ON THE SYMMETRY CLASSIFICATION OF TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS

by

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Abstract. — This is the text of my 50 minute talk. To read and understand it, you are advised to consult the enclosed slides.

1. Periodic Table

We have already heard several talks about topological insulators (and superconductors) at this meeting, so I hope it's okay for me not to give another introduction to the subject, but just remind you of some of the major examples as we go along.

Let me begin by showing the so-called "Periodic Table". Some four years ago, most if not all of the examples of topological insulators and superconductors known at that time were put into a table [of homotopy groups] by the Santa Barbara group, following Kitaev and their own earlier work. This included, among others, the quantum spin Hall insulator and the insight that He-3B is a topological superfluid. The Periodic Table caused quite a stir in the community; in any case, it has been reprinted in numerous articles including a major review by Hasan and Kane, and it has been discussed in a number of papers by various groups. Those of some importance for this talk are listed at the bottom of this slide. My goal here is to add some perspective to this striking table. At the end, I will sketch a new proof of the table based on homotopy theory, and nothing but homotopy theory. (In particular, there will be no K theory.)

2. Ten-Way Classification

My talk is organized in two major parts. In the first part, I will review a classification result dubbed the "Tenfold Way". While conceived for the symmetry classification of disordered fermions, an adaptation thereof serves the goal of classifying (symmetry-protected) topological ground states. In the second part, I will concentrate on systems with translation symmetry and look at the free-fermion ground states classified by the Periodic Table.

2.1. Quasi-particle vacua (I). — In this talk, we will go right to the heart of the matter! Thus I will not introduce any Hamiltonians and then follow the usual practice of flattening them, making them Dirac, or whatever, for the purpose of topological classification. Rather, we will look directly at the ground states, as follows.

Let $c_\alpha^\dagger$ denote the Fock operator creating a single particle in state α, and let c_α be the corresponding annihilation operator. The Fock vacuum is uniquely characterized by the property of being annihilated by all the annihilation operators:

$$c_\alpha \, |\text{vac}\rangle = 0 \qquad (\alpha = 1, 2, \ldots).$$

Now define the notion of *quasi-particle vacuum* by an analogous property,

$$\widetilde{c}_\alpha \, |\widetilde{\text{vac}}\rangle = 0 \qquad (\alpha = 1, 2, \ldots),$$

where the operators $\widetilde{c}$ are the result of making a Bogoliubov transformation:.

$$\widetilde{c}_\alpha = \sum_{\alpha'} \big(c_{\alpha'} \, u_{\alpha'\alpha} + c_{\alpha'}^\dagger \, v_{\alpha'\alpha} \big).$$

Such vacua are also referred to as many-body ground states in the Hartree-Fock-Bogoliubov mean-field approximation.

Note that the (large) family of quasi-particle vacua contains the family of n-particle Slater determinants as a special case. For this we set

$$\widetilde{c}_\alpha = c_\alpha^\dagger \quad (1 \le \alpha \le n), \quad \widetilde{c}_\alpha = c_\alpha \quad (\alpha > n).$$

2.2. Quasi-particle vacua (II). — Now, in order to handle the general situation with symmetries, we introduce the following notation.

Let U denote the vector space of operators annihilating a single particle, and let V be the corresponding space of creation operators:

$$\sum u_\alpha \, c_\alpha \in U \, , \quad \sum v_\alpha \, c_\alpha^\dagger \in V \, .$$

I use the word "Nambu space " for the orthogonal sum

$$W = U \oplus V.$$

This vector space carries a Hermitian scalar product defined by

$$\langle \psi \mid \psi' \rangle := \{ \psi^\dagger, \psi' \}, \quad \psi = \sum \big(u_\alpha \, c_\alpha + v_\alpha \, c_\alpha^\dagger \big) \in W,$$

where † means (taking) the adjoint operator and the curly brackets denote the anti-commutator. Thus W is what is called a Hermitian vector space.

