Bott Periodicity and the “Periodic Table” of Topological Insulators and Superconductors

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Kitaev's Periodic Table of topological insulators/ superconductors

from Hasan & Kane, Rev. Mod. Phys. (2011):

	Symmetry			d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

QSHI = Quantum Spin Hall Insulator

Quantum Hall Effect

He-3 (B phase)

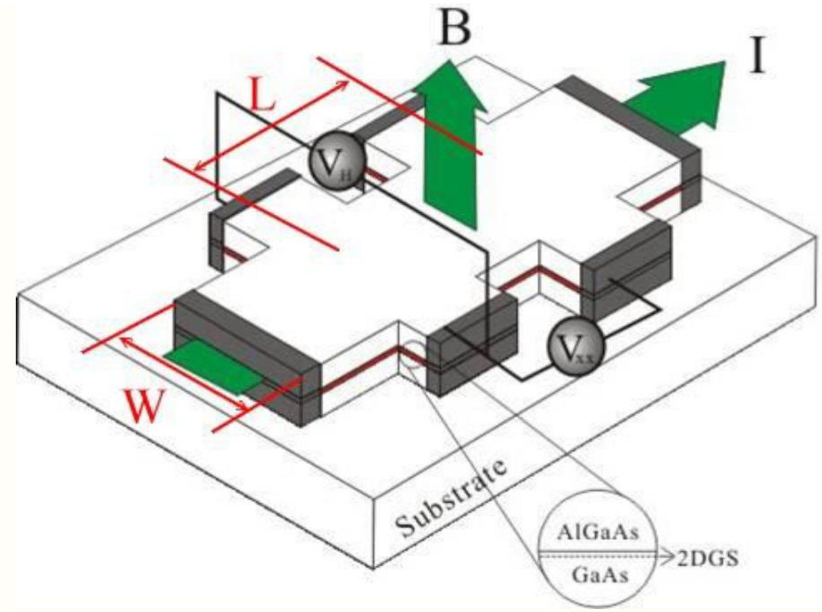
QSHI: HgTe

Majorana

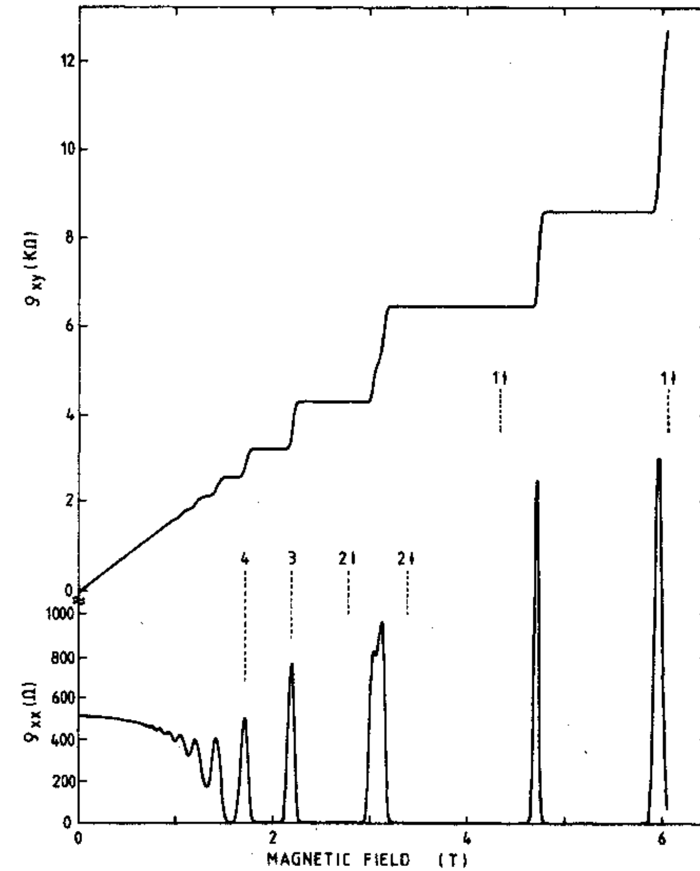
$Bi_2 Se_3$

Schnyder, Ryu, Furusaki, Ludwig (2008); Kitaev (2009); Teo & Kane (2010); Stone, Chiu, Roy (2011); Freedman, Hastings, Nayak, Qi, Walker, Wang (2011); Abramovici & Kalugin (2012); Freed & Moore (2013)

