BOTT PERIODICITY FOR $\mathbb{Z}_2$ SYMMETRIC GROUND STATES OF GAPPED FREE-FERMION SYSTEMS WITH DISORDER

by

R. Kennedy & M.R. Zirnbauer

Abstract. — In this 30min talk I give a gentle introduction to joint work with Ricardo Kennedy where, building on the symmetry classification of disordered fermions, we give a proof of the proposal by Kitaev for a "Bott clock" topological classification of free-fermion ground states of gapped systems with symmetries. Our approach differs from previous ones in that (i) we work in the standard framework of Hermitian quantum mechanics over the complex numbers, (ii) we directly formulate a mathematical model for ground states rather than spectrally flattened Hamiltonians, and (iii) we use homotopy-theoretic tools rather than K-theory. Basic to our proof is a natural transformation relating the ground state of a d-dimensional system in symmetry class s to the ground state of a $(d+1)$-dimensional system in symmetry class $s+1$. This relation gives a new vantage point on topological insulators and superconductors.

1. Introduction

1.1. Periodic Table. — Let me begin by putting up the so-called "Periodic Table" copied from the review article by Hasan and Kane. Some five or six years ago, most if not all of the examples of topological insulators and superconductors known at that time were put into this systematic table by Kitaev, stimulated by work of the Santa Barbara group around Ludwig. The quantum Hall effect was placed into a sub-table of period 2. The quantum spin Hall insulator, the Majorana, He-3B, etc., all went into a second sub-table, which exhibits an 8-fold periodicity.

A certain amount of foundational work on this striking table has been done and, in particular, efforts have been made to unveil the mysterious mechanism behind the periodic structure. The papers of most importance to my story are listed at the bottom of this slide.

Now the constancy of the groups along the diagonal begs the question: is there a so-called "diagonal map", which takes a symmetry-protected topological phase in d dimensions and transforms it into another such phase in one more dimension and in the neighboring symmetry class? Note that this map applied to the Majorana chain should give a 2d time-reversal invariant superconductor, and applied again, a 3d topological insulator. Starting from the Kitaev chain, its application should give a 2d chiral p-wave superconductor, and applied again something like He-3 B; and so on – you get the idea.

Such a mapping was written down by Teo and Kane some time ago. Their first formula says how to go from a chiral Hamiltonian to a non-chiral one. The second formula takes you from non-chiral back to chiral by tensoring with a quasi-spin degree of freedom, and it comes with some case-dependent instructions as to which Pauli matrices to use.

In this talk, I am asking a more ambitious question: is there a "master" diagonal map that handles all cases at once, by a single and universal principle? Can one make a convincing argument that the map indeed gives a one-to-one correspondence between symmetry-protected topological phases? (As a mathematical physicist one would like to formulate and prove a theorem.) And can one specify the precise conditions under which the Table holds, and when it fails? My modest goal in this 30min talk is to assure you that the master diagonal map exists and to explain how it works.

1.2. Distinctive features of our approach. — Here are the highlights and special points that distinguish our approach in Cologne (joint work with R. Kennedy) from what's done in the published literature.

