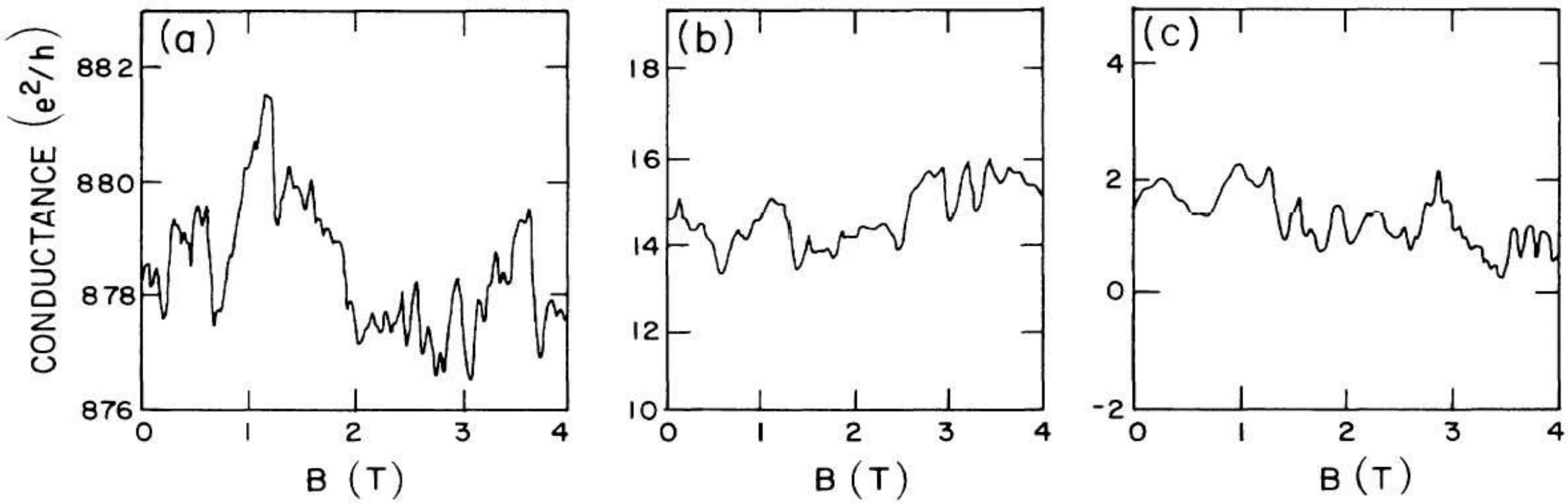
Of symmetries, symmetry classes and symmetric spaces: from disorder and quantum chaos to topological insulators

Martin R. Zirnbauer

Berlin (March 27, 2012)

Universal Conductance Fluctuations (UCF)

From Lee, Stone & Fukuyama (1987):



0.8 μm-diameter gold ring | Quasi-1D silicon MOSFET | Anderson model (numerical)

Compare with fluctuations of slow neutron scattering cross sections on nuclear targets

Symmetries

Symmetries in quantum mechanics

Q: What's a symmetry in quantum mechanics?

A: An operator $T : \mathscr{R}\psi_1 \mapsto \mathscr{R}\psi_2$ on Hilbert rays that preserves all transition probabilities: $\left|\langle T\mathscr{R}\psi_2 , T\mathscr{R}\psi_1\rangle\right|^2 = \left|\langle \mathscr{R}\psi_2 , \mathscr{R}\psi_1\rangle\right|^2$.

Wigner's Theorem:

A symmetry T in quantum mechanics can always be represented on Hilbert space by an operator $\hat{T}$ which is either unitary or anti-unitary.

