PARTICLE-HOLE SYMMETRIES IN CONDENSED MATTER

by

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Abstract. — This is the text outline for a plenary talk (40 minutes) at the DPG Spring Meeting in Munich (2019).

1. Overture/Introduction

First of all, I wish to thank my "Sektion" (foremost its chairman, Professor Wipf) for inviting me to speak to a big crowd at this DPG Spring Meeting. It is a somewhat emotional affair for me, as my last time here (in the heart of the campus) was 42 years ago, when I was a second-year physics student at the Technical University of Munich.

What I want to tell today is an exciting story from condensed matter physics, and I will try to tell it in a way that conveys at least some of the excitement to my broad audience from the physics of hadrons, plasmas, gravitation, cosmos, etc.

1.1. Quantum Hall Effect. — By way of motivation and introduction, let me begin with one of the prime examples of particle-hole symmetry in condensed matter. The setting here is that of a two-dimensional electron gas at low temperature and in a strong magnetic field, which are the conditions for the quantum Hall effect to be observed: a current driven through the sample results in a voltage exactly perpendicular to the current direction.

For a very strong magnetic field, all electrons occupy single-particle states in the lowest Landau level. The number of available states is determined by simply counting the number of magnetic flux quanta through the sample. With ν denoting the fraction of occupied states, the operation of exchanging particles and holes (or particle-hole conjugation) sends ν to $1-\nu$. If the cyclotron energy is the largest relevant energy scale and all electron-electron interactions are two-body, then one can show that the Hamiltonian is particle-hole symmetric, with the consequence that physical observables such as the longitudinal conductivity σ_{xx} are even under ν going to $1-\nu$. While particle-hole conjugation in general relates systems at different filling fractions, a special situation occurs at $\nu = 1/2$ (or half filling), where one may have a ground state that is particle-hole symmetric.

1.2. Girvin. — Let me make it clear that particle-hole symmetry in the lowest Landau level is not a recent discovery. It was Steve Girvin who first explored (in 1984) the use of particle-hole symmetry in the study of the fractional quantum Hall effect.

1.3. Experiment. — It is a fact, however, that the powerful notion of particle-hole symmetry in the lowest Landau level was mostly ignored for more than 3 decades by theory and experiment alike. It was only very recently that experimentalists began to take the notion seriously. What you see here is the longitudinal conductivity as a function of the filling fraction, for various strengths of the magnetic field. The measured data are plotted as black lines. The red lines are just to guide the eye; they are the mirror image of the measured data (by reflection at half filling). Thus by design, particle-hole symmetry holds to the extent that the black and red lines lie on top of each other. As you can see, that is not the case at 1.5 Tesla, but the agreement gets better and better with increasing field strength, and it is almost perfect at 12 Tesla.

1.4. Son. — You may ask why the dormant concept of particle-hole symmetry came to life 30 years after its inception. The answer is a brilliant paper published in 2015 by Dam Son, proposing a particle-hole symmetric theory of the Fermi-liquid ground state for the half-filled lowest Landau level. It is one of my goals in this talk to give you a flavor of Son's spectacular proposal. (As an aside, I believe that Son is well known in high-energy hadron physics for his work, using gauge-gravity duality, on the minimal viscosity of the quark-gluon plasma.)

1.5. Outline. — The rest of my talk is divided into three parts. As a theorist, I want to begin with some theoretical background on the meaning of particle-hole symmetry. Then I will turn to two major examples: first, the Nobel-Prize winning Haldane phase of anti-ferromagnetic quantum spin chains and, second, Son's particle-hole symmetric theory of the half-filled lowest Landau level.

2. Particle-Hole Symmetry

2.1. Motivation: Tenfold Way. — This slide is a brief digression to indicate why the subject of particle-hole symmetry in condensed matter is dear to me. There exists a classification scheme which I call the Tenfold Way of disordered free fermions, especially electrons, where the basic setting is a symmetry group acting on a fermionic Fock space. In that setting, we studied the set of Hamiltonians that commute with all the given symmetries. Restricting the Hamiltonians to be quadratic in the creation and annihilation operators (i.e., working in the most general version of Hartree-Fock-Bogoliubov mean-field theory) we found that there are exactly 10 types of building blocks; it has become customary to label them by their names in the Cartan classification of symmetric spaces – a kind of *lingua franca* in the area of topological quantum matter. As an important corollary, it follows that the momentum-space Hamiltonians $H(k)$ that arise in the Fourier-Bloch theory for translation-invariant systems, are always of one of those 10 very types.