Integer Quantum Hall Effect



2D disordered electrons; high *B*, low *T*

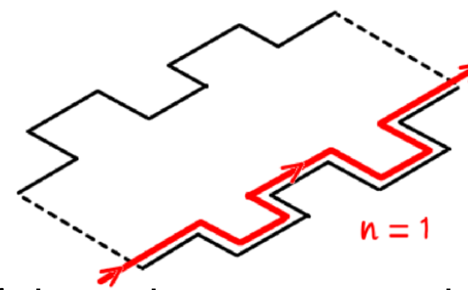


D.J. Thouless (1982-5; Nobel Prize Physics 2016):

Ground state = complex line bundle $\mathscr{A}$ over T^2

Hall conductance $= I/V_{\rm H} = n\,e^2/h$

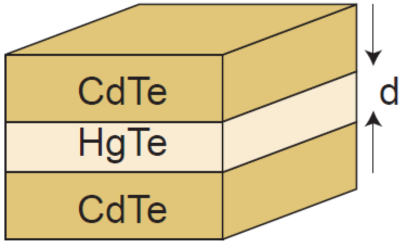
n = (integral of) first Chern class of $\mathscr{A}$



bulk-boundary correspondence

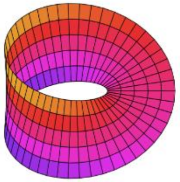
Quantum Spin Hall Insulator

(Kane & Mele, 2005)



Strong spin-orbit scattering (preserves time-reversal invariance) causes band inversion $\curvearrowright$ the bundle of Fermi projections is twisted.

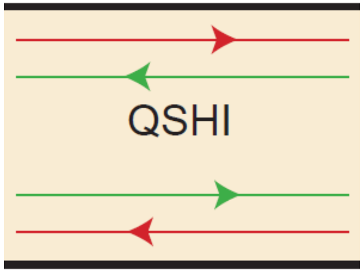
This twisting is detected by the Kane-Mele index.



Bulk-boundary correspondence:

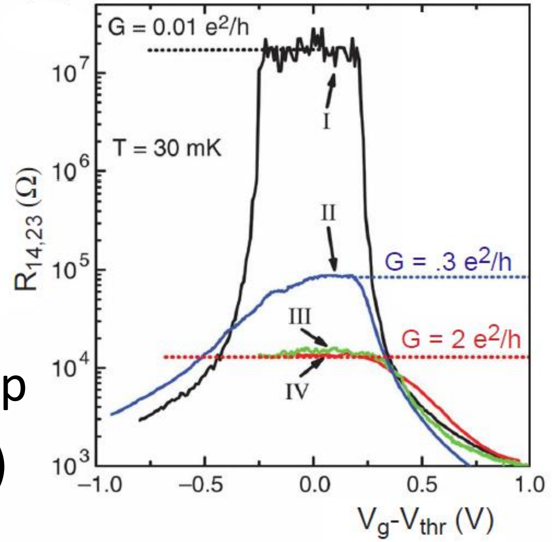
non-trivial bulk topological invariant

→ perfectly conducting surface mode



Molenkamp group

(Würzburg, 2007)



Majorana chain (gapped 1d superconductor)

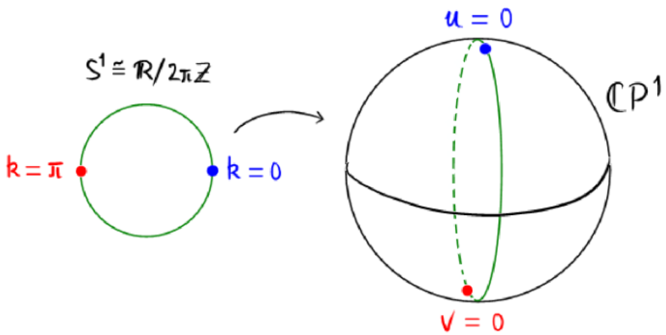
No symmetries, "spinless fermions", single band; momentum k

