PHYSICS COLLOQUIUM TALK: MAXWELL IN CHAINS

by

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Abstract. — In standard text books on electrodynamics Maxwell's laws are often illustrated by using field lines – a notion lacking relativistic covariance and obscuring the superposition principle (see e.g. R. Feynman: The Feynman Lectures on Physics, Vol. II, Chapter 1-5). In this colloquium I am going to present an improved variant of the field line picture: the discrete approximation of the electromagnetic field by chains (B and D are modeled by 1-chains, E and H by 2-chains). The language of chains offers a simple, direct and intuitive approach to Maxwell's electrodynamics, which does no suffer from having to compromise on physical and mathematical correctness. Fundamental aspects such as parity invariance, or the transcription of the theory to curved space-time, are especially transparent in this approach. In my talk I will illustrate the chain picture at a number of examples: (i) Aharonov-Casher effect, (ii) equivalence between problems of magneto- and electrostatics, (iii) emission of an electromagnetic signal by the discharge of a capacitor, (iv) magnetic excitation due to a point charge in motion.

1. Feynman

Good Afternoon! As you have probably inferred from the announcement, today's colloquium won't be about research but about a topic in physics teaching: I am going to give you an introduction to the course on classical electrodynamics which I've been teaching at Cologne for over a decade now.

I'd like to start with one of the cornerstones of 20^{th} century physics teaching: the Feynman Lectures on Physics. In Volume II, Chapter 1-5 (with the title: "What are the fields?") Feynman in his inimitable way pokes fun at the notion of field lines often invoked by physicists. In short, Feynman criticizes two deficiencies: i) the field line picture does not reflect the superposition principle – which he says is the deepest principle of electrodynamics – and ii) field lines lack relativistic covariance (they disappear in certain reference frames).

Here is what Feynman says about his second point of criticism: "Suppose that you finally succeeded in making up a picture of the magnetic field in terms of some kind of lines or of gear wheels running through space ..." And then comes the punch line: "The gear wheels or lines disappear when you ride along with the object!" (Put differently:

if you move along with the charges, then the current vanishes and so do the magnetic field and its field lines.)

Feynman's conclusion is that the field line picture only helps you in special situations, and in general you have no choice but to work with the abstract field idea.

Feynman is right, of course, but then he isn't. What he demolishes is the classical field line picture of textbook physics. In today's colloquium I will present to you a modern alternative, which withstands Feynman's criticism and serves as the intuitive basis for the electrodynamics course taught at Cologne University.

2. Contents

Here is what I intend to do. I will start with a simple introduction to the chain picture of electrodynamics. Once we comprehend the boundary operator on chains, we can formulate Maxwell's equations. Afterwards we shall introduce the metric, for the purpose of formulating the constitutive laws. These complete the system of basic equations of electrodynamics, and we can then move on to applications.

First, I will discuss a relativistic phenomenon, the so-called Aharonov-Casher effect, at the simple example of a coil in motion. Second, there will be a neat duality between magneto- and electrostatics. Third, I will turn to a dynamical problem: discharge of a capacitor or, rather, the electrodynamic signal radiated off by such a process. And finally, I will revisit Feynman's problem of disappearing field lines – you might be curious to see how the chain formalism deals with that issue.

3. Atomistic Picture: Chains

All of my talk will be based on the reasonable *assumption* that the "continuous can be approximated by the discrete". Thus we'll take an atomistic point of view, replacing smooth configurations of the electromagnetic field by spiky ones, not because we believe that the fundamental constituents of the field are discrete, but because it serves the purpose of better visualization.

Such an approach is legitimate if we can get arbitrarily close to the true configuration by refining the discretization. This, then, is what you must bear in mind when contemplating the pictures I'll show: that they are to be refined and that ultimately a limit is taken to recover the continuum.

3.1. Charge density.— Having stated the rules of the game – namely, approximation of the continuum by the discrete, with a continuum limit to be taken at the end – let's start discretizing electrodynamics. Let's have a bunch of charges q_i located at points p_i, to build up the charge density ρ. (Actually, the discrete model for ρ is not just an approximation but is in fact an accurate representation of classical reality, as electric charge for all we know is concentrated in point particles each carrying one fundamental unit of charge. This model is of course modified by the quantum mechanical description of charged particles by a wave function.)

For the book keeping, it is useful to form the linear combination of the points, each weighted by its charge. Such a formal sum $\rho = \sum_i q_i p_i$ is called a 0-chain as the objects being combined, the points, are zero-dimensional.