The classification statement of the Tenfold Way holds under the assumption of a symmetry group G built from an arbitrary subgroup G_0 of unitary symmetries together with two, one or none of two distinguished anti-unitary generators; these are time-reversal and/or particle-hole symmetry. The present talk grew out of a review article (soon to be finished) which highlights and clarifies the role of particle-hole symmetry in the Tenfold Way.

2.2. Example 1: cosine band. — Perhaps the simplest example of a particle-hole symmetric Hamiltonian is that of a cosine band of electrons with translation-invariant hopping along a chain of atomic sites. The Hamiltonian is diagonal in the momentum representation, and if the chemical potential is tuned to half filling, it has a particle-hole symmetry. Indeed, if one shifts the wave number by π (or half the Brillouin zone) and exchanges creation with annihilation operators, the sign change from the dispersion relation is canceled by the sign change from restoring the normal order of operators.

Note that particle-hole conjugation takes the imaginary unit to its negative; we say that C is complex anti-linear.

2.3. Example 2: Hubbard model. — Let's turn to a second example, the Hubbard model. We may view it as an extension of the previous one. For simplicity (not by necessity) our particles still move in one dimension, along a chain of sites indexed by the integers, but they now carry spin $1/2$. Very importantly, they repel each other by a two-body interaction ($U > 0$) – the square of the local charge, normal ordered with respect to half filling. The operator C is the same particle-hole transformation as before; in fact, the switch from momentum to position space converts the momentum shift by π to multiplication by an alternating sign. Thus C is still a symmetry of the kinetic term. Moreover, it is also a symmetry of the interaction term. This is because C simply reverses the sign of the local charge, which enters into the Hubbard interaction as the square. Altogether, we see that the Hubbard Hamiltonian is particle-hole symmetric. At half filling, the particle-hole symmetric Hamiltonian has a particle-hole symmetric ground state.

In the condensed matter community, it is well known that the low-energy physics of the Hubbard model at half filling and large U is governed by the famous Heisenberg anti-ferromagnetic quantum spin chain. It follows that the latter is particle-hole symmetric. One can also see the symmetry directly by observing that C sends each component of the local spin operator to its negative.

2.4. Example 3: Dirac fermions. — Thirdly, let me mention a closely related example not from condensed matter physics: relativistic Dirac fermions. In first quantization, the Dirac operator (written here in terms of the usual α and β matrices) changes sign under an operation called "charge conjugation", which has the characteristic property of being anti-linear (the bar means complex conjugation). What may be confusing to the novice is that in second quantization, after redefining the vacuum to be the Dirac sea of filled negative-energy states, charge conjugation becomes a *complex linear* operation. In fact, it is a unitary symmetry of the quantum field theory Hamiltonian.

2.5. Gapped systems (insulators). — After these examples, let us inject some theoretical background. In particular, let us monitor the fate of charge conjugation between first and second quantization.

The notion of particle-hole symmetry is tied to a chemical potential (or Fermi energy) separating the conduction bands, which are empty in the free-fermion ground state, from the valence bands, which are filled. The Fock space is then built up by creating multiple excitations on top of the filled Fermi sea: single particles in the conduction bands and single holes in the valence bands.

Now we need two maps: (one) the Dirac ket-to-bra bijection, known in mathematics as the Fréchet-Riesz isomorphism, say γ, which takes "kets" to "bras" or vectors in the single-particle Hilbert space to co-vectors; (two) an isomorphism K that exchanges the single-particle states of positive and negative energy. Given these, we can form a two-step composite: first, we map Fock space to a fictitious Fock space by letting K exchange V_+ and V_-; second, we return to the physical Fock space by applying Fréchet-Riesz. [Def.:] The resulting map C is called a *particle-hole symmetry* if it commutes with the Hamiltonian.