Fact. There is a one-to-one correspondence between quasi-particle vacua and Hermitian subspaces $A \subset W$ subject to the conditions

$$\{A, A\} = 0 \, , \quad \dim A = \tfrac{1}{2} \dim W.$$

Thus A is a space of generalized annihilation operators. (Indeed, the anti-commutator of any pair of operators in A vanishes, and A has the largest possible dimension for this property to hold.)

Now in the presence of a group G of symmetries we require

$$g \cdot A = A \quad (\text{for all } g \in G).$$

We then ask: can a classification of G-invariant quasi-particle vacua be given?

2.3. What's a symmetry?— Let's start at the very beginning and recall what is meant by a symmetry in the context of quantum mechanics. Following Wigner (who was awarded the 1963 Physics Nobel Prize for his foundational work on symmetry principles and their application to nuclear and particle physics) a symmetry in quantum mechanics is primarily a transformation on the rays of Hilbert space, with the property that all transition probabilities are preserved.

Now one corner stone of our subject is a theorem (attributed to Wigner) stating that any quantum symmetry T lifts to Hilbert space as a linear operator, say $\hat{T}$, which is *either unitary, or anti-unitary*. In the former case the Hermitian scalar product of Hilbert space is preserved, in the latter case it is preserved up to complex conjugation.

I wish to make two remarks here. First comes the obvious statement that symmetries always form a group, G. Indeed, if two operators are symmetries, then so is their composition. The second remark is that in order for an operator $\hat{T}$ to be a symmetry of a quantum system with Hamiltonian H, we require that $\hat{T}$ *commutes* with H. Thus so-called chiral symmetries, which *anti-commute* with the Hamiltonian – a prominent example is the chirality operator γ_5 which anti-commutes with the massless Dirac operator – such "chiral symmetries" aren't symmetries, at least not in the sense of this talk. (I must emphasize this point because my work on symmetry classification is often cited in an abridged form that overlooks or ignores this crucial aspect.)

2.4. Setting: Fock space & and symmetries. — Next we specify the mathematical framework in which to carry out the classification; our setting is a refinement of that used by Freeman Dyson in his famous Threefold Way of random matrix theory.

2.4.1. Fock space. — First of all, we replace Dyson's general Hilbert space by the more elaborate structure of a Fock space, $\mathscr{F}$, built from a single-particle Hilbert space V. Since we are dealing with fermions, Fock space is formed by taking exterior powers; note that $\mathscr{F}_n = \wedge^n(V)$ denotes the n-particle sector. And, of course, the Fock operators creating and annihilating particles satisfy the canonical anti-commutation relations,

$$c_\alpha^\dagger c_\beta + c_\beta c_\alpha^\dagger = \delta_{\alpha\beta}\,, \quad c_\alpha c_\beta + c_\beta c_\alpha = 0 = c_\alpha^\dagger c_\beta^\dagger + c_\beta^\dagger c_\alpha^\dagger\,,$$

referred to as CAR for short.

2.4.2. Symmetries. — Following Dyson, we adopt the setting of a symmetry group G acting on Fock space by unitary and anti-unitary operators. We require the group of unitary symmetries to be defined on the single-particle space V and extend it to Fock space F in the natural way. [This requirement, I should say, excludes Yangian and other quantum group symmetries, which arise at the many-particle level.] There is no further restriction; thus the group of unitary symmetries may be quite arbitrary.

As for the anti-unitaries, we allow for the possible presence of time-reversal symmetry, which is defined on the single-particle Hilbert space and extends to Fock space in the usual way. Moreover, the structure of Fock space opens the possibility for another anti-unitary operation to be a symmetry; this is (twisted) particle-hole conjugation, C, which transforms a state of n particles into a state of n holes.