From experience I know that many in the audience are surprised at this stage. Let me therefore add that chains are best viewed as elements of a *vector space*. In the case at hand, the points p_i correspond to the basis vectors and the charges q_i to the components of the "vector" $\rho = \sum q_i p_i$. If that still looks weird, you might find if helpful to use traditional calculus to write for ρ the following expression:

$$\rho = \sum_i q_i \, \delta(\mathbf{x} - \mathbf{x}_i) \, d^3x ,$$

a sum of δ-functions located at the points p_i with position vectors $\mathbf{x}_i$ and weighted by the charges q_i. (This anticipates the mathematical fact that Poincaré duality relates 0-chains to densities.)

3.2. Electric excitation.— The next picture should look familiar, too: we imagine the electric excitation D, also known as the electric displacement field, to consist of lines γ_i carrying electric fluxes ϕ_i. Thus D is approximated by a 1-chain: a linear combination of one-dimensional objects γ_i weighted by real numbers ϕ_i. Note that the electric flux lines carry a sense of direction – we call this an *inner orientation*.

3.3. Electric field strength.— In addition to the 1-chain model of the electric field, we will need its (Hodge-)dual description as a 2-chain. Imagine a collection of surface elements S_i, each associated with a voltage V_i. Form the linear combination of the S_i's multiplied by the V_i's; what you get is a 2-chain called the *electric field strength E*.

We are all familiar with the physical meaning of this 2-chain in the static limit: in that case the surface elements of E piece together to closed surfaces, the so-called equipotential surfaces. Yet the 2-chain model of E makes sense not only in the static limit but also in dynamical situations; this important aspect of the formalism will be exploited in the sequel.

3.4. Magnetic field strength and excitation. — Moving on to the magnetic analog of D, let's again consider a set of lines γ_i with fluxes ϕ_i, now of magnetic type. The 1-chain $\sum_i \phi_i \gamma_i$ is called the *magnetic field strength B*.

In contrast with its electric partner, B carries an *outer orientation*, i.e. its lines have a sense of circulation instead of a sense of direction. Traditionally speaking, D is a polar vector, whereas B is an axial vector; this expresses the fact that a parity transformation reverses the sign of D but not the sign of B (and, indeed, a sense of circulation remains invariant under space reflections).

Finally, there exists the dual model of the magnetic field as a 2-chain: the magnetic excitation, H, consists of surface elements carrying an inner orientation. Please note that what's an outer orientation of a line is an inner orientation of the surface perpendicular to the line.

4. Boundary Operator

To formulate Maxwell's equations, we need one differential-type operator. This is neither the divergence operator, nor the curl, nor the gradient, but a metric-free ancestor of all of these, the *boundary operator*: a linear operator that maps k-chains into $(k-1)$-chains, by taking the boundary while preserving the type of orientation.

4.1. Example 1. — Let's see how this works by looking at some examples. The boundary of a line element γ with inner orientation is the final point of the line minus the initial point. In the picture, the green piece of line γ has for its boundary the points shown in red, with the final point p_f counted as plus and the initial point p_i as minus. In formulas: the boundary of the 1-chain γ is the 0-chain $p_f - p_i$ (not to be confused with the difference vector of two points in an affine space).

4.2. Example 2. — As a second example, consider a surface element S with outer orientation given, say, by a normal vector. The boundary of S is the closed line γ surrounding it, with an outer orientation (or sense of circulation) obtained by following the vector of S and curving outward. (Note that unlike the vector-differential operator "curl" the boundary operator does not depend on a choice of right-hand or left-hand rule; in order to determine the orientation of γ, all we need is the distinction between the interior and the exterior.)

So much for the boundary operator, and now, here we go:

5. Homogeneous Maxwell Equations

5.1. No magnetic monopoles. — The first law of the homogeneous Maxwell equations states that the 1-chain of B is closed – we say that magnetic flux lines have no beginning and no end. This holds true as long as there are no magnetic monopoles. Actually, our formalism could easily accommodate magnetic monopoles, but they have never been observed experimentally, and so I shall not consider them.

Thus the magnetic field strength has zero boundary.

5.2. Law of induction. — The other homogeneous law, namely Faraday's law of induction, states that the electric field strength E **does** have a boundary, if B is time-dependent. More precisely, when adding the 1-chain of the boundary of E and the 1-chain of the time derivative "B–dot", one obtains zero.