- First of all, we start from the tenet that a symmetry is a unitary (or anti-unitary) transformation that *commutes* with the Hamiltonian. To make sure you're with me, let me make the point that all of the entries below Ξ in the famous Table are operations that *anti-commute* with the Hamiltonian. For a number of reasons we do **not** accept such operations as true symmetries. One reason is that the associated symmetry indices $[0, 1, -1]$ give no clue as to why the classes are arranged in the particular order they are. In our work, armed with the tight notion of symmetry, we deduce this so-called Kitaev sequence from first principles (so to speak).
- Secondly, to the extent that only the static properties (as opposed to the dynamical response) of the physical system are under investigation, the classification problem at hand is a problem of classifying *ground states*. Therefore, once the symmetry class has been determined, the Hamiltonian leaves the scene and does not reappear in our approach. Thus we work directly with a mathematical model for the ground state and, in particular, we have no need for the commonly invoked process of "spectral flattening" of the Hamiltonian.
- Thirdly, and most importantly, our work follows a different principle of topological classification than usual. Starting with Kitaev, the community has largely relied on the algebraic tools of K-theory to define and compute topological invariants. In contrast, in our work we use tools from homotopy theory. Let me stress that homotopy classes are finer and carry more information than do K-theory classes, in general.
- Last but not least, we work in the standard framework of Hermitian quantum mechanics over the complex numbers. While this may sound like a "no-brainer", there actually exist statements in the literature that one is better off working over the real numbers, like Dyson did in his Threefold Way. The thinking behind this claim was presumably that the 8-fold periodicity of the Table is reminiscent of a periodicity phenomenon for real Clifford algebras. While that is certainly true, it turned out to be most revealing for us to keep the real structure *flexible*. In fact, to get the best perspective of our "diagonal map", we employ two *different* operations of taking the complex conjugate.

2. Universal Model for Free Fermion Ground States

This talk is in two parts. The second part, introducing the Diagonal Map, is based on the first part, where I describe a universal model for free-fermion ground states of gapped systems with symmetries.

2.1. Ground states as vector bundles. — First, a quick word on notation: we denote fermion annihilation operators by c and creation operators by $c^\dagger$, as usual. The symbol M stands for momentum space and momenta are denoted by k. (For simplicity, let me assume translation invariance for now and comment on the disordered situation later).

It is then a basic fact of second quantization that a free-fermion ground state $|\text{g.s.}\rangle$ is uniquely determined by specifying for each momentum k the quasi-particle operators $\widetilde{c}_1(k), \ldots, \widetilde{c}_n(k)$ that annihilate it:

$$\widetilde{c}_j(k)|\text{g.s.}\rangle = 0 \quad (j = 1, \ldots, n).$$

(n is the total number of bands.) In other words, free-fermion ground states are in one-to-one correspondence with collections of vector spaces $\overset{1:1}{\leftrightarrow} \{A_k\}_{k\in M}$ where we take

$$A_k = \mathrm{span}_{\mathbb{C}}\{\widetilde{c}_1(k), \ldots, \widetilde{c}_n(k)\}$$

to be spanned by the annihilation operators at k; more precisely by those **lowering** the momentum by k.

To give an example, consider a system with conserved particle number (or charge) and two bands, one conduction and one valence band. In this case the annihilation space A_k is spanned by two operators: the one removing a particle in the conduction band at momentum k and another one creating a particle in the valence band at $-k$:

$$A_k = \mathrm{span}_{\mathbb{C}}\{c_p(k), c_h^\dagger(-k)\}.$$

(I must admit that for charge-conserving systems, our description is somewhat redundant, but this is the price to be paid for a comprehensive model that encompasses all cases, including superconductors.)

If the physical system were a metal, the annihilation space A_k as a function of k would jump at the Fermi surface, but since we consider gapped systems the assignment $k \mapsto A_k$ has the good feature of being continuous everywhere, giving a vector bundle $\{A_k\}_{k\in M}$.

Now we observe that even in the absence of any symmetries (other than translations), the vector spaces A_k are constrained by the condition that any pair of annihilation operators must have vanishing anti-commutator by the canonical anti-commutation relations. I write this condition as

$$\{A_k, A_{-k}\} = 0,$$

and refer to it as the Fermi constraint. Returning to the Periodic Table, let me point out that what is commonly known as the "particle-hole symmetry" of class D is nothing but the Fermi constraint relating opposite fibers (A_k with A_{-k}) in our model. Another comment directed at the experts in the audience is that our vector bundle $\{A_k\}_{k\in M}$ is complex and cannot be viewed as a real vector bundle in any sense. (Indeed, while the Fermi constraint determines A_{-k} from A_k, it does not give rise to any complex linear or anti-linear mapping between the vectors of these two spaces.)