Bogoliubov transformation: $\gamma_k = u(k)\, c_k + v(k)\, c^\dagger_{-k} \quad (k \in \mathbb{R}/2\pi\mathbb{Z})$

Fermi constraint: $\{\gamma_k, \gamma_{-k}\} = 0$

$\curvearrowright \quad u(k)\, v(-k) + v(k)\, u(-k) = 0\,.$

Weak pairing: $u(0) = v(\pi) = 0$ (topol. invt.)



Weak pairing: bulk-boundary correspondence → gapless edge state

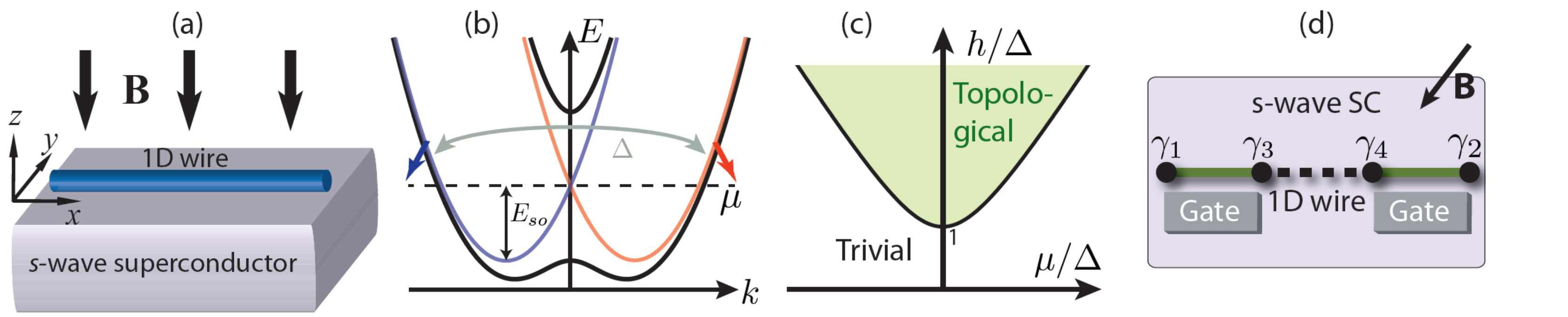
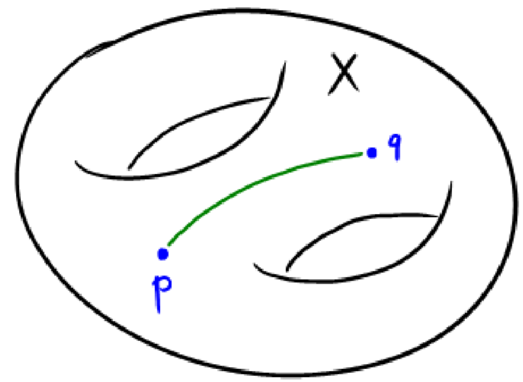


FIG. 6. (a) Basic architecture required to stabilize a topological superconducting state in a 1D spin-orbit-coupled wire. (b) Band structure for the wire when time-reversal symmetry is present (red and blue curves) and broken by a magnetic field (black curves). When the chemical potential lies within the field-induced gap at $k = 0$, the wire appears 'spinless'. Incorporating the pairing induced by the proximate superconductor leads to the phase diagram in (c). The endpoints of topological (green) segments of the wire host localized, zero-energy Majorana


"He is not only a good conductor, he is a Superconductor!"
WATCH YOUR STEP
Claus Grupen 2013

Bott Periodicity

Bott Periodicity (I)