As a remark, note that Fréchet-Riesz is anti-linear by definition. Therefore, if K is anti-linear, then the resulting symmetry operation C is linear – that's the situation with charge conjugation of relativistic Dirac fermions. In contrast, our condensed matter examples realize the opposite scenario. Indeed, recall C for the cosine band [formula]. Two operations were involved: (i) a shift of the momentum by π (that's the effect of a linear operator K) and (ii) the exchange of creation and annihilation operators (that's the effect of γ).

2.6. Gapless systems. — Let us now turn to the more interesting case of gap*less* systems. The QCD community knows that gauge backgrounds with non-trivial topological charge give rise to fermionic zero modes, by a variant of the Atiyah-Singer index theorem. Similarly, the condensed matter community has learned that topological insulators host protected zero modes or gapless states at their boundaries. So, we pose the question how to think about particle-hole symmetry for systems with zero modes. Without too much loss, we'll assume the number (N) of zero modes to be finite.

In what follows, by *particle-hole conjugation* we mean the anti-linear algebra automorphism that extends the operation of exchanging creators with annihilators [γ]. We then ask: does particle-hole conjugation lift to Fock space; i.e., does there exist an anti-linear operator, say Ξ, which sends a state with n occupied zero modes to one with n vacant zero modes, and which induces the exchange of creation and annihilation operators?

2.7. Particle-Hole Conjugation Lifted. — The answer is yes, as follows. Recall the Fréchet-Riesz isomorphism, and let it act on the n-particle sector of Fock space. Now, fix some state of total occupation. Then we get another mapping, called "Wedge", into the n-hole sector by testing against n-particle states and comparing with the duality pairing by the choice of totally occupied state.

Particle-hole conjugation, Ξ, is the composition of Fréchet-Riesz with Wedge. While Ξ is invariantly defined, it is most easily pictured in the occupation number basis, where it simply swaps empty with occupied single-particle states. As usual, there is an induced action of Ξ on the operator algebra; this is the desired automorphism that exchanges creation with annihilation operators. It is anti-linear because so is Fréchet-Riesz while Wedge is linear.

2.8. Two facts. — Here come two important facts. Firstly, any one-body operator X is odd under particle-conjugation if it is self-adjoint and Weyl-ordered; this covers the operators of momentum and spin, and energy (more precisely, the traceless one-body Hamiltonians).

It follows that particle-hole conjugation can never be a symmetry of any Fermi liquid, where the one-body part of the Hamiltonian dominates. To get a symmetry, one needs to compose Ξ with another spectrum-reversing operation (K).

Secondly, Ξ squares to the identity times a sign that depends on the total number of zero modes. This characteristic feature sets it apart from time reversal, T, which squares to the identity times a sign alternating with the particle number.

2.9. Symmetry protection of zero modes (class $A\mathrm{III}$). — Now, although the notion of particle-hole symmetry arises at the free-fermion level, it continues to make sense for interacting systems (e.g., the Hubbard model). So, let's make a definition: we say that a Hamiltonian (for free or interacting fermions) is of class $A\mathrm{III}$ if it has a particle-hole symmetry and conserves charge (thus, superconductors are excluded).

Then a good question to ask is whether zero modes are protected in class $A\mathrm{III}$. The answer is that there exist 4 distinct scenarios:

1. The boring case of a unique ground state (i.e., no zero modes).
2. The more interesting case of a Kramers pair (needs C squaring to -1).
3. (and 4.) C may act as a supersymmetry, i.e., mediate between ground states of opposite fermion parity (one bosonic, and the other fermionic).

As a side remark for the experts: by a principle known as bulk-boundary correspondence, this fourfold scenario of zero modes directly leads to the $\mathbb{Z}_4$-classification of $A\mathrm{III}$-protected topological phases in one space dimension. In particular, it gives a new perspective on the so-called Haldane phase, which I now turn to as my first major example.

3. From free fermions to the Haldane phase

3.1. Antiferromagnetic quantum spin chains. — In the early 1980's Haldane made a conjecture that greatly confounded the experts.Yet, over time it was found to be correct and deep, and it earned him 1/3 of the Physics Nobel Prize of two years ago.