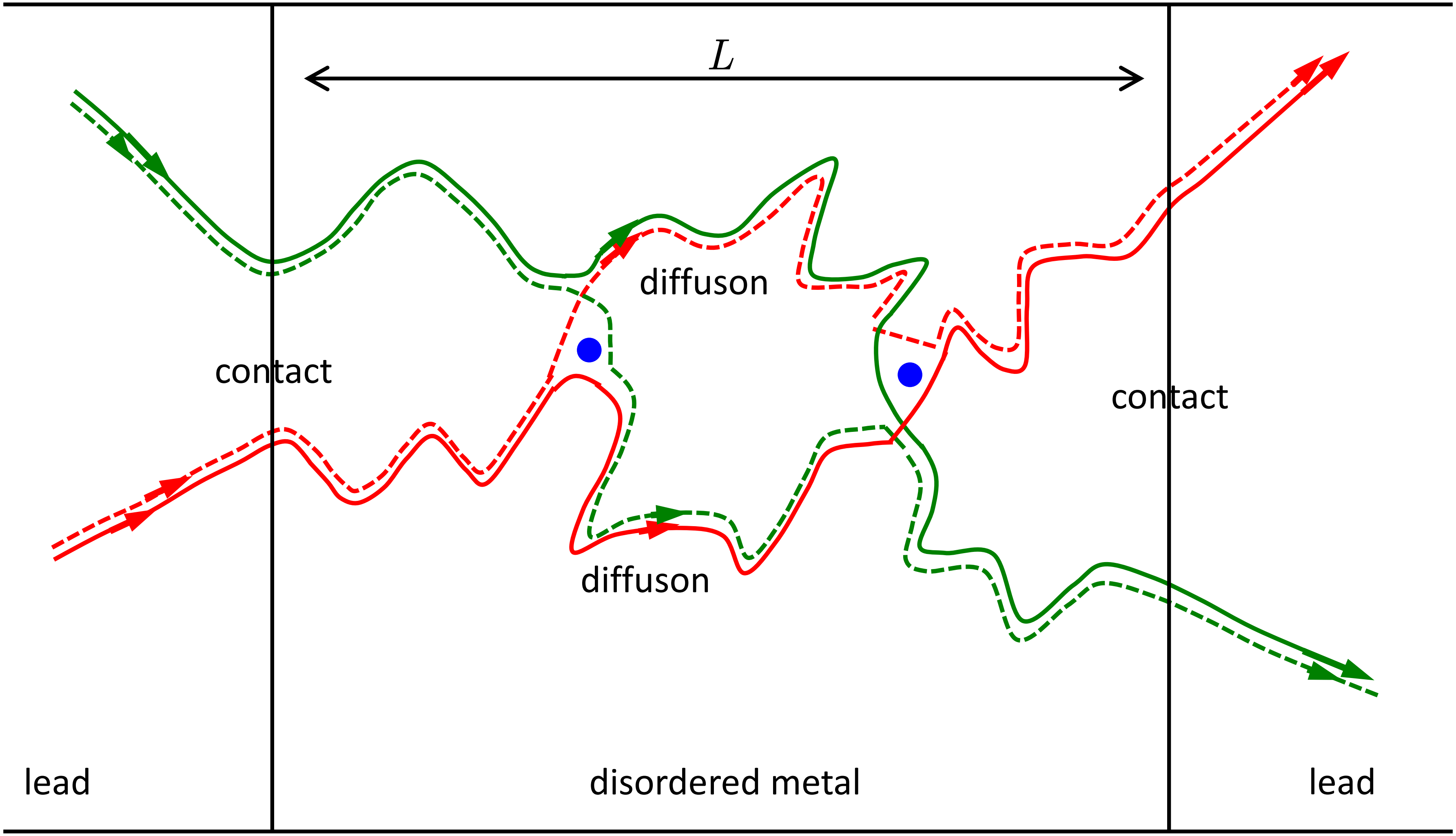
$$\langle \hat{T}\psi_2 | \hat{T}\psi_1 \rangle = \overline{\langle \psi_2 | \psi_1 \rangle}$$

Eugene P. Wigner

Remark 1: The symmetries form a group, G.

Remark 2: Symmetries commute with the Hamiltonian ($\hat{T}H = H\hat{T}$). Thus "chiral symmetry" ($\gamma_5 D \gamma_5 = -D$) is not a symmetry.

Example 1: diffuson contribution to UCF



$$(e^2/h)^{-2}\,\mathrm{var}\big(G(L)\big) \;=\; L^{d-4}\int \frac{d^dq}{(q^2)^2} \;\propto\; \mathrm{O}(1)$$

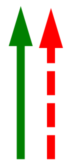
The role of symmetries: massless modes

Green (or resolvent) operator: $G(z) = (z - H)^{-1}$

Resolvent identity: $(z-w)\,G(z)\,G(w) = G(w) - G(z)$

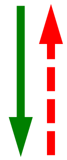
For $G^{\pm} := (E \pm \mathrm{i}\varepsilon - H)^{-1}$ the U(1) symmetry of particle number conservation entails $\overline{G^{+}_{ab}} = G^{-}_{ba}$. This gives rise to a sum rule: $\sum_b |G^{+}_{ab}|^2 = (2\mathrm{i}\varepsilon)^{-1}(G^{-}_{aa} - G^{+}_{aa})$. Hence, Fourier transform is diffusive:

$$\mathscr{F}_q\left(|G^{+}_{ab}|^2\right) \propto (Dq^2 + \mathrm{i}\varepsilon)^{-1}$$