X compact Riemannian manifold

ν triple $(p, q; h)$

X^ν space of minimal geodesics of class ν

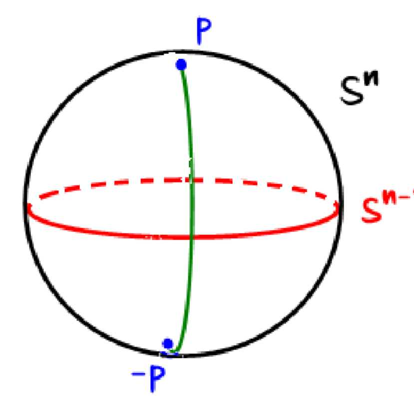
$|\nu|$ minimum over all non-minimal geodesics of class ν of no. of negative eigenvalues of Hessian of length function

Thm (Bott, 1959). If X is a symmetric space, then so is X^ν, and $\pi_d(X^\nu) = \pi_{d+1}(X) \quad \text{for} \quad 0 < d < |\nu| - 1\,.$

Example. $X = S^n, \quad X^{(p,-p)} = S^{n-1}, \quad |\nu| = 2(n-1)$

$\Longrightarrow \pi_d(S^{n-1}) = \pi_{d+1}(S^n) \quad \text{for} \quad 0 < d < 2n-3$

(Freudenthal suspension theorem).



What's a symmetric space?

Riemann tensor: $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm}$

Def.: A (locally) symmetric space is a Riemannian manifold $X = U/K$ with covariantly constant curvature: $\nabla R = 0$.

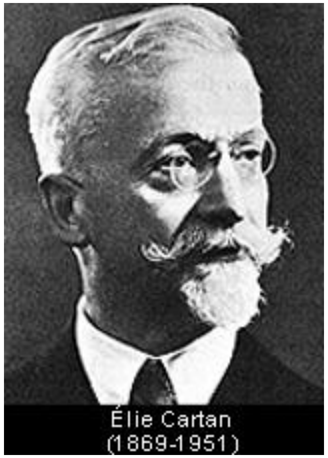
Ex. 1: the round two-sphere $X = \mathrm{S}^2$, $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

Ex. 2: the set $X = \mathrm{Gr}_n(\mathbb{C}^N) = \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$ of all subspaces $\mathbb{C}^n \simeq V \subset \mathbb{C}^N$

Classification:

Globally symmetric spaces classified by E. Cartan (1926)

10 large families: *A*, *A*I, *A*II, *A*III, *BD*, *BD*I, *C*, *C*I, *C*II, *D*III



Bott Periodicity (II)

ν–sequences: complex and real

Raoul Bott (1923-2005)

		π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
C_{even}	$\mathrm{U}/\mathrm{U}\times\mathrm{U}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C_{odd}	U	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R_0	O/U	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R_1	U/Sp	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
R_2	$\mathrm{Sp}/\mathrm{Sp}\times\mathrm{Sp}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
R_3	Sp	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
R_4	Sp/U	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
R_5	U/O	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
R_6	$\mathrm{O}/\mathrm{O}\times\mathrm{O}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R_7	O	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1. Bott Periodic Table of the stable homotopy groups of symmetric spaces

Bott Periodicity (III): Morse Theory

Manifold M, function $f: M \to \mathbb{R}$ (smooth and proper).

$$M^a := \{x \in M \mid f(x) \leq a\}.$$

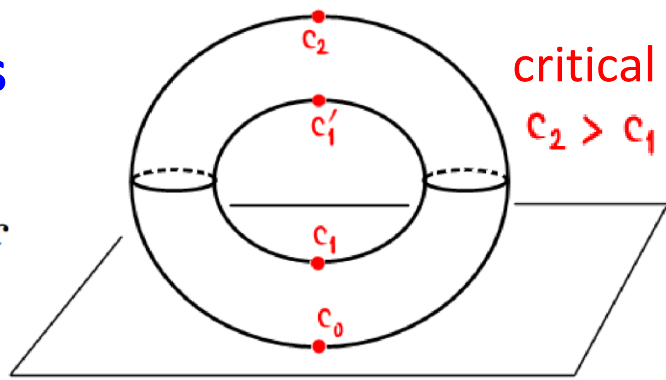
homotopy-equivalent

Thm 1. If f has no critical values in $[a,b]$, then $M^a \sim M^b$.