2.2. Universal model (including symmetries). — Our next step is to incorporate symmetries into the mathematical model of free-fermion ground states as vector bundles. In that process, as outlined by Kitaev, true physical symmetries are converted into "pseudo-symmetries". Let me write down the outcome first and do the explaining afterwards.

For a system with symmetry index s one has a Clifford algebra of s pseudo-symmetries:

$$J_l J_m + J_m J_l = -2\delta_{lm}\mathbf{1},$$

where the $J_1, \ldots, J_s$ are complex-linear operators on the sum of A_k with its orthogonal complement, A_k^{c}. Since that sum comprises all operators (creation as well as annihilation), it is independent of the momentum: $A_k \oplus A_k^{\mathrm{c}} \equiv \mathbb{C}^{2n}$.

A concise summary of the effect of imposing symmetries is the following definition.

Definition 2.1. — *A translation-invariant free-fermion ground state of a gapped system of symmetry class s is a rank-n vector sub-bundle $\{A_k\}_{k\in M}$ with fibers $A_k \subset \mathbb{C}^{2n}$ subject to*

1. Fermi constraint : $\{A_k, A_{-k}\} = 0,$

2. pseudo-symmetries : $J_1 A_k = \ldots = J_s A_k = A_k^{\mathrm{c}},$

for all $k \in M$.

We speak of "pseudo-symmetries" because each operator of the set $J_1, \ldots, J_s$ sends A_k to its orthogonal complement, whereas a true unitary symmetry would map A_k to itself.

Example 2.1. — Let the gapped system (and hence the ground state) be time-reversal invariant. Then, applying the time-reversal operator to any annihilation operator at momentum k we get an annihilation operator at the opposite momentum, $-k$. If we carry on by applying the operator γ of Hermitian conjugation ($\gamma: \ c \leftrightarrow c^\dagger$), we end up with a creation operator back at $+k$. Thus the composition $J_1 = \gamma \circ T$ sends A_k to its orthogonal complement A_k^{c}, which yields the first pseudo-symmetry relation. Moreover, being the product of two anti-unitary operators, J_1 is unitary; and assuming the case of fermions with half-integer spin, T squares to minus one, J_1 does the same, and we have identified the first generator of the Clifford algebra of pseudo-symmetries.

2.3. Kitaev sequence. — How does this story continue? While Kitaev wrote down only the first two steps, in the following table I present the whole answer.

class	symmetries	s	pseudo-syms
D	none	0	CAR
DIII	T (time reversal)	1	$J_1 = \gamma T$
AII	T, Q (charge)	2	$J_2 = \mathrm{i}\gamma T Q$
CII	T, Q, C (ph-conj)	3	$J_3 = \mathrm{i}\gamma C Q$
C	S_1, S_2, S_3 (spin rot)	4	see below
CI	S_1, S_2, S_3, T	5	
AI	S_1, S_2, S_3, T, Q	6	
BDI	S_1, S_2, S_3, T, Q, C	7	

Remember that even if there are no symmetries [beyond translations], we still have the Fermi constraint due to Fermi statistics; that characterizes class D [housing the Majorana chain], or $s = 0$. As we have seen, imposing T gives one pseudo-symmmetry J_1, which puts us in the superconducting class DIII, or $s = 1$. Next, we add the requirement that particle number (or charge Q) be conserved. This allows us to form a second pseudo-symmetry $J_2 = \mathrm{i}QJ_1$, which is readily seen to anti-commute with J_1 and square to minus the identity. We have now arrived in class AII [housing the quantum spin Hall insulator], or $s = 2$. Next, to move on to class CII or $s = 3$, we include particle-hole conjugation (C) as a third symmetry – more precisely, as a true physical symmetry which commutes with the Hamiltonian. To continue even further, we reset the clock (so to say) by erasing all symmetries T, Q, and C, and we demand instead that the spin-rotation generators S_1, S_2, S_3 be symmetries. The rest of the story is a repetition of what happened at the beginning.