Faradays's law is not so easily pictured, as it involves the operation of taking a derivative, a notion tied to the continuum. But having discretized in space, let's go all the way and make time discrete, too, by replacing the derivative $\dot{B}$ by a quotient of differences with time step Δt. Then, after reorganizing the equation a little bit, we can draw the following informative picture:

Imagine a magnetic flux line with a kink in it, and let the kink relocate from one corner of a rectangle S to the opposite corner in a time interval Δt; in other words, the flux line straddles S by its two configurations at $t - \Delta t/2$ and $t + \Delta t/2$. (Such a process may, for example, occur for a Josephson vortex in a layered superconductor when the magnetic field axis is tilted by a small angle with respect to the plane of the layers.) Faraday's law then states that the 2-chain of the electric field strength E at time t must contain some surface element S', $\partial S' = \partial S$, carrying voltage $V = \phi/\Delta t$ (where ϕ is the magnetic flux of the line). Indeed, the boundary of the 2-chain $E(t) = V \cdot S'$ matches the negative time derivative of the flux line configuration if the outer orientation of S' is given by the sense of direction in which the surface element S' is pierced by the circulation of the magnetic flux line at the *earlier* time $t - \Delta t/2$.

5.3. Intuitive picture. — While that's all we can say based on Faraday's law alone, the full set of laws of electrodynamics determine the dynamics of the electromagnetic field completely (provided that the sources are given), and by noting that these laws are local we arrive at a stronger statement: the world surfaces swept out by moving magnetic flux lines consist of instantaneous surface elements of E. Or, to make this intuitive notion even more visual, we could say, for fun, that the vacuum is a fluorescent medium (what a ridiculous concept!), and as magnetic flux lines move through it, the medium lights up, and what shines (and instantly returns to darkness) are surface elements of E.

Having said the above, let me stress that Faraday's law alone makes *no* statement about the zero-boundary part of E; there may always exist some closed surfaces – of the equipotential type – whether B is time-dependent or not.

5.4. A simple application. — To reinforce the intuitive picture of magnetic flux lines sweeping out surfaces (or surface elements) of E, let's look at a simple example:

What you see here is the cross section of a long, thin coil. In the wire of the coil, an electric current flows, so (as we know) there is magnetic flux in the interior of the coil, with the dots indicating the points of intersection of the flux lines with the plane of the drawing.

Now let the current carrying coil be in motion with velocity v. Then the moving lines of B sweep across infinitesimal surfaces which piece together to the surfaces of E shown in green. These surfaces have their boundaries at the location where the time derivative of B is nonzero (in agreement with Faraday's law).

We can put this result in a formula:

$$E = \iota(v)B\,,$$

by introducing an operator "iota(v)" which attaches the direction of v to the lines of B. The resulting surface elements are weighted by $1/\Delta t$, and finally we send $\Delta t \to 0$.

So $\iota(v)$ increases the degree of a chain by one (in the present instance from 1 to 2), while the boundary operator introduced before decreases the degree by one unit. There is some similarity between the iota-operation and the well-known vector product in $\mathbb{R}^3$. An important difference, however, is that the vector product needs the right-hand rule and measurement of angles, whereas $\iota(v)$ needs no such thing.

5.5. Atomistic Picture: Chains (V). — The iota-operation is also useful for some other purposes, e.g. for describing the electric current density j.

Here you see once more, in the simplest situation, exactly how $\iota(v)$ acts. Take a point p and attach the velocity vector v multiplied by a time step Δt. The resulting line segment is weighted by the inverse time step. In the limit $\Delta t \to 0$ you get an infinitesimal line segment γ, which we write as $\iota(v)p$. By linear extension this defines a linear and local operator $\iota(v)$ taking 0-chains to 1-chains.

Let me add for the experts that, strictly speaking, $\iota(v)p$ is no longer a 1-chain but rather a 1-current (in the sense of deRham). Being physicists, however, we call limits of functions still functions (e.g. we call Dirac's delta-distribution the δ-"function"), and by the same token I call something a 1-chain even though it is really a singular limit of 1-chains.

Now let there be a system of point charges moving with velocity v. The electrical current density then is the 1-chain j whose line segments are obtained by attaching the velocity v to the points of the charge density ρ. The charges of the points are picked up as weights in the process.