Haldane's conjecture concerns one-dimensional quantum systems, so-called anti-ferromagnetic quantum spin chains. My cartoon shows a snap shot of an array of spinning degrees of freedom. The axis of the spinning motion varies slowly as we move along the chain, but the sense of circulation changes rapidly, alternating between sites (as is characteristic of an anti-ferromagnet). The simplest Hamiltonian couples only nearest neighbors, with positive exchange coupling to give a tendency for local anti-ferromagnetic order.

The most "quantum" of all spin chains is the one for spin $1/2$. Research on it goes back to the early days of quantum mechanics and is associated with the names of Heisenberg and

Bethe (who famously invented what became known as the Bethe Ansatz for the ground-state wave function of the Heisenberg chain). In the limit of infinite chain length the spin $1/2$ chain has excitations of arbitrarily small energy; we say that it is gapless. This fact was first proved by Lieb-Schultz-Mattis in the early 1960's.

So it came as a surprise when Haldane (twenty years later) argued that the gapless feature was specific to half-integer spin and that chains with integer spin had a disordered ground state with an energy gap for excitations.

3.2. Haldane phase. — Thus the Haldane conjecture says that AF quantum spin chains with integer spin are gapped (at least generically). What became clear over the years (but was not yet in Haldane's original work) is that the story doesn't end here: among the gapped systems with integer spin, there still exists a subtle difference between odd integer and even integer spin. In the former case (and, in particular, for $S = 1$) there exists a kind of hidden *topological order*, which becomes manifest in finite systems through the existence of gapless excitations or zero modes localized at the boundary of the spin chain. Moreover, these boundary excitations carry spin $1/2$ (!), even though the spin chain itself consists of spins 1 and nothing but spins 1 – an example of what is called "fractionalization".

This striking phenomenon can be understood in complete detail for a specific Hamiltonian put forward by Affleck-Kennedy-Lieb-Tasaki (AKLT, for short). They found that if one adds a suitably chosen next-nearest neighbor coupling, then one can get an explicit expression for the ground state as a so-called matrix-product state.

According to our current understanding, the AKLT state is an exactly solvable example of a large class of systems referred to collectively as the "Haldane phase", and the latter is considered as a paradigm for a new type of order that transcends the traditional notion due to Landau, Ginzburg, and Wilson.

3.3. Haldane phase as an SPT phase. — Now, topological order of robust type was first discovered and studied in the fractional quantum Hall effect. In comparison, the topological order of the Haldane phase has the weaker property of being stable only under a family of deformations restricted by symmetry (not just any deformation) – one therefore speaks of a symmetry-protected topological phase.

The question now is: protection by which symmetry? The answer evolved over time. While the details are quite interesting, I'll skip them today but for the acknowledgment that Frank Pollmann (of this university) has been the main player in the game. His latest proposal (with Verresen and Moessner) is that the Haldane phase is protected by a particle-hole symmetry. I had announced that result in various talks in 2016.

3.4. Definitions and statement. — To be clear about what the statement is, let's make a couple of definitions. We say that two Hamiltonians are in the same topological phase if they are connected by a homotopy (or continuous deformation) subject to the condition that every Hamiltonian along the path of the homotopy has a unique ground state with an energy gap for excitations.

Second, recall what it means for a Hamiltonian to be of class $A\mathrm{III}$. Our claim then is that the Haldane phase is connected by homotopy in class $A\mathrm{III}$ to a well-known and well-studied free-fermion topological phase. The argument goes roughly as follows.

3.5. Su-Schrieffer-Heeger model. — Our starting point on the free fermion side is the Su-Schrieffer-Heeger model. That's none other than the cosine band of before, but now with hopping amplitudes that alternate between strong and weak bonds. To account for the doubling of the unit cell, one halves the Brillouin zone and "backfolds" the band. The breaking of translational invariance by the alternating hopping opens an energy gap, giving a particle-hole symmetric insulator at half filling. It is easy to verify that the valence band carries a non-trivial topological invariant, which is reflected by a zero mode localized at the boundary.

