From Random Matrices to Supermanifolds

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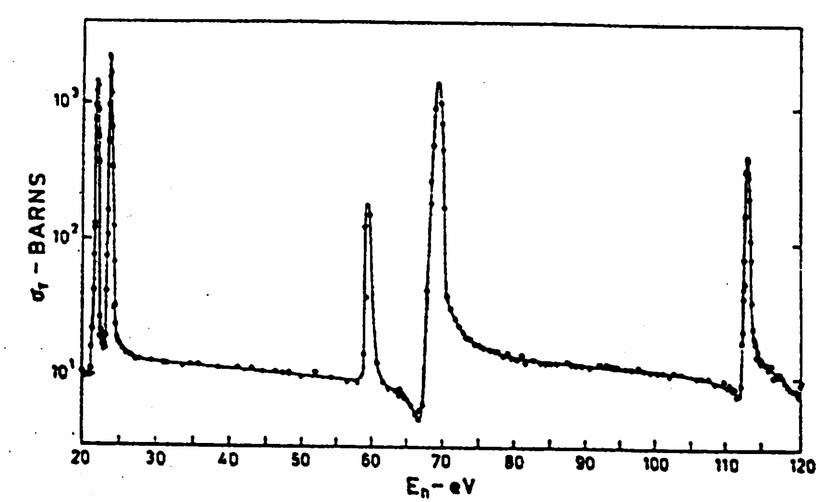
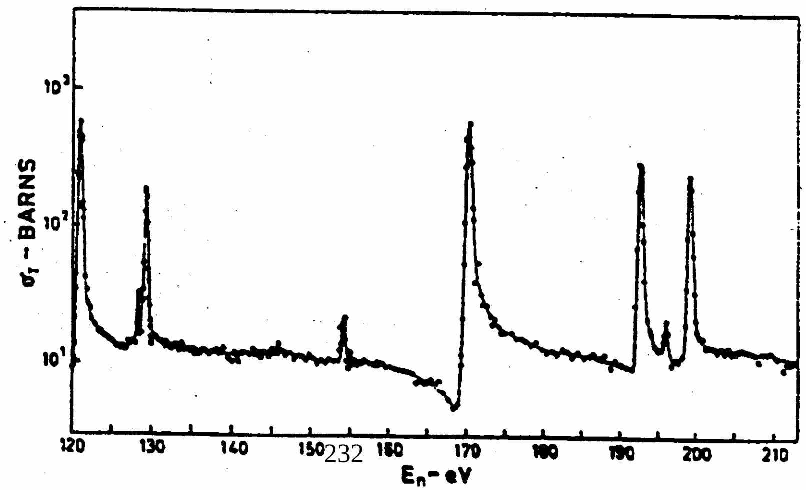
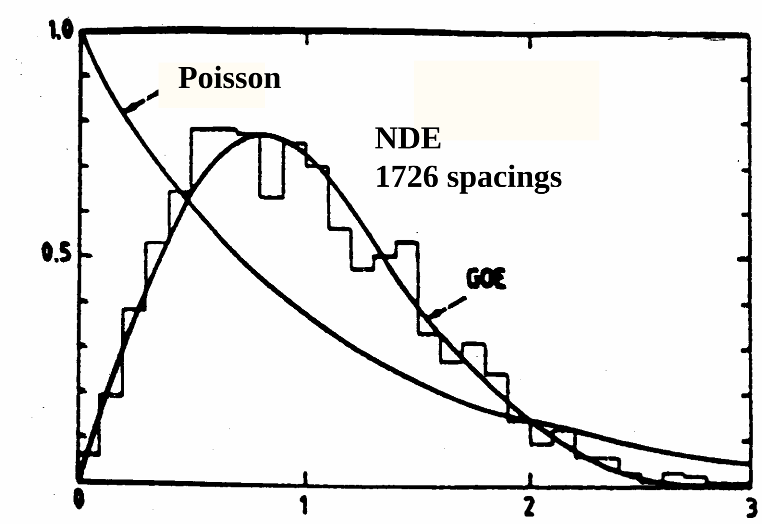
- Why random matrices? What random matrices?
- Which supermanifolds?
- Random matrix problems lead to questions about which supersymmetric field theories?
- Some results: spontaneous symmetry breaking, …

Nuclear Data Ensemble





Total cross section versus c.m. energy for scattering of neutrons on 232Th.
The resonances all have the same spin 1/2 and positive parity.