2.5. Step of reduction. — Recall that we want to classify ground states (more precisely: quasi-particle vacua). The main step in this quest is a reduction process by which the subgroup G_0 of unitary symmetries is essentially removed. To carry out this reduction, we first observe that the G-action on Fock space induces a G-action on Nambu space, and then we identify in W all irreducible representations of G_0. Let λ be an index that runs over the isomorphism classes of irreps that occur. All irreps in the same class can be viewed as identical copies of a standardized representation, say R_λ. They are gathered in a so-called G_0-isotypic component, which is the tensor product of R_λ with a factor accounting for multiplicity:

$$W = \bigoplus\nolimits_\lambda W_\lambda , \quad W_\lambda = \mathscr{H}_\lambda \otimes R_\lambda .$$

For a perfect mathematical formulation, one thinks of each representation of class λ as a linear mapping (or homomorphism) that takes the standard representation space R_λ and sends it as a G_0-representation into W:

$$\mathscr{H}_\lambda = \mathrm{Hom}_{G_0}(R_\lambda , W) \simeq \mathbb{C}^{m_\lambda} .$$

Phrased in simple terms, each multiplicity space $\mathscr{H}_\lambda$ is just $\mathbb{C}^{m_\lambda}$, where m_λ is the number of times the irreducible G_0-representation R_λ occurs in W. Let me illustrate this with two examples.

Example 1. Take G_0 to be a group Γ of space translations, which is Abelian. Then λ is the quantum number for momentum ($\lambda = k$), and all representation spaces are one-dimensional ($R_\lambda = R_k \simeq \mathbb{C}$). Our multiplicity space here is the space of Fock operators that lower the momentum by k:

$$\mathscr{H}_k = U_k \oplus V_{-k} ;$$

it is spanned by the annihilation operators at momentum k and the creation operators at momentum $-k$.

Example 2. For another example, take G_0 to be the (non-Abelian) group of SU_2 spin rotations. Then λ is the spin j of a single particle, and R_λ is the standard spin representation space of dimension $2j+1$. Here one has to deal with the fact that creating spin j isn't the same as annihilating spin j; therefore, some SU_2-equivariant isomorphism is employed in order to bring the decomposition into the desired form.

[For more examples and a leisurely account of this story, you may wish to consult my Oxford Handbook Article on *Symmetry classes* (arXiv:1001.0722).]

2.6. Ten-Way Classification. — We are now ready to make a statement. Let $G = G_0 \cup G_1$ be any group of unitary and anti-unitary symmetries acting on Fock space (as described above) and hence on Nambu space. We also recall that each quasi-particle vacuum is identified with its space A of annihilation operators. The statement then is

Theorem. Every G-invariant quasi-particle vacuum decomposes as an orthogonal sum:

$$A = \bigoplus\nolimits_\lambda A(\lambda), \quad A(\lambda) = x(\lambda) \otimes R_\lambda ,$$

where each vector space $x(\lambda)$ lies in some classifying space X_λ, of which there exist 10 different types. The latter are in bijection with the 10 large families of symmetric spaces.

Remark. This is an immediate corollary of a classification result proved by Heinzner, Huckleberry, and myself. But let me go slowly, and take some time to explain the meaning of the statement and the idea of its proof.

2.7. What's a symmetric space?— In Riemannian geometry there exists something called the Riemann curvature tensor. In a coordinate basis, it has the well-known expression

$$R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm} \, .$$

A (locally) symmetric space is defined to be a Riemannian manifold $X = U/K$ with a Riemann tensor which is covariantly constant:

$$\nabla R = 0 \, .$$

The simplest example of a such a space is the round two-sphere S^2,

$$X = \mathrm{S}^2 \, , \quad ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \, ,$$

with line element induced by the Euclidean distance of three-dimensional space. Another example is the set $\mathbb{C}^n \simeq V \subset \mathbb{C}^N$ of all complex n-dimensional subspaces in N dimensions. Such a space is called a Grassmann manifold, or a Grassmannian for short. Note that any two n-dimensional subspaces can be mapped into each other by a unitary transformation of the full space. Since nothing changes when we just transform within a fixed subspace and its orthogonal complement, one has the identification

$$X = \mathrm{Gr}_n(\mathbb{C}^N) = \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$$

of the Grassmannian with a quotient of unitary groups.