2.3.1. Going beyond $s = 3$. — It remains to explain why the three generators of the spin-rotation group together with $s-4$ symmetries (taken from T, Q, C) amount to s pseudo-symmetries. To that end, we must introduce the so-called $(1,1)$ doubling isomorphism, as follows. Let

$$C_s(n) := \{A \in \mathrm{Gr}_n(\mathbb{C}^{2n}) \mid J_1 A = \ldots = J_s A = A^{\mathrm{c}}\}$$

denote the classifying space for class s, i.e. the space of all annihilation n-planes $A \equiv A_k$ allowed by s pseudo-symmetries (we suppress the momentum k for now). Inside it, we have the subspace $R_s(n)$ of all annihilation planes A that also satisfy the Fermi constraint:

$$R_s(n) := \{A \in C_s(n) \mid \{A, A\} = 0\}.$$

Now we double the number of bands, going from n to $2n$ [formally: $\mathbb{C}^{2n}$ to $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$], and on the doubled space we introduce the operators

$$I = \begin{pmatrix} 0 & \mathbf{1}_{2n} \\ -\mathbf{1}_{2n} & 0 \end{pmatrix}, \quad K = \mathrm{i}\begin{pmatrix} \mathbf{1}_{2n} & 0 \\ 0 & -\mathbf{1}_{2n} \end{pmatrix}, \quad \widetilde{J}_l = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \quad (l = 1, \ldots, s).$$

This extends the Clifford algebra of $J_1, \ldots, J_s$ by two extra generators, I and K, with one important subtlety: the distinguished generator K reverses the sign of the anti-commutator bracket: $\{Kw, Kw'\} = -\{w, w'\}$, whereas all others preserve it. We call such a generator K "imaginary", while I and the J's are called "real".

In this setting one has two isomorphisms, each relating the classifying space determined by the original Clifford algebra to the corresponding classifying space for the extended algebra:

$$C_s(n) \simeq C_{s+2}(2n), \quad R_s(n) \simeq R_{s+1,1}(2n).$$

(The proof is an exercise in linear algebra.) Thus if we double the number of bands and add two pseudo-symmetries, one real and one imaginary, then the situation remains unchanged.

This lemma lets us settle the issue in question. Given the spin-rotation generators S_1, S_2, S_3 and $s-4$ pseudo-symmetries $J_5, \ldots, J_s$, we put the former on the diagonal blocks of the doubled space and the latter on the diagonal blocks:

$$\widetilde{J}_l := \begin{pmatrix} \mathrm{i}S_l & 0 \\ 0 & -\mathrm{i}S_l \end{pmatrix} \quad (l \leq 3), \quad \widetilde{J}_4 := I, \quad \widetilde{J}_l := \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \quad (l \geq 5).$$

We identify $\widetilde{J}_4 \equiv I$ and note that $K = \mathrm{i}\widetilde{J}_1\widetilde{J}_2\widetilde{J}_3$. All operators $\widetilde{J}_1, \ldots, \widetilde{J}_s$ preserve the anti-commutator bracket transferred to the doubled space in the natural way.

Now, on physical grounds, the spin-rotation generators commute with the pseudo-symmetries (J_5, J_6, J_7) drawn from (T, Q, C). It follows that the new operators $\widetilde{J}_1, \ldots, \widetilde{J}_s$ satisfy the standard Clifford relations, and by an easy argument using the $(1,1)$ doubling isomorphism, one verifies that the s pseudo-symmetries $\widetilde{J}_1, \ldots, \widetilde{J}_s$ are indeed equivalent to the $s-4$ pseudo-symmetries $J_5, \ldots, J_s$ in conjunction with three spin-rotation symmetries S_1, S_2, S_3.

In summary, there is a reason from first-principles why one wants to put the 8 "real" classes in this very order [new table], starting with class D at $s=0$ and going down to class BDI at $s=7$.