Wigner 1955

Nearest-neighbor spacing distribution for the ``Nuclear Data Ensemble´´ comprising 1726 spacings. For comparison, the RMT prediction labelled GOE and the result for a Poisson distribution are also shown.

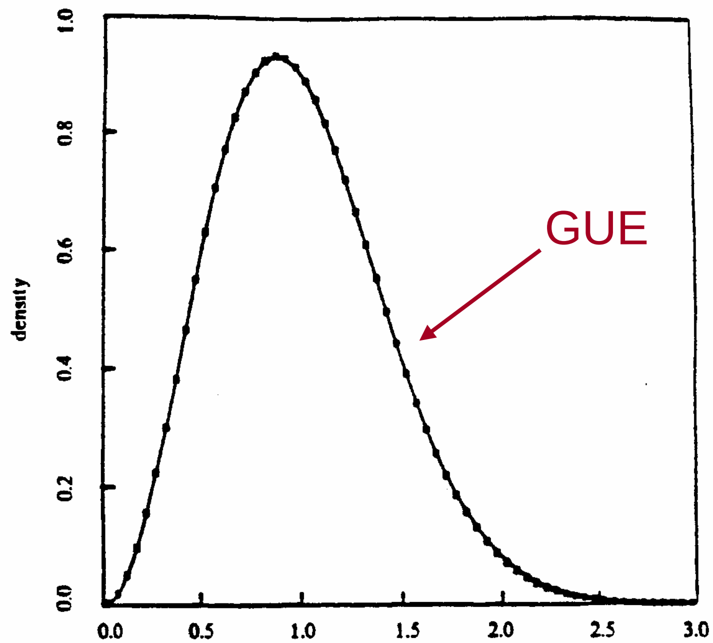
Universality of spectral fluctuations

In the spectrum of the Schrödinger, wave, or Dirac operator for a large variety of physical systems, such as

- atomic nuclei (neutron resonances),
- disordered metallic grains,
- chaotic billiards (Sinai, Bunimovich),
- microwaves in a cavity,
- acoustic modes of a vibrating solid,
- quarks in a nonabelian gauge field,
- zeroes of the Riemann zeta function,

one observes fluctuations that obey the laws given by random matrix theory for the appropriate Wigner-Dyson class and in the ergodic limit.

Spacing distribution of the Riemann zeroes



from A. Odlyzko (1987)

FIGURE 3: NORMALIZED SPACINGS (NEAREST NEIGHBOUR BETWEEN THE ZEROS OF $\zeta(s)$). IN THE NOTATION OF (2.12), μ_1 (RIEMANN, 10^{20}, 70×10^6) VERSUS μ_1 (CLASSICAL)

Wigner-Dyson universality

Wigner-Dyson symmetry classes:

- A : complex Hermitian matrices (‘unitary class’, GUE)
- AI : real symmetric matrices (‘orthogonal class’, GOE)
- AII : quaternion self-dual matrices (‘symplectic class’, GSE)

Dyson (1962, The 3-fold way): ``The most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of three known types.’’

This classification has proved fundamental to various areas of theoretical physics, including the statistical theory of complex many-body systems, mesoscopic physics, disordered electron systems, and the field of quantum chaos.