An important and well-known fact about symmetric spaces is that they were completely classified [the globally symmetric ones] by the French geometer Elie Cartan. Apart from a finite number of exceptional spaces, they come in 10 large families, which Cartan called A, AI, AII, AIII, BD, BDI, C, CI, CII, and DIII.

2.8. Quasi-particle vacua & symmetric spaces (I). — Next, for those of you who are not familiar with this story, I am going to give two examples of the correspondence between quasi-particle vacua and symmetric spaces.

The first one is the example with no symmetries whatsoever. In that case, denoting the dimension of single-particle Hilbert space by N, the space of all quasi-particle vacua is a symmetric space $X = \mathrm{O}(2N)/\mathrm{U}(N)$ (of Cartan type DIII).

To see this, take the Fock vacuum, say A_0 (spanned by the "bare" annihilation operators), and apply to it some element of the large group of Bogoliubov transformations, $\mathrm{O}(2N)$. The action of the Bogoliubov group $\mathrm{O}(2N)$ is transitive, so you'll be able to reach the most general quasi-particle vacuum from the Fock vacuum. Since the latter is invariant under the $\mathrm{U}(N)$ subgroup which simply transforms the c's amongst themselves, you get the claimed statement.

Let me remark that there exist several equivalent ways of looking at this symmetric space X. First of all, $X = \mathrm{O}(2N)/\mathrm{U}(N)$ may be viewed as the space of complex structures of $\mathbb{R}^{2N}$. In physical terms, this says that every point of X corresponds to a distinct way of decomposing $2N$ Majorana fermions into creation and annihilation parts. Second, X has 2 connected components. These correspond to quasi-particle vacua with even or odd fermion number. Third, the connected component with even fermion number is the space of BCS-states (Bardeen, Cooper, Schrieffer) of the most general form:

$$|\mathrm{BCS}\rangle = \exp\left(\sum Z_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger\right) |\mathrm{vac}\rangle .$$

2.9. Quasi-particle vacua & symmetric spaces (II). — For another example, let the symmetry group G be the $\mathrm{U}(1)$ group underlying charge (or particle number) conservation. In this case, quasi-particle vacua are Hartree-Fock ground states, also known as Slater determinants. To phrase this in our language, note that the decomposition into isotypic components now is the decomposition

$$W = U \oplus V$$

into annihilation and creation operators. $A = A(-) \oplus A(+)$ here is a sum spanned by some annihilation operators $A(-) \subset U$ and creation operators $A(+) \subset V$. If $\dim V = N$ and $\dim A(+) = n$ (that's the particle number), then $A(+)$ is a point of the Grassmannian $X = \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$. Physically speaking, $A(+)$ may be identified with the space of occupied states, and $A(-)$ with the space of empty states.

2.9.1. Riemannian structure?— Since I defined symmetric spaces as Riemannian manifolds, you may ask how one puts a Riemannian geometry on the space of quasi-particle vacua. The answer is that one takes the geodesic distance between two such vacua to be the positive number t if the ray of one is obtained from the ray of the other by applying the exponential of an anti-Hermitian particle-hole type operator of norm t:

$$\mathrm{dist}(x_1, x_2) = t \iff \mathscr{R}\,|\widetilde{\mathrm{vac}}_2\rangle = \mathscr{R}\,\mathrm{e}^S |\widetilde{\mathrm{vac}}_1\rangle, \quad \|S\| = t.$$

Here I am using the notation "little x" ($x \equiv \mathscr{R}\,|\widetilde{\mathrm{vac}}\rangle$) for the ray of the ground state (omitting the irrelevant overall phase factor).

2.10. Proof of classification theorem. — So much for background underlying the classification theorem of HHZ. A brief sketch of its proof is as follows:

1. Recall the decomposition of Nambu space into isotypic components with respect to the subgroup of unitary symmetries G_0:

$$W = \bigoplus\nolimits_\lambda W_\lambda, \quad W_\lambda = \mathscr{H}_\lambda \otimes R_\lambda .$$

2. Transfer all remaining structure (CAR, T, C) to the multiplicity blocks $\mathscr{H}_\lambda$.
3. Show that, in the process, the only change that may occur is a change of the type of involution, i.e. $T = T_{\mathrm{eff}} \otimes \beta$ with $\beta^2 = \pm 1$, etc.
4. Enumerate the possible cases.