2.4. Disorder. — Having assumed translation invariance so far, let me now say a few words about disordered systems. Disorder of course kills momentum conservation, and one might worry that our vector bundle formalism based on conserved momentum goes down the drain. However, this is not so. From the large body of work on the quantum Hall effect, one in fact knows how to proceed. What one does is to tweak the disordered system by introducing a flat connection (θ) or by twisting the boundary conditions, and one then studies the vector bundle $\{A_\theta\}_{\theta \in \mathrm{T}^d}$ of ground states over the torus (T^d) of twist parameters. What I have said up to now carries over to this setting, and so does the diagonal map to be described below. This remark is crucial for the grand perspective, as it is the stability with respect to disorder that singles out the ten classes of the Periodic Table from the plethora of topological crystalline insulators.

This brings us to the question: why are there exactly ten classes and not more? Well, I have written and spoken about this theme at length [arxiv-ref; see also colloquium talks on my homepage], so it shouldn't cause too much grief if there is no time to go into any detail here. Let me just say this. The Kitaev sequence introduces the symmetries in a certain fixed order, and you might get the idea that new classes would come up if you deviated from that order or modified the symmetries. Well, don't waste your time! It is a theorem (dubbed the Tenfold Way) that no matter what group of unitary and anti-unitary transformations you start from – be it discrete or continuous –, you must always end up with a classifying space of one of the ten known types, provided that your symmetries are true physical ones.

Of course, to reach the count of ten one augments the 8 "real" classes I discussed by 2 "complex" ones.

class	symmetries	pseudo-syms
A	Q	none
AIII	Q, C	$J_1 = \mathrm{i}\gamma C$

To realize class A, one takes charge (Q) to be conserved. This effectively cancels the Fermi constraint and one is left with no pseudo-symmetries at all. Ground states in this case are just plain complex vector bundles. Finally, by imposing an additional symmetry of particle-hole conjugation, one arrives at class AIII, featuring one pseudo-symmetry.

3. The Diagonal Map

So much for symmetries. I am now getting to the main point of the talk, but first I wish to make another remark about the basic principles of our game.

3.1. How to classify?— There exist several notions of topological equivalence for vector bundles, and they are not the same. The finest classification is by homotopy – two of our vector bundles belong to the same homotopy class if they are adiabatically connected, i.e., can be transformed into each other by a continuous sequence of infinitesimal deformations. Another classification is by isomorphy – two vector bundles are said to be isomorphic if an isomorphism (a single, possibly large transformation) takes one into the other. Isomorphy is coarser than homotopy. Indeed, two vector bundles in the same isomorphism class need not be in the same homotopy class, unless the number of conduction bands is sufficiently large. Third, if not only the conduction bands but also the valence bands are taken to be very numerous, isomorphism classes stabilize as so-called K-theory classes.

One should of course discuss which of these notions is most appropriate to the physics of topological insulators, yet I am not aware of any such discussion in the current literature. It seems to me that most authors do acknowledge homotopy to be the relevant classification principle, but in practice perform their computations assuming K-theory.

To give an example, for systems in symmetry class D in 3 dimensions, two of them momentum-like and one space-like (associated with a defect), the K-theory analysis of Teo and Kane predicts a $\mathbb{Z}_2$-classification. While this is the correct answer in the K-theory limit of infinitely many bands, it is not correct for a small band number. In fact, the minimal case $(n = 1)$ can be shown to be classified by the integers ($\mathbb{Z}$). Such discrepancies are of course expected, as homotopy groups are known to be vastly more intricate than K-theory groups. (In view of such examples as the homotopy groups of the spheres, it actually looks like a miracle that the homotopy groups of the Periodic Table could possibly be as simple as they are.)