Outline

- Motivation: universality of disordered spectra
- Symmetry classes of disordered fermions: 10-fold way
- Riemannian symmetric superspaces (an example)
- Recent results: diffusion in a SUSY hyperbolic sigma model

Symmetry Classes of Disordered Fermions

- Heinzner, Huckleberry, MRZ,
  Commun. Math. Phys. 257 (2005) 725
- MRZ, Encylopedia of Mathematical Physics,
  vol.5, 151-160 (Elsevier, 2006)

Symmetry classes: setting & motivation

Consider one – particle Hamiltonians (fermions) :

$$H = \tfrac{1}{2}\sum W_{\alpha\beta}\,(c_\alpha^* c_\beta - c_\beta\, c_\alpha^*) + \tfrac{1}{2}\sum (Z_{\alpha\beta}\, c_\alpha^* c_\beta^* + \overline{Z}_{\alpha\beta}\, c_\beta\, c_\alpha)$$

Canonical anticommutation relations : $c_\alpha^*\, c_\beta + c_\beta\, c_\alpha^* = \delta_{\alpha\beta}$

Applications/examples :

- Hartree – Fock – Bogoliubov theory of superconductors
- Dirac equation for relativistic spin 1/2 particles

Following Dyson, classify such Hamiltonians according to symmetries! What are the irreducible blocks that occur?

Conjecture (Altland & MRZ, 1996) :

Classification by large families of symmetric spaces

Proof of conjecture

Disordered fermion systems with quadratic Hamiltonians:

$$H = \tfrac{1}{2}\sum W_{ij}\, c_i^* c_j + \tfrac{1}{2}\sum Z_{ij}\, c_i^* c_j^* + \text{h.c.}$$

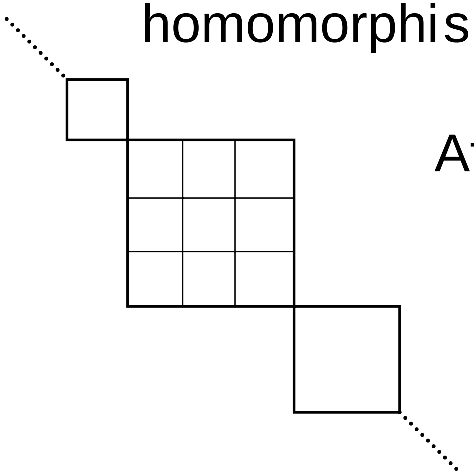
$$= \tfrac{1}{2}\begin{pmatrix} c^* & c \end{pmatrix}\begin{pmatrix} W & Z \\ Z^* & -W^{\mathrm{t}} \end{pmatrix}\begin{pmatrix} c \\ c^* \end{pmatrix} \qquad (W^* = W,\ Z = -Z^{\mathrm{t}}).$$

Nambu space $V \oplus V^*$ with symmetric bilinear form by C.A.R.

Unitary and antiunitary symmetries: $G = U \cup TU$ (U compact).

Decompose into irreducible blocks of U – equivariant

homomorphisms $R \to V$ and transfer structure (C.A.R. $+\, T$)



After transfer, every set of block data specifies
a classical irreducible symmetric space,
and every classical irreducible symmetric
space occurs in this way.
(Heinzner, Huckleberry, MRZ; 2005)

The ten large families of symmetric spaces

family	symmetric space	form of $H = \begin{pmatrix} W & Z \\ Z^* & -W^t \end{pmatrix}$
A	$\mathrm{U}(N)$	complex Hermitian
$A\mathrm{I}$	$\mathrm{U}(N)/\mathrm{O}(N)$	real symmetric
$A\mathrm{II}$	$\mathrm{U}(2N)/\mathrm{USp}(2N)$	quaternion self – adjoint
C	$\mathrm{USp}(2N)$	Z complex symmetric, $W = W^*$
$C\mathrm{I}$	$\mathrm{USp}(2N)/\mathrm{U}(N)$	Z complex symmetric, $W = 0$
D	$\mathrm{SO}(2N)$	Z complex skew, $W = W^*$
$D\mathrm{III}$	$\mathrm{SO}(2N)/\mathrm{U}(N)$	Z complex skew, $W = 0$
$A\mathrm{III}$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	Z complex $p\times q$, $W = 0$
$BD\mathrm{I}$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	Z real $p\times q$, $W = 0$
$C\mathrm{II}$	$\mathrm{USp}(2p+2q)/\mathrm{USp}(2p)\times\mathrm{USp}(2q)$	Z quaternion $2p\times 2q$, $W = 0$