It should be emphasized that the process of reduction by the unitary symmetries does modify, in general, the CAR structure and/or the algebraic properties of the anti-unitary symmetry operations.

Example 1. Let G_0 be the group $\mathrm{SU}(2)$ of spin rotations, with fundamental representation $R_\lambda = (\mathbb{C}^2)_{\mathrm{spin}}$. In this case, the reduction process eliminating the spin degree of freedom effectively replaces CAR by CCR (canonical commutation relations) — a kind of "fermion-boson transmutation"!

Example 2. Let the setting be the same as in the first example, but now augmented by the anti-unitary symmetry of time reversal T. If $T^2 = -1$ (as is the case for spinful fermions), then $T_{\mathrm{eff}}^2 = +1$ after spin reduction.

3. Free-fermion ground states

3.1. Quasi-particle vacua with translation symmetry. — We begin with some notation. Let Γ denote a group of translations, and let $\widehat{\Gamma} \equiv M$ be its Fourier dual, i.e., momentum space (Brillouin zone). As before, U_k (V_{-k}) denotes the space of single-particle annihilation (resp. creation) operators lowering the momentum by $k \in M$. For each k we form the Hermitian vector space $W_k = U_k \oplus V_{-k}$ with dimension $\dim W_k = 2N$ and scalar product

$$\langle \psi \mid \psi' \rangle = \{\psi^\dagger, \psi'\}.$$

Adapting a fact stated in Section 2.2 we have:

Fact. An insulator ground state of free fermions is a rank-N complex vector bundle

$$\mathscr{A} \xrightarrow{\pi} M, \quad \pi^{-1}(k) \equiv A(k) \subset W_k$$

subject to the CAR constraint

$$\{A(k), A(-k)\} = 0 \quad (\text{for all } k \in M).$$

Recall that this condition says that the anti-commutator for any pair of single-particle annihilation operators must vanish. In the presence of symmetries other than translations there will be complex (anti-)linear conditions on the fibers $A(k)$.

3.2. Quantum Spin Hall Insulator (*A*II). — Let us illustrate the general statement by an example of great physical interest: the so-called quantum spin Hall insulator.

We assume that particle number is conserved and take the symmetry group G to be generated by time reversal (for spin-1/2 electrons, so that $T^2 = -1$) and by a group Γ of translations. In this case, our quasi-particle vacua are Hartree-Fock mean-field ground states (with an even number n of occupied bands). Because momentum k is conserved due to translation symmetry, we may describe these ground states by assigning to each k-value the corresponding vector space, $V(k) \subset \mathbb{C}^N$, of occupied states (or, using the language of solid state physics, the vector space spanned by the *valence band* states at momentum k). If the system is an insulator, i.e., the Fermi energy sits in a gap, this assignment determines what is called a vector bundle: a family of vector spaces that varies continuously with k. The presence of T-symmetry implies that the time-reversal operator applied to the valence states of momentum k yields the valence states of momentum $-k$:

$$T\,V(k) = V(-k).$$

For the special case of a time-reversal invariant momentum $k_0 = -k_0$ it follows that the vector space of valence states is T-invariant: $T\,V(k_0) = V(k_0)$. By the properties of the anti-unitary operator T this implies that the vectors of $V(k_0)$ organize into so-called Kramers pairs.

For $M = \mathrm{S}^2$, the case of a two-dimensional fluid where $k_0 = 0$ and ∞ are the only T-invariant momenta), such vector bundles fall into two topological classes. Physically speaking, these are those of a trivial band insulator and of the quantum spin Hall phase (and every one of our translation-invariant Hartree-Fock mean-field ground states belongs to one of the two).