In formulas: j is the result of operating with $\iota(v)$ on the charge density ρ, or explicitly: $j = \iota(v)\rho = \sum_i q_i \gamma_i$ where $\gamma_i = \iota(v)p_i$.

6. Inhomogeneous Maxwell Equations (I)

Now on to the inhomogeneous Maxwell equations.

We'll do again the easy part first. Gauss's law states that the 0-chain of ρ added to the boundary of the 1-chain of D yields zero; or in familiar language: charges are sources of electric excitation.

To illustrate, consider a single line of D, which begins somewhere and ends somewhere else. Then, to satisfy Gauss's law you must place a positive charge (of magnitude equal to the electric flux of the line) at the beginning, and a negative one at the end.

Please note the obvious fact that, given D, Gauss's law determines ρ, but the converse is not true. Indeed, from one solution of Gauss's law (for fixed ρ), we can produce any number of solutions by adding closed lines to D. (In particular, by adding closed lines, I can turn the shown configuration into a static dipole field.)

6.1. Inhomogeneous Maxwell's Equations (II). — Finally, let's formulate the Ampere-Maxwell law in chain language. It says:

The boundary of the 2-chain of the magnetic excitation H agrees with the sum of the 1-chains of the electric current j and the displacement current $\dot{D}$.

To draw a picture, let's again make time discrete (with time step Δt) and approximate $\dot{D}$ by a quotient of differences. Then, if in time Δt a piece of electric flux line moves from left to right (like so), there first of all had better be charges – recall Gauss's law – attached to both ends, the positive one giving charge flow from left to right, the negative one from right to left. Now, reversing the direction of the left line segment (taking into account the minus sign in the formula) we see that what we have is a closed curve: Ampere-Maxwell says that there must exist some surface element of H which has this closed curve for its boundary.

(As an aside: an immediate consequence of the Ampere-Maxwell law is charge conservation. Indeed, by applying the boundary operator to both sides and using the obvious fact that the boundary of a boundary always vanishes, we infer that

$$0 = \partial^2 H = \partial j + \partial \dot{D} = \partial j + \dot{\rho}\,,$$

which is the so-called continuity equation phrased in chain language.)

7. Metric: Star Operator

That completes the list of Maxwell equations, but it does not complete the formulation of Maxwell's electrodynamics. For one thing, simple counting shows that the system of equations at hand is underdetermined; for another, we haven't used any metric operations (such as measuring the length of a curve, or the angle between intersecting curves); all of Maxwell's laws (as we presented them) were of a differential or even topological nature.

But electrodynamics *does* know about the geometry of space-time, and now the point has come for us to introduce it.

The metric is injected into the formalism by means of a linear operator, called the star operator, $\star$, which converts a surface element into the line element *perpendicular* to it, and vice versa. More generally, $\star$ turns k-chains into $(3-k)$-chains.

To be more precise, $\star$ is a *local* operator; in the picture shown here the left-hand side is normalized by the area of the surface element S, and the right-hand side by the length of the line segment γ; true equality is achieved in the limit where both (the area of S and the length of γ) are sent to zero. (For the experts: $\star$ sends chains not to chains but to deRham currents.)

The example demonstrates that $\star$ deals with orientation in the natural way: outer is sent to inner, and inner to outer.

This was the case of the orientation being given by a sense of direction; it works the same way if the orientation is given by a sense of circulation.

7.1. Constitutive Laws. — Using the star operator, we formulate the so-called constitutive laws relating field strengths E, B to excitations D, H:

$\star$ applied to the surface elements of E produces the line elements of D (on multiplying by the dielectric constant ε_0). Similarly, $\star$ applied to the surface elements of H produces the lines of B (on renormalizing by the magnetic permeability μ_0).

Maxwell's equations together with the constitutive laws form a consistent and beautiful system of equations which determine the electromagnetic field completely, if the charge density ρ and the current density j are given. Before putting that system to work in a few examples, I want to drive home a few comments:

8. Discussion

I advertise the chain picture for the intuitive approach it offers to electrodynamics. From my extensive teaching experience I can say that students easily grasp the notion of chains and the operations on them. In fact, after the initial shock – no textbooks are available! – they love it.

Some things not so transparent in the traditional vector formulation are *really* clear; for example: parity invariance is totally manifest! (Indeed, we never had to use the right hand. In contrast, in the traditional calculus one constantly uses the right-hand rule – since there is no curl without the right-hand rule – only to discover at the end that nothing depends on it.)