3.2. Diagonal map $(d,s) \to (d+1,s+1)$. — The goal now is to take a d-dimensional ground state of class s and turn it into a $(d+1)$-dimensional ground state of class $s+1$. So, let there be s pseudo-symmetries $\widetilde{J}_1,\ldots,\widetilde{J}_s$ and a vector bundle $\{\widetilde{A}_k\}_{k\in M_d}$ of class s over d-dimensional momentum space M_d. Our first step is to jack up these data by applying the $(1,1)$ doubling isomorphism. Thus, doubling the number of bands, we extend the Clifford algebra of pseudo-symmetries by two generators, I and K, one of which is imaginary (K), and we lift the vector bundle isomorphically to the doubled band space.

Then, introducing a coordinate t for the extra dimension to be added, we make the fibers A_k (of the given vector bundle) t-dependent by applying a one-parameter group of unitary transformations:

$$A_{k,t} := \mathrm{e}^{(t/2)KJ(A_k)} \cdot A_k\,,$$

where $J(A) = \mathrm{i}(\Pi_A - \Pi_{A^\mathrm{c}})$ is multiplication by i on A and $-\mathrm{i}$ on the orthogonal complement A^c. By investing the algebraic properties at hand, one verifies that this definition has all the right properties:

1. the Fermi constraint is satisfied:
$$\{A_{k,t}\,, A_{-k,-t}\} = 0\,;$$
2. the pseudo-symmetry relations hold:
$$J_1 A_{k,t} = \ldots = J_s A_{k,t} = A_{k,t}^{\mathrm{c}} = I A_{k,t}\,;$$
3. and the fibers depend periodically on t:
$$A_{k,t+2\pi} = A_{k,t}\,.$$

Thus we have attached to momentum space a circle parametrized by $t \in \mathbb{R}/2\pi\mathbb{Z}$, and we added one pseudo-symmetry given by the real generator I. The finished product is a vector bundle $\{A_{k,t}\}_{(k,t)\in M_{d+1}}$ ($M_{d+1} = M_d \times \mathrm{S}^1$) of class $s+1$ in dimension $d+1$.

Let me illustrate the procedure by two examples.

3.3. Example 1. — In the first example, we start from data of class D in zero dimension and apply our diagonal map to manufacture a time-reversal invariant ground state for the superconducting class DIII in one dimension. Taking the simple case of only one state ($n = 1$), we have a real classifying space consisting of just two points, the line of the empty state and that of the occupied state:
$$n = 1:\; R_0(1) = \{\mathbb{C}\cdot c, \mathbb{C}\cdot c^\dagger\}.$$
The procedure of $(1,1)$ doubling here amounts to forming the tensor product with two-dimensional spin space, $(\mathbb{C}^2)_{\mathrm{spin}}$. The operator I is to be identified with the first pseudo-symmetry of the Kitaev sequence:
$$I \equiv J_1 = \gamma T = (\sigma_1)_{\mathrm{BdG}} \otimes (\mathrm{i}\sigma_2)_{\mathrm{spin}}\,.$$
The simplest choice of imaginary generator K is this:
$$K = \mathrm{i}(\sigma_1)_{\mathrm{BdG}} \otimes (\sigma_1)_{\mathrm{spin}}\,,$$
and for the reference plane A we take the state with both spin states occupied:
$$A = \mathrm{span}_{\mathbb{C}}\{c_\uparrow^\dagger, c_\downarrow^\dagger\} \cong |\mathrm{full}\rangle.$$
We then apply the one-parameter group of our diagonal map to find a vector bundle spanned by two quasi-particle operators:
$$A_k = \mathrm{e}^{(k/2)KJ(A)}\cdot A = \mathrm{span}_{\mathbb{C}}\left\{c_\sigma^\dagger(-k)\cos(k/2) - c_{-\sigma}(k)\sin(k/2)\right\}_{\sigma=\uparrow,\downarrow}.$$
This may look more familiar when written as a BCS ground state:
$$|\mathrm{g.s.}\rangle = \mathrm{e}^{\sum_k \cot(k/2) P_k}|\mathrm{vac}\rangle\,, \quad P_k = c_\uparrow^\dagger(k) c_\downarrow^\dagger(-k)\,.$$
For a more general choice of $K = K(\alpha)$ the Cooper pair operator takes the form
$$P_k = c_\uparrow^\dagger(k)c_\downarrow^\dagger(-k)\,\cos\alpha + \left(c_\uparrow^\dagger(k)c_\uparrow^\dagger(-k) - c_\downarrow^\dagger(k)c_\downarrow^\dagger(-k)\right)\,\sin\alpha\,,$$
which clearly displays the spin-triplet pairing of the superconductor at hand. The system is in a symmetry-protected topological phase since you cannot undo the winding in its ground state without breaking time-reversal invariance.