Thm 2. Let $x \in M$ be a non-degenerate critical point of f of index n. If x is the only critical point in $f^{-1}[f(x)-\varepsilon, f(x)+\varepsilon]$, then $M^{f(x)+\varepsilon} \sim M^{f(x)-\varepsilon} \cup e_n$ (n-cell e_n).

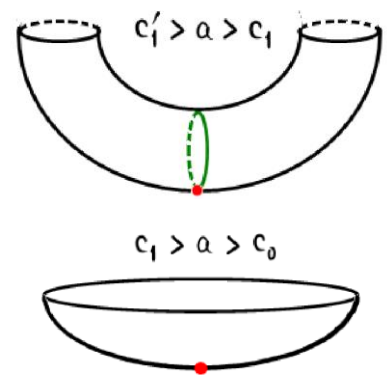
Example: 2-torus

height function f



critical values

$c_2 > c_1 > c_1' > c_0$



Fact. Functions f of the needed kind (Morse functions) do exist.

Bott Periodicity (IV): Idea of proof

$M = \Omega^{\nu} X$ (space of paths in X from p to q of homology class h),
Morse function $f \equiv$ length of path.

$M^{a} = \emptyset$ if $a < L \equiv$ geodesic distance from p to q,

$M^{a} = X^{\nu}$ if $L < a < L + \varepsilon$,

$\vdots$

$\Omega^{\nu} X = X^{\nu} \cup e_n \cup e_n' \cup \ldots$ where $\dim e_n \geq |\nu|$.

Hence $\pi_{d+1}(X) = \pi_d(\Omega^{\nu} X) = \pi_d(X^{\nu})$ if $0 < d < |\nu| - 1$.

Bott Periodicity (V): Clifford algebra

$W \cong \mathbb{C}^{2n}$ Hermitian vector space with compatible symmetric bilinear form $\{\cdot\,,\cdot\}: W \otimes W \to \mathbb{C}$.

$J_1, \ldots, J_s$ generators of Clifford algebra on W:

$$J_l J_m + J_m J_l = -2\delta_{lm} \mathrm{Id}_W, \quad (J_l^* = J_l^{-1} = J_l^T).$$

Note: $W = E_{+\mathrm{i}}(J) \oplus E_{-\mathrm{i}}(J)$.

Let $C_0(n) := \mathrm{Gr}_n(W)$ Grassmann m'fld of complex n-planes in W.
Define

$$C_s(n) := \{A \in C_0(n) \mid J_1 A = \ldots = J_s A = A^{\mathrm{c}}\},$$
$$R_s(n) := \{A \in C_s(n) \mid \{A, A\} = 0\}.$$

Lemma.

$C_0(n) \supset C_1(n) \supset C_2(n) \supset \ldots$ complex Bott $\nu-$ sequence,

$R_0(n) \supset R_1(n) \supset R_2(n) \supset \ldots$ real Bott $\nu-$ sequence.

More precisely,

s	$C_s(8r)$	$R_s(8r)$
0	$\cup_{p+q=16r}\mathrm{U}_{16r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\mathrm{O}_{16r}/\mathrm{U}_{8r}$
1	$(\mathrm{U}_{8r} \times \mathrm{U}_{8r})/\mathrm{U}_{8r}$	$\mathrm{U}_{8r}/\mathrm{Sp}_{8r}$
2	$\cup_{p+q=8r}\mathrm{U}_{8r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\cup_{p+q=4r}\mathrm{Sp}_{8r}/(\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q})$
3	$(\mathrm{U}_{4r} \times \mathrm{U}_{4r})/\mathrm{U}_{4r}$	$(\mathrm{Sp}_{4r} \times \mathrm{Sp}_{4r})/\mathrm{Sp}_{4r}$
4	$\cup_{p+q=4r}\mathrm{U}_{4r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\mathrm{Sp}_{4r}/\mathrm{U}_{2r}$
5	$(\mathrm{U}_{2r} \times \mathrm{U}_{2r})/\mathrm{U}_{2r}$	$\mathrm{U}_{2r}/\mathrm{O}_{2r}$
6	$\cup_{p+q=2r}\mathrm{U}_{2r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\cup_{p+q=2r}\mathrm{O}_{2r}/(\mathrm{O}_p \times \mathrm{O}_q)$
7	$(\mathrm{U}_r \times \mathrm{U}_r)/\mathrm{U}_r$	$(\mathrm{O}_r \times \mathrm{O}_r)/\mathrm{O}_r$

Sketch of proof for $R_0(n)$.