Students hear with satisfaction that the Maxwell equations they have made an effort to comprehend are the ultimate version. To pass from flat space electrodynamics to electrodynamics in the vicinity of a black hole, all you need to do is: go to the constitutive laws and replace the flat space $\star$ operator by the curved space $\star$ operator; **Maxwell's equations remain the same!**

A fourth statement is this: if you're ever in doubt about the true and precise nature of a physical observable, think about how to MEASURE it (and you'll find out.)

9. Discussion (II)

For example, the electric excitation or displacement field D can be measured by Maxwell's double plate experiment. (I assume that most of you know this, so I don't have to review the details.) By this procedure D is unambiguously identified as a 1-chain with inner orientation (or, by Poincaré duality, as a 2-form of odd type).

Here I've sketched a possible gedanken-measurement of the magnetic excitation H. To probe for H, use a solid superconductor, say a long thin cylinder. The superconductor screens the magnetic field (provided it is in the Meissner phase, which I'll assume) by activating surface supercurrents. Thus a cylindrical hole is punched into H, and by Ampere's law the boundary of H equals the lines of the supercurrent.

In short, by picking up the total current around the cylinder, you're measuring the line integral of H along the cylinder. Because what's measured is a line (integral), the observable must consist of surfaces [in 3 space dimensions lines are paired in an invariant and non-degenerate way with surfaces (and only with surfaces), by intersection of sets], and since the current circulates, a sense of circulation is naturally given to H.

10. Aharonov-Casher Effect

This is a picture we've already seen: it's the current currying coil, moving with speed v. I am going to use this picture to motivate the so-called Aharonov-Casher effect, which arises from the relativistic phenomenon of an electrically neutral particle (say, a neutron) interacting with an electric field through its spin-magnetic moment. (In nuclear and condensed matter physics this is also known as spin-orbit coupling.)

The next picture looks familiar, too: it illustrates the electric field strength E carried along by the moving coil. The green lines are the surfaces of E intersected with the plane of the drawing.

By the electrical constitutive law, the 2-chain of E is accompanied by the lines of the electric excitation D, which are perpendicular to the surfaces of E.

Gauss' law then implies that some charges are situated on the coil (which had been neutral when at rest). The neutral coil is charged all of a sudden – a relativistic effect!

The resulting charge distribution can be characterized by its multipole moments. If we're interested only in the leading component, namely the electric dipole moment, the calculation gets very simple.

In the dipole limit we may approximate the coil as a line with magnetic flux Φ and no transverse extension. According to Ampere's law and the magnetic constitutive law, Φ is the product of μ_0 times the magnetic dipole moment of the coil per unit of length (ν_m). En route to the charge density one multiplies by μ_0, ε_0 and v. Since $\varepsilon_0\mu_0 = 1/c^2$ (the inverse square of the speed of light), the electric dipole moment of the moving coil (again per unit of length) works out to be $\nu_\mathrm{e} = \nu_\mathrm{m} \times v/c^2$.

What's important here is that this is the *exact* result, even when the coil is moving at a speed close to the speed of light. While the relativistic effect of length contraction makes the cross section of the coil look oblate, the magnetic field strength is enlarged at the same time, so that the total magnetic flux Φ remains the same. (Magnetic flux is a relativistic invariant; if it wasn't, there would be serious trouble with the Aharonov-Bohm effect of quantum mechanics.)

Moreover, the result remains correct when the long thin coil is replaced by an elementary particle (say, a neutron) with a spin-magnetic moment. We then have: (effective) electric dipole moment = vector product of the particle velicity with the spin-magnetic moment, divided by c^2 — that's the classical physics behind the Aharonov-Casher effect. (In the quantum theory, the resulting interaction with an electric field leads to a geometric phase which can be made visible in interference experiments.)

11. ElectroMagnetostatics (I)

Next, let's do a little bit of electro-/magnetostatics, and let's consider an electric dipole layer S, with electric dipole moment per unit area given by ν. I claim that the charge density of the dipole layer with surface S is *minus* ν *times the boundary of* $\star S$. This is verified as follows:

Here is the surface S; we assume it comes with an *outer* orientation. Applying the star operator to S turns it into a "hairy" object: the line elements of the 1-chain $\star S$ are tiny pieces of hair perpendicular to the surface. Taking the boundary (end point minus initial point) gives an arrangement of dipoles; the minus sign places the positive pole below and the negative pole above the surface. On dimensional grounds, multiplication by the dipole moment per unit area turns this into a charge density.