3.4. Example 2. — For a second example, we start from the outcome of the previous one and progress to a two-dimensional band insulator in class AII. Here the effect of $(1,1)$ doubling is to introduce two bands (labeled by p and h) for the spinful system. To implement charge conservation (Q) directly, we first make a particle-hole type transformation on our class-DIII superconductor to turn it into a particle-number conserving reference bundle:

$$A_{-k_1} = \mathrm{span}_{\mathbb{C}} \left\{ c_{h\uparrow}^\dagger(k_1)\, \cos(k_1/2) + \mathrm{i} c_{p\downarrow}^\dagger(k_1)\, \sin(k_1/2);\ \&\uparrow\leftrightarrow\downarrow,\, \mathrm{i}\to -\mathrm{i} \right\}.$$

Then, applying again our one-parameter group (to flow now in the radial direction of the figure), we get a ground state with one valence band as shown explicitly ($k = (k_0, k_1)$):

$$A_{-k} = \mathrm{e}^{-(k_0/2)KJ(A_{-k_1})} \cdot A_{-k_1} = \mathrm{span}_{\mathbb{C}} \Big\{ \&\uparrow\leftrightarrow\downarrow,\, \mathrm{i}\to -\mathrm{i}\,;$$
$$\Big(c_{h\uparrow}^\dagger(k)\, \cos(k_1/2) + \mathrm{i} c_{p\downarrow}^\dagger(k)\, \sin(k_1/2) \Big) \cos(k_0/2)$$
$$+ \Big(c_{p\downarrow}^\dagger(k)\, \cos(k_1/2) + \mathrm{i} c_{h\uparrow}^\dagger(k)\, \sin(k_1/2) \Big) \sin(k_0/2) \Big\}$$

and another one obtained by flipping the spin and replacing i by $-$i. Note that at $k_0 = \pm\pi/2$ – the two poles of a sphere with polar coordinate $k_0 + \pi/2$ –, the k_1-dependence goes away:

$$A_{\pm\pi/2,k_1} = \mathrm{span}_{\mathbb{C}} \left\{ c_{h\uparrow}^\dagger \pm c_{p\downarrow}^\dagger\,,\, c_{h\downarrow}^\dagger \pm c_{p\uparrow}^\dagger \right\}.$$

These two points can be seen to be isolated zeroes of the Kane-Melé Pfaffian, which implies that our ground state has non-trivial Kane-Melé invariant and lies in the quantum spin Hall phase. In fact, it is one of the more transparent representations of that phase known to me.

This should suffice for examples, as it should be more or less clear how the story continues: one can get the topological phases for high (d,s) from those for low (d,s) by iterating one and the same universal construction, namely the diagonal master map explained here. Iterating the special case at hand, the next symmetry-protected topological phases in line are those of the 3d insulator in class CII, the 4d superconductor in class C, and so on.

The big question now is whether this mapping of vector bundles yields an isomorphism on homotopy classes or, physically speaking, a one-to-one correspondence of symmetry-protected topological phases along the diagonal (of the Periodic Table). The answer is yes, albeit with a trivial modification related to keeping base points fixed, and under two provisions: momentum space must be a sphere, and the number of bands (n) cannot be too small (as compared with the dimension d). The proof is beyond the scope of this talk (but you'll find it in a preprint to be posted very soon).

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R. KENNEDY & M.R. ZIRNBAUER