Physical realizations

AI	electrons in a disordered metal with conserved spin and with time reversal invariance invariance
A	same as AI, but with time reversal broken by a magnetic field or magnetic impurities
AII	same as AI, but with spin-orbit scatterers
CI	quasi-particle excitations in a disordered spin-singlet superconductor in the Meissner phase
C	same as CI but in the mixed phase with magnetic vortices
DIII	disordered spin-triplet superconductor
D	spin-triplet superconductor in the vortex phase, or with magnetic impurities
AIII	massless Dirac fermions in SU(N) gauge field background (N > 2)
BDI	same as AIII but with gauge group SU(2) or Sp(2N)
CII	same as AIII but with adjoint fermions, or gauge group SO(N)

Altland, Simons & MRZ, Phys. Rep. 359 (2002) 283

Random matrix methods

Methods based on the joint probability density for the eigenvalues of a random matrix:

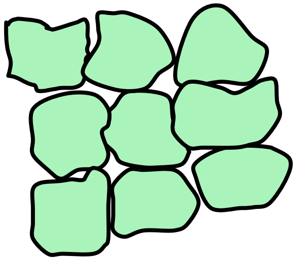
- Orthogonal polynomials + Riemann-Hilbert techniques
- (Scaling limit) reduction to integrable PDE's (Painleve-type)

In contrast, superanalytic methods apply to band random matrices, granular models, random Schrödinger operators etc.

- Hermitian (or Hamiltonian) disorder:
  Hubbard-Stratonovich Schäfer-Wegner method (1980) ,
  see MRZ, arXiv:math-ph/0404057 (EMP, Elsevier, 2006),
  superbosonization (2007)
- Unitary (scattering, time evolution) disorder:
  color-flavor transformation (1996), Howe duality (2004)

Wegner's N-orbital model (class A)

Hermitian random matrices H for a lattice Λ with N orbitals per site $i \in \Lambda$.



Hilbert space $V = \oplus_{i\in\Lambda} V_i\ , \qquad V_i = \mathbb{C}^N .$

Orthogonal projectors: $\Pi_i : V \to V_i\,.$

Fourier transform of probability measure $d\mu(H)$:

$$\int \exp(-\mathrm{i}\,\mathrm{Tr}\, HK)\, d\mu(H) \;=\; \prod \Omega_{ij}(K_{ij}), \quad K_{ij} = \Pi_i\, K\, \Pi_j\,,$$

local gauge invariance: $\mathrm{U}(V_1)\times \mathrm{U}(V_2)\times \ldots \times\, \mathrm{U}(V_{|\Lambda|})$.

Gaussian distribution as a special case:

$$\prod \Omega_{ij}(K_{ij}) = \exp\Big(-\tfrac{1}{2N}\sum\nolimits_{i,j} C_{ij}\,\mathrm{Tr}\,\Pi_i K\,\Pi_j K\Big)$$

$$i = j:\ \ C_{ii} = \mathcal{O}(N^0)\,, \qquad i \neq j:\ \ C_{ij} = \mathcal{O}(N^{-1}).$$

Riemannian symmetric superspaces

and the correspondence with supersymmetric nonlinear sigma models

Symmetric supermanifolds: an example

Hermitian vector space $U = \mathbb{C}^{p+q}$.

The space of all orthogonal decompositions

$$U = U^{+} \oplus U^{-} \cong \mathbb{C}^{p} \oplus \mathbb{C}^{q}$$

is a Grassmann manifold $\mathrm{U}_{p+q}/(\mathrm{U}_{p} \times \mathrm{U}_{q}) = M_1$.

Pseudo – hermitian vector space $V = \mathbb{C}^{p+q}$ of signature (p,q).