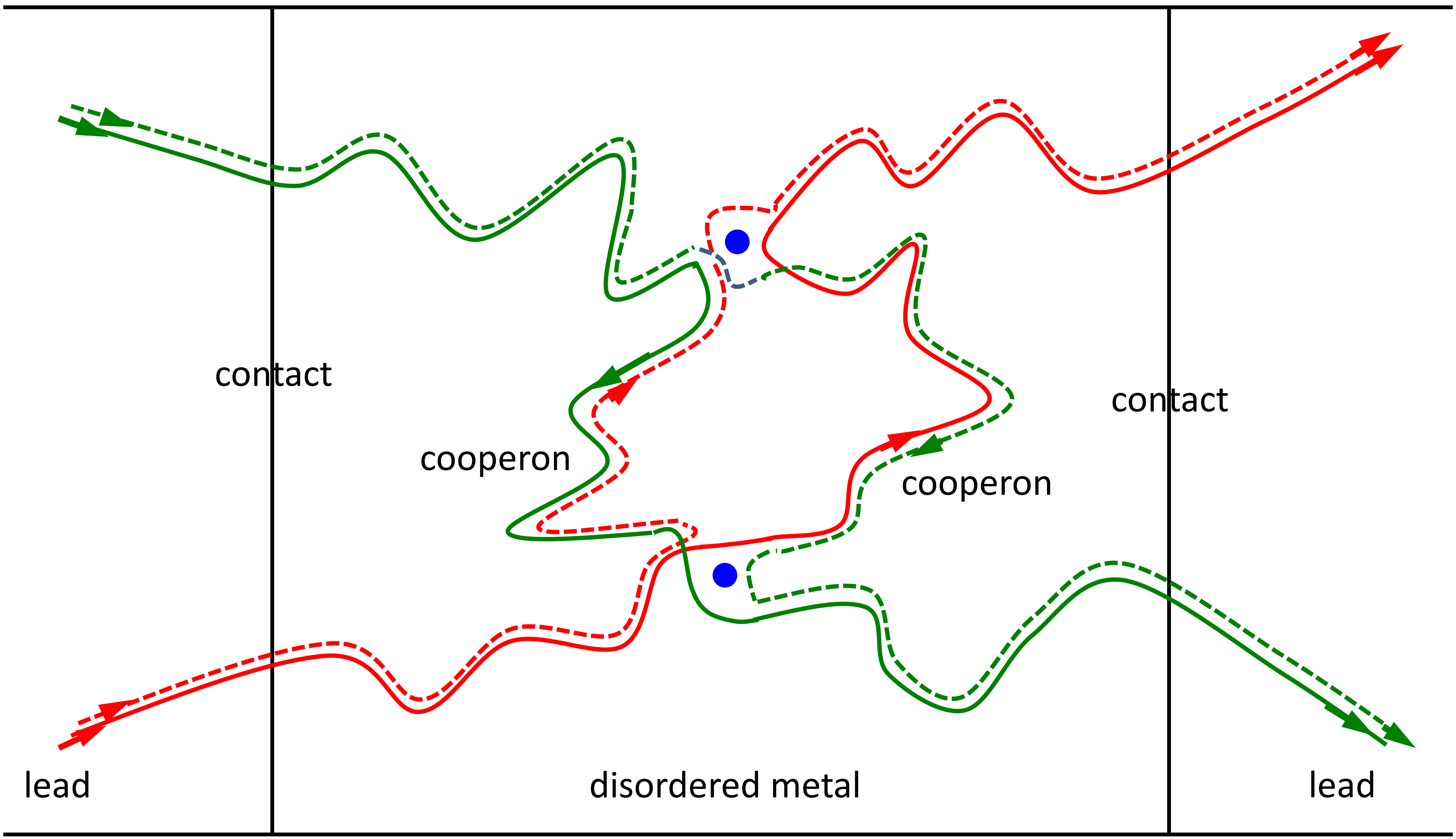
diffuson: 

Time-reversal symmetry gives: $G^{+}_{ab} = G^{+}_{Tb\,Ta}$

Sum rule: $\sum_b G^{+}_{Tb\,Ta}\,\overline{G^{+}_{ab}} = (2\mathrm{i}\varepsilon)^{-1}(G^{-}_{aa} - G^{+}_{aa})$

cooperon: 

The cooperon contribution to UCF



The presence of the cooperon mode doubles the variance.

Example 2: wires with symplectic symmetry

- "Symplectic" disordered wires (i.e. time-reversal inv. electrons subject to spin-orbit scattering in quasi-one dimension)

- Model: 1d Dirac fermions in class *A*II (→ Kramers degeneracy)

$$H = \begin{pmatrix} v_F \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x} + A(x) & B(x) \\ B^\dagger(x) & -v_F \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial x} + A^T(x) \end{pmatrix}$$

  2*N* channels coupled by *N* x *N* random matrices

  $A^\dagger = A\,,\ \ B^T = -B$ with i.i.d. Gaussian entries

- For large *N*, disorder averages are computable by the mapping to a nonlinear sigma model (Wegner, Efetov).

Super Fourier Analysis and Localization in Disordered Wires

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(Received 7 November 1991)

The problem of calculating the zero-temperature mean conductance $\langle c \rangle$ of a disordered thick metallic wire coupled at both ends to ideal leads is formulated as a diffusion problem on a Riemannian symmetric superspace **G/K** (Efetov). The problem is solved exactly by Fourier transforming the diffusion kernel. Although the solution agrees with known results for the case of orthogonal and unitary symmetry, it has the surprising feature that $\langle c \rangle$ never falls below the minimum value of $e^2/2h$ for long wires with symplectic symmetry, in leading order of the expansion around the metallic and thick limit.