3.6. From SSH to Haldane phase. — After this preparation, we consider two SSH chains for spinful electrons, amounting to a total of four chains. Note that the alternation of the hopping is shifted between the two subsystems.

Now, we turn on two types of interaction: a repulsive Hubbard interaction and a Hund's rule coupling; both are consistent with the particle-hole symmetry required for class $A\text{III}$. With the chemical potential set to half filling (that's two electrons per site, as there are four orbitals), we deform the Hamiltonian along some continuous path in class $A\text{III}$. An especially easy path to follow is sketched on the next slide.

3.7. Deformation. —

1. First, turn off the weak hopping; this gives four flat bands; the energy gap stays open.
2. Then, turn on the Hubbard repulsion; because we are at half filling, this forces two electrons onto every site; the energy gap stays open.
3. Finally, crank up the Hund's rule coupling; this aligns the spins of the two electrons per site to form a composite spin 1; the energy gap still stays open.

The outcome of the construction is exactly the famous AKLT matrix-product state. Thus we have established an adiabatic connection between the Haldane phase and the free fermions of the SSH model.

4. Half-filled Landau level

In the remaining time, I'll return to the half-filled lowest Landau level for my second major example.

4.1. Halperin-Lee-Read. — That story begins in 1994 with Halperin, Lee, and Read, who proposed a Fermi-liquid ground state for the lowest Landau level at half filling. Converting electrons into composite fermions by a procedure called flux attachment, they argued that the latter could form a Fermi sea; the idea is that the flux attachment cancels the magnetic field on average. In the field-theory Lagrangian, flux attachment is implemented by the introduction of a Chern-Simons (fictitious) gauge field, a.

Although the HLR theory was quite successful in fitting the observed phenomena, two bothersome issues remained: (i) there never emerged any plausible theory of the effective mass, and (ii) the HLR Lagrangian does not exhibit any sort of particle-hole symmetry. The latter is a serious problem because the Coulomb interaction projected to the lowest Landau level is ostensibly particle-hole symmetric. The new theory proposed by Son resolves both of these issues.

4.2. Wang-Senthil. — The composite fermion emerging from Son's theory is pictured as shown here. In the old HLR theory one assumed that one electron was bound to one hole of unit charge created by the insertion of a double flux (or double vortex). In the new picture, the two vortices are separated in space; one carries a charge of $-1/2$, the other one binds an electron and thus carries the opposite charge $(+1/2)$. The new picture is consistent with particle-hole symmetry; the HLR picture is not.

4.3. Son's logic. — Son starts from the observation that, for all practical purposes, the lowest Landau level can be realized as the zero-energy sector of a Dirac fermion in a homogeneous magnetic field. Given that starting point, he uses a particle-vortex transformation to pass to a dual description in terms of a fermionic vortex field ψ. The outcome is a theory with neither a mass term nor a Chern-Simons term for the emergent gauge field a, in stark contrast with HLR. (Son argues that HLR can be retrieved by non-relativistic reduction.)

And, getting to the punch line, he gives a clear answer to the long-standing question about the fate of particle-hole symmetry: it is realized as the anti-linear operation of CT – relativistic charge conjugation combined with time reversal.

4.4. Symmetry considerations. — The details are quite stunning, as indicated on my last slide. The electromagnetic gauge field is *time-twisted*: which means that under time reversal it transforms by negative pullback; but, of course, by standard pullback under parity. In contrast, the charge 3-current two-form is *space-twisted*; correspondingly, a transforms under time reversal by standard pullback and under parity by negative pullback.

The inverted transformation behavior of a (as compared with A) has a surprising effect on the discrete symmetries: the first-quantized Hamiltonian for the fermionic vortex field is odd under each of time reversal, parity, and charge conjugation. And, most strikingly, time reversal is now linear, whereas parity is anti-linear!

After second quantization, the product CT is anti-linear and does ***not*** swap creation with annihilation operators; this is how it manages to be a symmetry of a Fermi-liquid state.

4.5. Experimental data (again). — To conclude: I think I have reported to you an example of theoretical physics at its very best. A long-standing discrepancy was resolved by deep insight and compelling analysis and is now stimulating new experiments.

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