The pseudo – orthogonal decompositions

$$V = V^{+} \oplus V^{-} \cong \mathbb{C}^{p} \oplus \mathbb{C}^{q}$$

form a non – compact Grassmannian $\mathrm{U}_{p,q}/(\mathrm{U}_{p} \times \mathrm{U}_{q}) = M_0$.

Globally symmetric Riemannian manifold : $M = M_1 \times M_0$

(of type AIII)

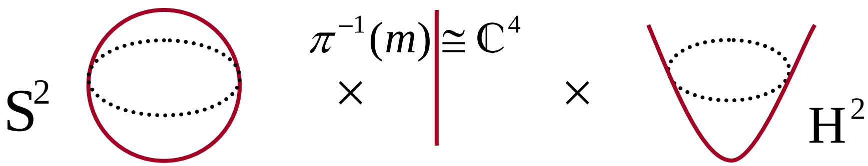
Example (cont'd)

Vector bundle $E \xrightarrow{\pi} M$.

A point $m \in M$ determines $U = U^+ \oplus U^-$, $V = V^+ \oplus V^-$.

Fibre
$$\pi^{-1}(m) = \mathrm{Hom}(U^+, V^-) \oplus \mathrm{Hom}(U^-, V^+) \oplus \mathrm{Hom}(V^-, U^+) \oplus \mathrm{Hom}(V^+, U^-) \,.$$

Minimal case:



The algebra of sections $\Gamma(M, \wedge E^*)$ carries a canonical action of the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(U \oplus V) \cong \mathfrak{gl}_{p+q|p+q}$.

Riemannian symmetric superspace $(M, \wedge E^*, \mathfrak{g})$.

Riemannian metric structure

Graded – commutative algebra of sections $\mathcal{A} = \Gamma(M, \wedge E^*)$

Invariance w.r.t. $\mathfrak{g}$ – action on $\mathcal{A}$ determines metric tensor

$$g: \mathrm{Der}\mathcal{A} \times \mathrm{Der}\mathcal{A} \to \mathcal{A}$$

Supersymmetric sigma model is functional integral of maps

$$\varphi: \mathbb{R}^d \supset \Sigma \to E$$

Action functional is given by the metric tensor in the usual way.

Riemannian structure is important for stability!

The 10-Way Table

Correspondence between random matrix models and supersymmetric nonlinear sigma models:

RME	A	AI	AII	C	CI	D	DIII	AIII	BDI	CII
noncomp.	AIII	BDI	CII	DIII	D	CI	C	A	AI	AII
susy NLsM										
compact	AIII	CII	BDI	CI	C	DIII	D	A	AII	AI

MRZ, J. Math. Phys. 37 (1996) 4986

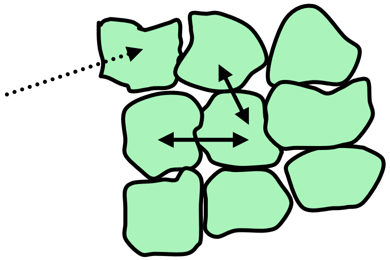
Some analytical estimates

for (super)symmetric nonlinear sigma models

Granular Model (deformed)

(Weakly coupled GUE's)

disordered metallic grain with N electron states



probability measure :

$$d\mu(H) = c\,|\,\mathrm{Det}\,(E + \mathrm{i}\varepsilon - H)\,|^{-2}\,\mathrm{e}^{-\sum_{i,j} J^{-1}_{ij}\,\mathrm{Tr}\,(\Pi_i\, H\, \Pi_j\, H)}\,dH$$

$$\left\langle |\,(E + \mathrm{i}\varepsilon - H)^{-1}(0,0)\,|^2 \right\rangle_{\mu} =$$

$$\int \mathrm{e}^{-\frac{1}{4}\sum_{i,j} J_{ij}\,\mathrm{Tr}\,(sQ_i\, sQ_j) - \sum_k \mathrm{Tr}\,(\varepsilon - \mathrm{i}sE)Q_k}\, Q_{0,11}\, Q_{0,22} \prod_{k\in\Lambda} \mathrm{Det}^N(Q_k)\, d\nu(Q_k),$$

2×2 Hermitean matrices $Q_k > 0$, $\qquad s = \mathrm{diag}\,(1,-1)$.