$$\langle c \rangle = T(i,1,1)/2 + \sum_{l \in 2\mathbb{N}+1} [T(i,l,l-2) + T(i,l-2,l)]/2$$

$$+ 2^4 \sum_{l_1,l_2 \in 2\mathbb{N}-1} \int_0^\infty d\lambda\, \lambda(\lambda^2+1)\tanh(\pi\lambda/2)\, l_1 l_2 (\lambda^2 + l_1^2 + l_2^2 - 1)$$

$$\times \prod_{\sigma,\sigma_1,\sigma_2 = \pm 1} (-1 + i\sigma\lambda + \sigma_1 l_1 + \sigma_2 l_2)^{-1} T(\lambda, l_1, l_2)$$

where $T(\lambda, l_1, l_2) = \exp\left(-(\lambda^2 + l_1^2 + l_2^2 - 1)(L/4\xi)\right)$

Y. Takane on wires with symplectic symmetry

J. Phys. Soc. Jpn. 73 (2004) 1430

Quantum Electron Transport in Disordered Wires with Symplectic Symmetry

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The conductance of disordered wires with symplectic symmetry is studied by the supersymmetric field theory. Special attention is focused on the case where the number of conducting channels is odd. Such a situation can be realized in metallic carbon nanotubes. The average dimensionless conductance $\langle g \rangle$ is obtained using Zirnbauer's super-Fourier analysis. It is shown that with increasing wire length, $\langle g \rangle \to 1$ in the odd-channel case, while $\langle g \rangle \to 0$ in the ordinary even-channel case. It should be emphasized that the so-called Zirnbauer's zero mode, which has been believed to be unphysical, is essential for describing the anomalous behavior in the odd-channel case.

The role of symmetries: symplectic zero mode

Split the expression:

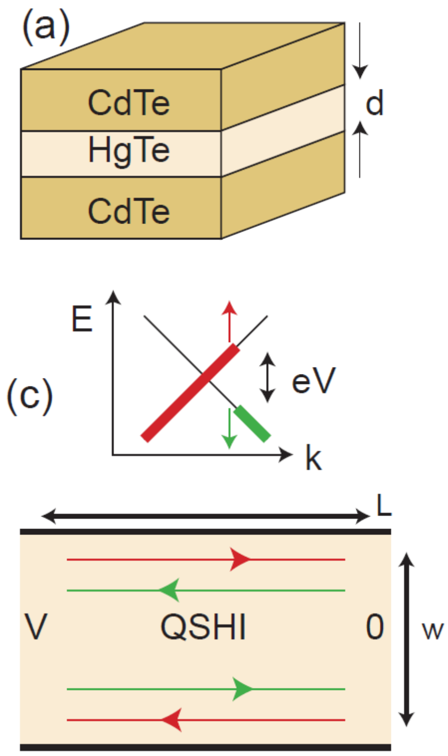
$$(e^2/h)^{-1}\langle c\rangle \ = \ \tfrac{1}{2}f_0(L)+\tfrac{1}{2}f_1(L)$$

$$\text{even } N: \quad f_0(L)=\frac{32}{9}\left(\frac{\pi\xi}{2L}\right)^{3/2}\mathrm{e}^{-L/2\xi}+\ldots$$ (Brouwer & Frahm, 1995)

$$\text{odd } N: \quad f_1(L)=1 + 2\mathrm{e}^{-4L/\xi}+\ldots$$ (Takane, 2004)

The zero mode for odd *N* is robust!

In fact, it is none other than the edge state of the quantum spin Hall insulator (protected by symmetry & topology).



Symmetry Classes

“Symmetry classes of disordered fermions and topological insulators”
http://www.uni-koeln.de/zirn

Universality of spectral fluctuations

In the spectrum of the Schrödinger, wave, or Dirac operator for a large variety of physical systems, such as

- atomic nuclei (neutron resonances),
- disordered metallic grains,
- chaotic billiards (Sinai, Bunimovich),
- microwaves in a cavity,
- acoustic modes of a vibrating solid,
- quarks in a nonabelian gauge field,
- zeroes of the Riemann zeta function,

one observes fluctuations that obey the laws of random matrix theory for the appropriate symmetry class and in the ergodic limit.