To distinguish between the two cases, one looks at a topological invariant that measures the twisting of the vector bundle — following Kane and Melé one associates with $V(k)$ the Pfaffian $p(k) := \mathrm{Pf}\,\tau_k$ of the skew-symmetric map

$$\tau_k :\ V(k) \to V(k)^* = (\mathbb{C}^N)^*/V(k)^\perp,\ \ v \mapsto \langle Tv, \cdot \rangle\,.$$

The phase of the complex-valued function $k \mapsto p(k)$ winds either an even or an odd number of times along a circle through 0 and ∞ in k-space.

3.3. From vector bundles to classifying maps. — We now change perspective and go to the equivalent description of a vector bundle by its classifying map. For concreteness, we do it again at the example of the quantum spin Hall insulator, and comment on the general situation at the end.

Recall that a choice of n-dimensional subspace $V(k) \simeq \mathbb{C}^n \subset \mathbb{C}^N$ determines a point x in a Grassmann manifold

$$x \in C := \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n).$$

(We now write "C", as in *classifying space*.) In the vector bundle picture we associate with each k-value a vector space $V(k)$, but we may equivalently associate with each k-value the corresponding point in C. Thus instead of a vector bundle $\{V(k)\}_{k\in M}$ we get a mapping $\psi :\ k \mapsto V(k) \in C$ from k-space into the Grassmannian C. The condition $TV(k) = V(-k)$ due to time-reversal symmetry translates into a condition on the map:

$$\widetilde{T}\,\psi(k) = \psi(-k).$$

At any T-invariant point $k_0 = -k_0$ this becomes a stability condition $\widetilde{T}\,\psi(k_0) = \psi(k_0)$ constraining ψ to take values in a certain subspace of C. That subspace, R, is the symplectic Grassmann manifold

$$R = \mathrm{Sp}(N)/\mathrm{Sp}(n) \times \mathrm{Sp}(N-n).$$

Its points are the fixed points of $\widetilde{T}$ (the induced action of time reversal) in C.

Turning to the general situation, G-invariant free-fermion ground states of gapped systems with translation symmetry ($\Gamma \subset G$) are described by classifying maps

$$\psi : M \to C$$

from momentum space into some symmetric space C subject to the condition

$$\widetilde{g} \cdot \psi(k) = \psi(g \cdot k) \quad (\text{for all } g \in G_{\text{red}})$$

for every element of the reduced group of symmetries which do not commute with the translations.

3.4. 1d superconductor, class D. — To give another example, let us consider the case of a single band of spinless fermions with no symmetries (a.k.a. class D) in one dimension. The vector bundle $\{A_k\}_{k \in M}$ of the (superconducting) ground state takes the form of

$$A(k) = \text{span}_{\mathbb{C}} \left(u(k)\, c_k + v(k)\, c_{-k}^{\dagger} \right),$$

where the function $v(k)/u(k) \equiv z(k) = -z(-k)$ is skew-symmetric by the CAR constraint $\{A(k), A(-k)\} = 0$. This ground state may look more familiar when written as a BCS state:

$$|\text{BCS}\rangle = \mathrm{e}^{\sum_k z(k)\, c_k^{\dagger} c_{-k}^{\dagger}} |\text{vac}\rangle.$$

In any case, the associated classifying map here is a map $\psi : M \to C$ from the momentum circle $M \simeq \mathrm{S}^1$ into a two-sphere:

$$C = \mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1) \simeq \mathrm{S}^2.$$

By the CAR constraint, the self-dual momenta $k_0 = -k_0$ get mapped into

$$R = \mathrm{O}(2)/\mathrm{U}(1) = \{|0\rangle, |1\rangle\},$$

which may be viewed as a set of two opposite points on S^2; physically speaking, one (say, the South Pole) corresponds to the vacuous state $|0\rangle$ and the other one (North Pole) to the fully occupied state $|1\rangle$.