Noncompact global symmetry (at $\varepsilon = 0$) :

$$Q_k \to TQ_kT^*, \qquad T^* = sT^{-1}s \in \mathrm{SU}(1,1).$$

Origin of Hyperbolic Symmetry

$$|\,\mathrm{Det}(E+\mathrm{i}\varepsilon-H)\,|^{-2}=\int \mathrm{e}^{+\mathrm{i}(\overline{\varphi}_1,\,(E+\mathrm{i}\varepsilon-H)\varphi_1)}\,\mathrm{e}^{-\mathrm{i}(\overline{\varphi}_2,\,(E-\mathrm{i}\varepsilon-H)\varphi_2)}$$

Note the sign change forced by the requirement of convergence of the integral. Hence we are dealing with a quadratic form of indefinite signature.

Sigma Model Approximation

SU(1,1) orbit $M = TT^* \equiv x$ is H^2 (2-hyperboloid)

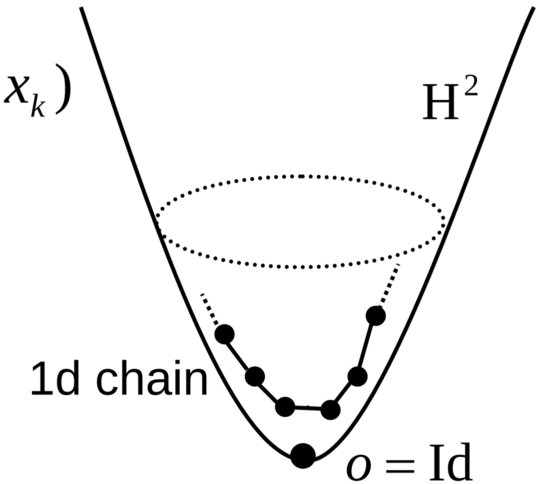
Restrict to critical manifold $\mathrm{H}^2 \times \ldots \times \mathrm{H}^2$

(by eliminating the massive modes).

$$\left\langle |(E + \mathrm{i}\varepsilon - H)^{-1}(0,0)|^2 \right\rangle_\mu \cong \left\langle \cosh^2 \circ \operatorname{dist}(x_0, o) \right\rangle_S$$

Gibbs measure $e^{-S} \prod_k d\,\mathrm{vol}(x_k)$

$$S = \beta \sum\nolimits_{ij}^{/} \cosh \circ \operatorname{dist}(x_i, x_j) + \varepsilon \sum\nolimits_k \cosh \circ \operatorname{dist}(x_k, o)$$



Spontaneous Symmetry Breaking

Theorem (Spencer and MRZ, CMP 252 (2004) 167):

$$\left\langle \cosh^2 \circ \operatorname{dist}(x_0, o) \right\rangle_S \leq \text{const} \quad \text{if} \quad \varepsilon \cdot \text{vol} \geq 1$$

and if $d \geq 3$ and β is not too small.

Remark: This bound means that the field stays near the origin. In the infinite-volume limit it implies regularity of the Green's function in ε (in sigma model approximation) and is consistent with the existence of extended states.

Proof: Use the Iwasawa decomposition $G = NAK$ for SU(1,1). Integrate out the nilpotent degrees of freedom, resulting in convex action for the torus variables. Apply the Brascamp-Lieb inequality.