Universality: quantum chaotic billiards

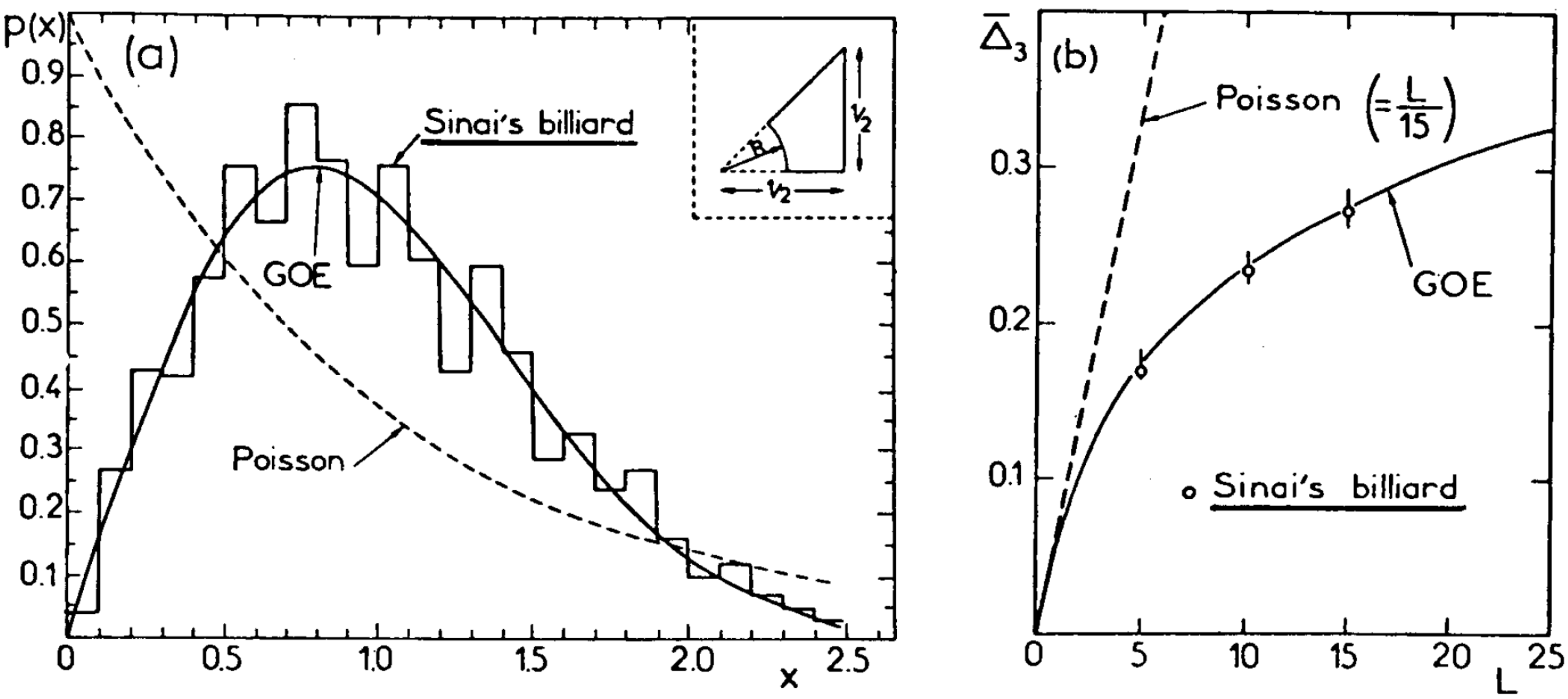


Fig. 12. Spectral fluctuations for Sinai's billiard (see fig. 11). (a) spacing distribution histogram; (b) Dyson–Mehta statistic. GOE and Poisson predictions are plotted for the sake of comparison (taken from [62]).

Random matrix conjecture by Bohigas, Giannoni, and Schmit (1984)

J. Math. Phys. 3 (1962) 1199

The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

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Institute for Advanced Study, Princeton, New Jersey
(Received June 22, 1962)

Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

Wigner-Dyson symmetry classes

- *A*   : complex Hermitian matrices (‘unitary class’, GUE)
- *A*I : real symmetric matrices (‘orthogonal class’, GOE)
- *A*II : quaternion self-dual matrices (‘symplectic class’, GSE)

This classification has proven fundamental to various areas of theoretical physics, including the statistical theory of complex many-body systems, mesoscopic physics, disordered electron systems, and the field of quantum chaos.

Symmetry classes: the need for an extension

Developments (going beyond Dyson) after 1990:

- Chiral classes: random matrix models for the Dirac operator (Gade & Wegner 1991, Verbaarschot & Zahed 1993)
- Novel interference phenomena due to Andreev reflection at interfaces between normal metals and superconductors (Oppermann 1990, Altland & MZ 1997)
- Universality in the statistics of the zeros of the Riemann zeta function and ensembles of related functions (Katz & Sarnak 1996)

Example: massless q.p. modes in superconductors

The Gorkov-Green operator $\mathscr{G}(z) = (z - \mathscr{H})^{-1} = \begin{pmatrix} G_{pp} & G_{ph} \\ G_{hp} & G_{hh} \end{pmatrix}$ satisfies

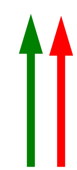
$$G_{pp}^T(z) = -G_{hh}(-z)\,, \quad G_{ph}^T(z) = -G_{ph}(-z)\,, \quad G_{hp}^T(z) = -G_{hp}(-z)$$

due to the canonical anti-commutation relations for fermions.