It's really a lot of fun playing with chains, and the $\star$ operator, the boundary operator, $\iota(v)$ etc.

12. EletroMagnetostatics (II)

Consider now a dipole layer in the form of an arbitrarily shaped surface S; it will excite some electric field D.

Compare this to the magnetostatic problem where a current is sent around the boundary loop of the surface, producing a magnetic field B.

I now pose to you the question: Is there any relation between the fields of the two problems? [You might be inclined to say NO, as the magnetic problem only knows about the location of the current loop, whereas the input to the electric problem is an entire surface.]

12.1. ElectroMagnetostatics (II) (again). — Well the answer, already known to Ampere, is that the fields are the same (apart from an obvious dimensional factor), *outside* the vanishingly small space inside the layer, i.e. the dipoles.

A method of proof is readily suggested by the chain picture. We make an ansatz for D as a sum of two terms: the tiny hair connecting the ends of the dipoles, plus some remainder D_1 .

Now we do a two-line calculation. In Gauss' law we replace ρ by the expression just explained ($\rho = -\nu\partial\star S$) and D by our ansatz. Two terms cancel, which motivates the ansatz, and it follows that the remainder D_1 has zero boundary.

But we are doing electrostatics, so the electric field strength E has zero boundary, too. E is proportional to the star of D (the constant of proportionality doesn't matter as $\partial E = 0$ is a null condition), so on inserting the ansatz we learn that the boundary of the star of D_1 is a factor times the boundary of S. Apparently, D_1 forgets the location of the surface S, and remembers only the boundary.

Turning to the magnetic side, recall that B is always closed. Since everything is static, Ampere's law applies, and in combination with the constitutive law relating B and H, we have that the boundary of the star of B is proportional to the boundary of S.

We see that D_1 and B satisfy the same set of equations. By the principle that the same equations have the same solutions the claim now follows, for the stray field (or the "leaky" part) of D, i.e. the part that's left after taking away the infinitesimal hairy stuff.

13. ElectroMagnetostatics (III)

This correspondence is not accidental but is a special case of a (little known) duality between magneto- and electrostatics. As a second case, let me mention the magnetostatics of a long thin coil. This problem is equivalent to the electrostatic problem of two monopoles placed at the end points of the coil. What we learn from this in particular is that the magnetic field outside a long thin coil knows only about the two end points, not about the exact whereabouts of the coil in between.

If the total electric charge vanishes, we can also go in the opposite direction, from electro- to magnetostatics. For example, the electric field of two parallel identical capacitor plates is the same (apart from a factor) as the magnetic field of the current carrying cylinder that connects the two plates.

14. Discharge of a capacitor (I)

Enough of statics! Let's get to a dynamical problem: discharge of a capacitor. We won't take the sketch here seriously, but will treat the charge distribution of the capacitor in dipole approximation (with electric dipole moment u). For later use, please note that o denotes the central point of the arrangement.

Now let the capacitor be discharged by accident or plan, at time $t = 0$. The process of discharging makes itself felt by the emission of an electromagnetic signal, and it is this signal that we wish to compute.

To prepare the solution, we write down the electric charge density ρ (prior to the discharge): attaching the dipole vector u to the point o and taking the boundary of the resulting (infinitesimal) line segment, we get an electric dipole, with the positive pole located below and the negative one above.

In the process of the discharge, the charge separation is undone; if we attach to o the negative of the dipole vector, we get the line segment of the time-integrated current pulse of the discharge.

15. Discharge of a capacitor (II)

After this preparation we tackle the calculation of the dynamical process. By simply combining the basic equations we easily find the inhomogeneous wave equation for the magnetic field strength: $\Box B = \mu_0 \partial \star j$. (Here $\Box$ is the d'Alembert or wave operator.)

Since the electric current density j vanishes at all times except zero, B satisfies the homogeneous wave equation $\Box B = 0$ for all times $t > 0$. What are the initial conditions for B? Since the *second* time derivative of B is to yield the singular (δ-type) current density, B itself is a continuous function of time at $t = 0$, and since it vanishes before the discharge it still does so immediately afterwards. However, the first time derivative, $\dot{B}$, must have a discontinuity in order for the second time derivative to yield the δ-profile of j; the jump of $\dot{B}$ is determined by the time-integrated current pulse just written down.