3.5. How to classify?— There exist several notions of topological equivalence for vector bundles, and they are not the same. The finest classification is by homotopy – two vector bundles (or the associated classifying map) belong to the same homotopy class if they can be transformed into each other by a continuous sequence of infinitesimal deformations. Another classification is by isomorphy – two vector bundles are said to be isomorphic if an isomorphism (a single, possibly large transformation) takes one into the other. Isomorphy is coarser than homotopy. Indeed, two vector bundles in the same isomorphism class need not be in the same homotopy class, unless the number of conduction bands is sufficiently large. Third, if both the number of conduction bands and valence bands are taken to infinity, isomorphism classes become stable equivalent classes; these are also known as K-theory classes.

Example 1. The "Hopf magnetic" insulator of Moore, Ran, and Wen (2008) has one valence band and conduction band each; hence $C = \mathrm{S}^2$. If $M = \mathrm{S}^3$ then

$$\mathbf{0} = \mathrm{Vect}_1^{\mathbb{C}}(M) \neq \pi_3(\mathrm{S}^2) = \mathbb{Z}.$$

Thus all complex line bundles over S^3 are isomorphic to the trivial bundle. On the other hand, there do exist infinitely many homotopy classes of maps from S^3 to S^2 (they are classified by the Hopf invariant).

Example 2. If $M = \mathrm{S}^1$ then

$$\mathbb{Z}_2 = \mathrm{Vect}_1^{\mathbb{R}}(M) \neq \pi_1(\mathrm{S}^1) = \mathbb{Z}.$$

This says that there exist two isomorphism classes of real line bundles over S^1 (the trivial bundle and the Möbius bundle), whereas the homotopy classes of maps from S^1 to S^1 are in bijection with $\mathbb{Z}$.

3.6. Notation for classifying spaces. — Here comes again the Periodic Table (of Kitaev, and Ludwig et al.) for the case of $G_{\mathrm{red}} = \mathbb{Z}_2$. Our notation for classifying spaces is as shown. (More precisely, the Grassmann manifolds in the table come with an extra $\mathbb{Z}$-factor; physically speaking, this integer corresponds to the number of occupied bands.) The periodicity statement of the ("real" part of the) Periodic Table then reads

$$[S^d, C_s(n)]_{\mathbb{Z}_2} \cong [S^{d+1}, C_{s+1}(2n)]_{\mathbb{Z}_2} \quad (n \gg d),$$

where $[S^d, C_s(n)]_{\mathbb{Z}_2}$ stands for the set of homotopy classes of $\mathbb{Z}_2$-equivariant maps from S^d into $C_s(n)$ (and similar on the RHS).

3.7. Homotopy-theoretic proof of the Periodic Table. — With R. Kennedy we are close to finishing a proof of the Periodic Table using methods of homotopy theory only. The proof is too technical to be suitable for presentation to a broad physics audience, so I'll just give a very brief outline. There are three main ingredients:

1. The Bott isomorphism $[S^D, R_s] \cong [S^{D+1}, R_{s-1}]$.
2. The Whitehead Theorem in G-equivariant homotopy (for $G = \mathbb{Z}_2$). This is needed to link the Bott isomorphism to our setting of G-equivariant homotopy.
3. A certain fiber bundle, whose projection map (a kind of "square") sends the relative homotopy groups of our G-equivariant setting to the homotopy groups governed by Bott periodicity.

Unfortunately, the requisite fiber bundle exists in only 2 of the 8 cases to consider. Therefore, to make the proof work, we are forced to consider the more general situation envisaged by Teo & Kane (2011), where $k \in M$ has D space-like and d momentum-like components. Then the argument goes as follows:

$$[S^{D,d}, C_s(n)]_{\mathbb{Z}_2} \overset{1,2}{\cong} [S^{D+s,d}, C_0(n)]_{\mathbb{Z}_2} \overset{3}{\cong} [S^{D+s,d+1}, C_1(2n)]_{\mathbb{Z}_2} \overset{1,2}{\cong} [S^{D,d+1}, C_{s+1}(2n)]_{\mathbb{Z}_2}.$$

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