Sum rule (*C* exchanges particles and holes):

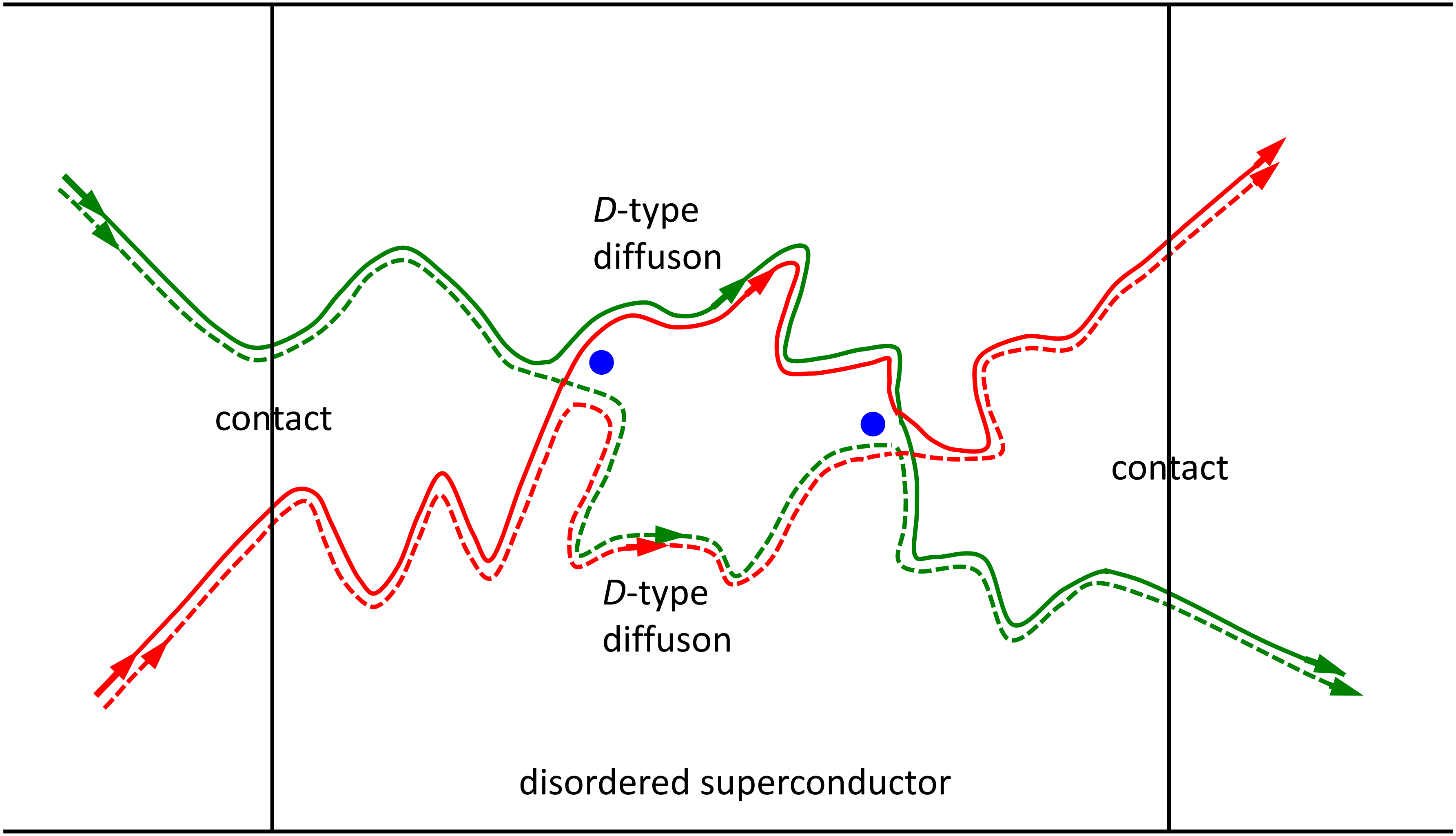
$$\sum_\beta \mathscr{G}^+_{\alpha\beta}(E)\,\mathscr{G}^+_{C\alpha,C\beta}(E) = -\sum_\beta \mathscr{G}^+_{\alpha\beta}(E)\,\mathscr{G}^-_{\beta\alpha}(-E) = \frac{\mathscr{G}^+_{\alpha\alpha}(E) - \mathscr{G}^-_{\alpha\alpha}(-E)}{2E}$$

gives rise to the *D*-type diffuson (Altland & MZ, 1997):



(*D*-type diffuson)

D-type diffuson in UCF



→ fluctuations in spin and heat transport (quasi-particles)

The Tenfold Way

MZ (1996), Altland & MZ (1997), Heinzner, Huckleberry & MZ (2004)

Our setting: Fock space with symmetries

Single-particle Hilbert space V, $\dim V = N$

- Fock space for (identical) fermions:

$$F = F_0 \oplus F_1 \oplus F_2 \oplus \ldots \oplus F_n \oplus \ldots \oplus F_N \,, \qquad F_n = \wedge^n(V)$$

  Particle creation $(c^\dagger)$ and annihilation operators (c) satisfy CAR,

$$c_\alpha^\dagger c_\beta + c_\beta c_\alpha^\dagger = \delta_{\alpha\beta}$$

- Unitary symmetries:
  any group of unitary operators defined on V and extended to F in the natural way.