In this way we reformulate the problem as an initial-value problem for the homogeneous wave equation. To solve the latter, we use the Green's function of the wave equation. By this I mean (in the present context) a 0-chain, i.e. an object consisting of points, Σ_t. The points of Σ_t all lie on the surface of a sphere centered at o and expanding with the speed of light (radius ct). The total "mass" (i.e., the sum of the weight factors of all points) of the 0-chain is t, i.e. in a discretization with N points every point carries weight t/N.

The 0-chain so defined can be shown to satisfy the homogeneous wave equation. (More precisely, Σ_t is a weak solution of the wave equation; since Σ_t is not differentiable in the usual sense, $\Box\Sigma_t$ must be defined in the sense of distributions.)

The short-time asymptotics of Σ_t is easily stated: the linear time dependence of the total mass makes $\lim_{t\to 0+}\Sigma_t$ vanish; in the first time derivative $\dot\Sigma_t$ this time dependence is differentiated away, hence $\lim_{t\to 0+}\dot\Sigma_t = o$ (i.e. the degenerate 0-chain whose points are all collapsed into the center o).

Comparing with the initial value problem for B, we see that B and Σ_t satisfy equations of identical structure; to get from $\dot\Sigma_{t=0+}$ to $\dot B|_{t=0+}$, we just need to apply the operator $-\varepsilon_0^{-1}\partial \star \iota(u)$.

16. Discharge of a capacitor (III)

Since that operator commutes with the wave operator $\Box$, the solution for B is immediately obtained in the form

$$B|_t = -\varepsilon_0^{-1}\partial \star \iota(u)\Sigma_t \,.$$

So we've already found the solution. Let us now illustrate it by a picture. To keep the picture transparent, I'll show only the contribution from a circle of constant latitude.

In the first step, we apply $\iota(u)$: what results is a fence of line segments parallel to the dipole vector u. Next is the star operator, turning the fence of line segments into the ribbon of perpendicular surface elements. Finally, the boundary has to be taken; the inner boundaries all cancel and what remains are two closed lines (one just a little inside and the other just a little outside the sphere of radius ct). After multiplication by the proper weight factor, these are to be interpreted as the flux lines of the magnetic field strength to be computed.

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Remark: The original version of this talk ended with an
illustration of the quantum Hall effect in the chain picture.
See the final slides of the slide show.
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17. Feynman's problem revisited

Let us close the circle by returning to our starting point: the disappearing magnetic lines. How does the chain picture deal with the problem Feynman complains about?

Consider a positive point charge at rest. By the laws of electrostatics, half lines of electric flux emanate from it. These lines are static, and there is no magnetic field.

Now let the point charge be in motion (with constant velocity v, for simplicity). The lines of D then sweep out world surfaces, which consist of infinitesimal surface elements of the magnetic excitation H in such a way that the boundary of H equals the closed 1-chain made from the sum of the electric current density j and the electric displacement current $\dot D$. The [slide] shows some of these surfaces of $H = -\iota(v)D$.

The magnetic constitutive law ($B = \mu_0 \star H$) implies the existence of lines of B perpendicular to the surfaces of H, as shown.

This rough picture can be made quantitative by refining the discretization, arranging the electric flux lines in an isotropic way (that's for low speeds; for high speeds there is a Lorentz contraction in the direction of v), and translating the hedge-hog configuration of D with velocity v to produce the infinitesimal surfaces of H. All this is consistent and correct if not fully intuitive.

Recalling the quote from Feynman: "The gear wheels or lines disappear when you ride along with the object", our answer is this: well of course they do, as they owe their very existence (as lines perpendicular to world surfaces swept out by moving electric flux lines) to the motion of the charged object.

17.1. Summary. — Within the traditional vector calculus of electrodynamics, one encounters difficulties when trying to draw consistent pictures of the electromagnetic field using geometric objects. These difficulties are unnecessary; they arise from (i) giving up the important distinction between D (H) and E (B) and (ii) modeling electric and magnetic fields as vector fields. The difficulties go away when the fields are modeled as chains as described in this talk. A crucial aspect is that E and H must be modeled as 2-chains, whereas B and D are 1-chains.

As a final remark, note that Feynman's other complaint (that the ideas of field lines do not contain the superposition principle) is immaterial. In fact, our chain picture perfectly fits with the superposition principle: chains, by definition, can be added and subtracted just like vectors.

[Here the story ends in the sound of Feynman's drumming ...]

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