SUSY hyperbolic nonlinear sigma model

Advantage of toy model :

Grassmann variables can be integrated out,

and effective action action is real,

so probabilistic methods apply.

$$Z_\Lambda(\beta,\varepsilon) = \int e^{-\beta\sum_{<i,j>} \cosh(t_i - t_j)} \operatorname{Det}^{1/2} D_\Lambda(t) \prod_{k\in\Lambda} e^{-t_k - \varepsilon \cosh t_k} dt_k \,,$$

$$[s; D_\Lambda(t)\, s] = \beta \sum_{<i,j>} e^{t_i + t_j} (s_i - s_j)^2 + \varepsilon \sum_{k\in\Lambda} e^{t_k} s_k^2 \;.$$

Equivalence to random walk in correlated random environment (edge reinforced random walk).

SUSY Ward identities

$\mathrm{OSp}(2|2)$ symmetry gives nontrivial relations, e.g.,

$$1 = \left\langle B_{xy}^{m}\,(1-\tfrac{m}{\beta}G_{xy})\right\rangle_{\Lambda,\beta,\varepsilon} ,$$

$$B_{xy} = \cosh(t_x - t_y) + \mathrm{e}^{t_x+t_y}(s_x - s_y)^2 ,$$

$$G_{xy} = \frac{\mathrm{e}^{t_x+t_y}}{B_{xy}}\,[(\delta_x - \delta_y); D^{-1}(\delta_x - \delta_y)] .$$

Main result

Theorem (Disertori, Spencer, MRZ; June 2008):

1. The fluctuations of the t field are small at all scales:

$$\left\langle \cosh^m (t_x - t_y) \right\rangle_{\Lambda,\beta,\varepsilon} \leq \text{ const },$$

uniformly in x, y and Λ as long as $\beta \gg m \gg 1$.

2. The average of the t field is bounded:

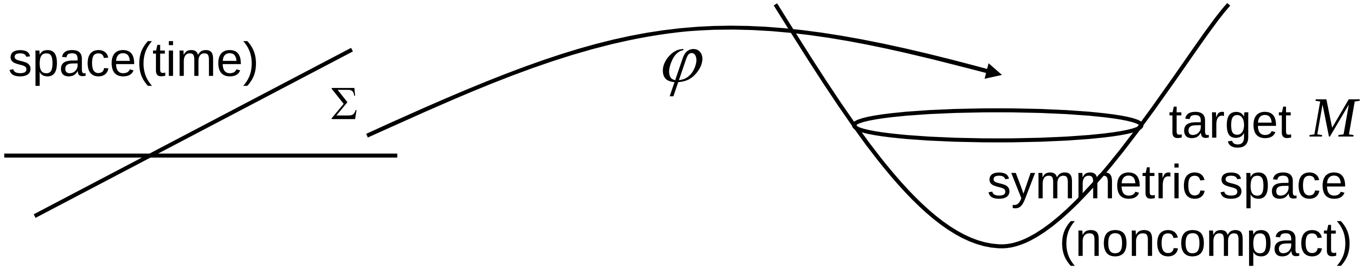
$$\left\langle e^{m t_x} \right\rangle_{\Lambda,\beta,\varepsilon} \leq \text{ const },$$

as long as $\varepsilon |\Lambda| = O(L^\alpha)$,

where $|\Lambda| = L^3$, $\beta \gg m \gg 1$. $\alpha > 0$.

Corollary: noncompact symmetry is spontaneously broken $\rightarrow$ diffusion (extended states)!

Noncompact nonlinear sigma models



Energy (action) function: $S = \int_\Sigma |D\varphi|^2 = \int d^d x \; g^{\mu\nu}_{(\Sigma)} \partial_\mu \varphi^a \partial_\nu \varphi^b g^{(M)}_{ab}$

Regularization: lattice $\Lambda \subset \Sigma$

Targets M : $\mathrm{SO}_{2,1}/\mathrm{SO}_2$, $\mathrm{U}_{p,q}/\mathrm{U}_p \times \mathrm{U}_q$, ...

d = 1: M. Niedermaier, E. Seiler, arXiv:hep/th-0312293

d = 2: Duncan, Niedermaier & Seiler, Nucl. Phys. B 720 (2005) 235

d = 3: Spencer & MRZ, Commun. Math. Phys. 252 (2004) 167