- Anti-unitary symmetries:
  1. Time reversal $T: V \to V$, extended to $T: F_n \to F_n$
  2. Particle-hole conjugation $C: F_n \to F_{N-n}$

Statement of problem

G = arbitrary symmetry group made from generators as described above

F = fermionic Fock space with a G-action

$\mathscr{H}$:= set of all (polynomials in) G-invariant one-body operators, i.e., operators which commute with all symmetry generators and are quadratic in particle creation and annihilation operators:

$$H = \sum_{\alpha\beta} h_{\alpha\beta}\, c_\alpha^\dagger\, c_\beta + \sum_{\alpha<\beta} \left(\Delta_{\alpha\beta}\, c_\alpha^\dagger\, c_\beta^\dagger + \overline{\Delta}_{\alpha\beta}\, c_\beta\, c_\alpha\right)$$

Q: What types of irreducible block occur in this setting?

Communications in
Mathematical Physics

Symmetry Classes of Disordered Fermions

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Abstract: Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

The primary data of the classification are a Nambu space of fermionic field operators which carry a representation of some symmetry group. Our approach is to eliminate all

Theorem (HHZ):

Every irreducible block (of the Hamiltonians) occurring in this setting corresponds to a classical irreducible symmetric space,

and conversely,

every classical irreducible symmetric space occurs in this way.

Example: class *D*

Majorana operators: $\psi_{\alpha,1} = \frac{1}{\sqrt{2}}(c_\alpha + c_\alpha^\dagger)\,, \quad \psi_{\alpha,2} = \frac{\mathrm{i}}{\sqrt{2}}(c_\alpha - c_\alpha^\dagger)$

Hamiltonian in Majorana basis: $H = \sum_{a<b}^{2N} X_{ab}\,\psi_a\psi_b\,, \quad X_{ab} = -X_{ba} = \overline{X}_{ba}$

Time evolutions $\mathrm{e}^{-\mathrm{i}tH/\hbar}$ are in $\mathrm{SO}(2N)$.

Realizations:

disordered superconductors with spin-triplet pairing and *T*-breaking *p*-wave order ($\mathrm{Sr_2RuO_4}$); *A*-phase of superfluid $^3\mathrm{He}$.

Topological quantum computing

Symmetric Spaces

What's a symmetric space?

Riemann tensor: $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}$

Def.: A (locally) symmetric space is a Riemannian manifold $X = G/K$ with covariantly constant curvature: $\nabla R = 0$.

Ex. 1: the round two-sphere $X = \mathrm{S}^2$, $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

Ex. 2: the set $X = \mathrm{Gr}_n(\mathbb{C}^{m+n}) = \mathrm{U}(m+n)/\mathrm{U}(m) \times \mathrm{U}(n)$ of all subspaces $\mathbb{C}^n \simeq V \subset \mathbb{C}^{m+n}$

Some facts:

- Complete classification by E. Cartan (1926)
  Large families: *A*, *A*I, *A*II, *A*III, *BD*, *BD*I, *C*, *C*I, *C*II, *D*III
- Metric tensor g_{ij} is the only G-invariant rank-2 tensor on X

Symmetric spaces in physics

- Symmetric spaces are natural candidates for order parameter spaces G/K (spontaneous symmetry breaking).

- Symmetric spaces are target spaces for nonlinear sigma models. One-parameter renormalizability (in 2 dimensions):

$$\beta_{ij} := \frac{d}{d\ln a}\, g_{ij} = -\lambda g_{ij}$$

- The large families of symmetric spaces are in one-to-one correspondence with symmetry classes for disordered fermions.

- Many-body ground states in the mean-field approximation (Hartree-Fock-Bogoliubov) organize into symmetric spaces.

Mean-field ground states

Fock operators: $c_\alpha^\dagger$ (creation), c_α (annihilation)

Fock vacuum: $c_\alpha \, |\mathrm{vac}\rangle = 0 \qquad (\alpha = 1, 2, \ldots)$

Quasi-particle vacuum: $\widetilde{c}_\alpha \, |\widetilde{\mathrm{vac}}\rangle = 0 \qquad (\alpha = 1, 2, \ldots)$

where $\widetilde{c}_\alpha = \sum_{\alpha'} \left(c_{\alpha'} u_{\alpha'\alpha} + c_{\alpha'}^\dagger v_{\alpha'\alpha} \right)$

Remark: q.p. vacua are also referred to as ground states in the Hartree-Fock-Bogoliubov mean-field approximation.

Special case (*N*-particle Slater determinant):

$$\widetilde{c}_\alpha = c_\alpha^\dagger \quad (1 \le \alpha \le N), \quad \widetilde{c}_\alpha = c_\alpha \quad (\alpha > N)$$

In the presence of a group G of symmetries, we require

$$g|\widetilde{\mathrm{vac}}\rangle = |\widetilde{\mathrm{vac}}\rangle \, \mathrm{e}^{\mathrm{i}\varphi} \quad (\text{for all } g \in G)$$

Mean-field ground states & symmetric spaces

Fix a symmetry group G.

As the (mean-field) Hamiltonian varies, so does $x \equiv \mathscr{R}\,|\widetilde{\mathrm{vac}}\rangle$.

Note: our variable q.p. vacua constitute a Riemannian manifold X by the (geodesic) distance function

$$\mathrm{dist}(x_1, x_2) = t \iff \mathscr{R}\,|\widetilde{\mathrm{vac}}_2\rangle = \mathscr{R}\,\mathrm{e}^A|\widetilde{\mathrm{vac}}_1\rangle, \ \ \|A\| = t.$$

Ex. ($G = \mathrm{U}(1)$): the space of *N*-particle Slater determinants in $\mathbb{C}^{M+N}$ is a Riemannian manifold $X = \mathrm{Gr}_N(\mathbb{C}^{M+N}) = \mathrm{U}(M+N)/\mathrm{U}(M) \times \mathrm{U}(N)$.

Q: What can be said about the structure of X in general?

A: For any symmetry group G, the manifold X of G-invariant quasi-particle vacua is a symmetric space (corollary of HHZ).

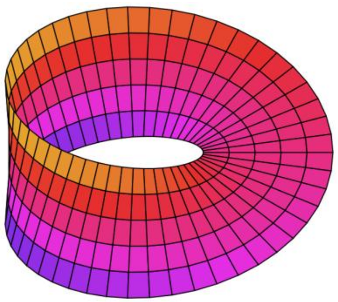
Quantum Spin Hall Insulator (*A*II)

Particle number conserved → q.p. vacua are Hartree-Fock states.

Γ:= translation group; $\widehat{\Gamma}$:= Brillouin zone (momentum space).

Let G be generated by Γ and time reversal ($T^2 = -1$).

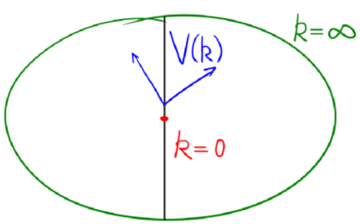
Fact: the HF ground state of a band insulator is a vector bundle $\widehat{\Gamma} \ni k \mapsto V(k) \in \mathbb{C}^{m+n}$, $V(k) \simeq \mathbb{C}^n$:= vector space of valence states.



Time-reversal symmetry implies $T\,V(k) = V(-k)$.

At T-invariant momenta $k_0 = -k_0$ one has $T\,V(k_0) = V(k_0)$.

K-theory (for class *A*II and $\widehat{\Gamma} = \mathrm{S}^2$) → there exist 2 isomorphism classes of such vector bundles (Kane & Mele, 2005).



Alternative view (symmetric spaces)

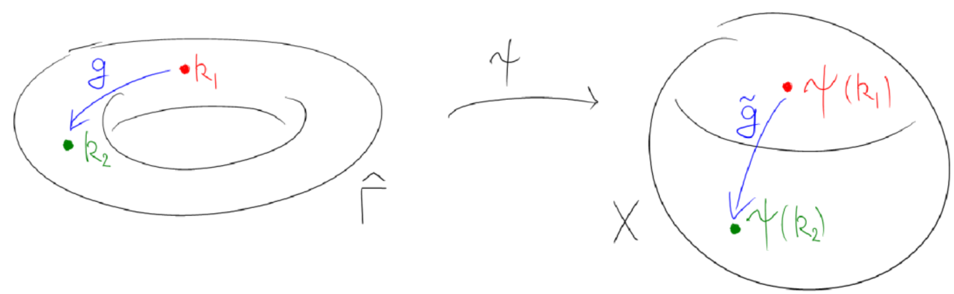
Recall: $V(k) \simeq \mathbb{C}^n \subset \mathbb{C}^{m+n}$ determines $x \in X := \mathrm{U}(m+n)/\mathrm{U}(m) \times \mathrm{U}(n)$.

Thus $\{k \mapsto V(k)\}$ determines a mapping $\{k \mapsto \psi(k) \in X\}$.

Constraint $T\,V(k) = V(-k) \quad \Longrightarrow \quad \widetilde{T}\,\psi(k) = \psi(-k)$,

and $T\,V(k_0) = V(k_0) \quad \Longrightarrow \quad \psi(k_0) \in X_0 = \mathrm{Sp}(m+n)/\mathrm{Sp}(m) \times \mathrm{Sp}(n)$.

General picture:



Topological phases (mean field) are given by homotopy classes of mappings into a symmetric space, $\psi : \widehat{\Gamma} \to X$, subject to an equivariance condition $\widetilde{g} \cdot \psi(k) = \psi(g \cdot k)$ for all $g \in G_{\mathrm{red}}$.

“Periodic Table”

Symmetry				*d*							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

Quantum Hall Effect

He-3 (B phase)

QSH: HgTe

$Bi_{1-x}Sb_x$

Kitaev (2008),

Ludwig et al. (2009)

from Hasan & Kane, Rev. Mod. Phys. (2011)

Remark: Table is incomplete because

- only special symmetry groups are considered,
- *K*-theory may miss some finer points of topology.

The End