Let $W = A \oplus A^c$ and $\{A, A\} = 0$.

Then for $g \in \mathrm{U}(W) \cap \mathrm{O}(W) \cong \mathrm{O}_{2n}$ it follows that

$$W = gA \oplus g \cdot A^c, \text{ and } \{gA, gA\} = 0.$$

The O_{2n}-action on $R_0(n)$ is transitive.

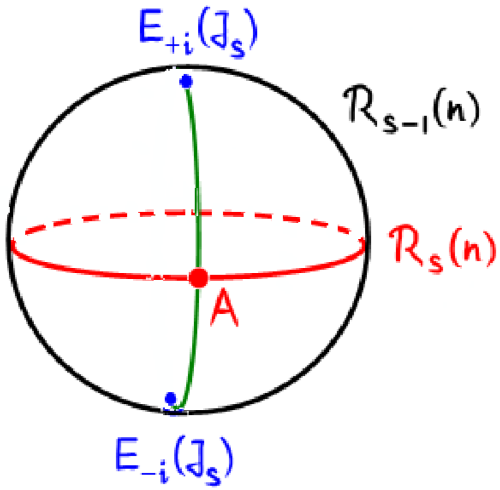
The stabilizer of A is $\mathrm{U}(A) \cong \mathrm{U}_n \subset \mathrm{O}_{2n}$. Hence $R_0(n) \cong \mathrm{O}_{2n}/\mathrm{U}_n$.

Bott Periodicity (VI): Bott Map

Make the identifications $X \equiv R_{s-1}(n)$, $X^{\nu} \equiv R_s(n)$;
and $\Omega^{\nu} X \equiv$ paths from $E_{+\mathrm{i}}(J_s)$ to $E_{-\mathrm{i}}(J_s)$.

Given $A \in R_s(n)$,
assign to A a minimal geodesic $\beta : [-\pi/2, \pi/2] \to R_{s-1}(n)$
by $t \mapsto \beta_t(A) = \mathrm{e}^{(t/2)J_s J(A)} \cdot A, \qquad J(A) = \mathrm{i}(\Pi_A - \Pi_{A^c})$.

This is the Bott Map ...



Comment. Same for $C_s(n)$ instead of $R_s(n)$.

Bott Periodicity (II')

ν−sequences: complex and real

		π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
C_{even}	$\mathrm{U}/\mathrm{U}\times\mathrm{U}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C_{odd}	U	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R_0	O/U	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R_1	U/Sp	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
R_2	$\mathrm{Sp}/\mathrm{Sp}\times\mathrm{Sp}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
R_3	Sp	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
R_4	Sp/U	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
R_5	U/O	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
R_6	$\mathrm{O}/\mathrm{O}\times\mathrm{O}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R_7	O	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Bott Map

Table 1. Bott Periodic Table of the stable homotopy groups of symmetric spaces

Free-Fermion Ground States

Universal model (notation/setting)

Quasi-particle vacuum $\equiv$ free-fermion ground state
$\equiv$ Hartree-Fock-Bogoliubov mean-field ground state

Single-particle annihilation (creation) operators $c_\alpha \ (c_\alpha^\dagger)$

$$U = \mathrm{span}_{\mathbb{C}}\{c_1, \ldots, c_n\}, \quad V = \mathrm{span}_{\mathbb{C}}\{c_1^\dagger, \ldots, c_n^\dagger\}$$

$W = U \oplus V$ space of field operators (Nambu space)

Structure on W: CAR bilinear form $\{\cdot,\cdot\} : W \otimes W \to \mathbb{C}$

Hermitian conjugation $\gamma : W \to W$

Fact. Free-fermion ground states $|\mathrm{g.s.}\rangle \overset{1:1}{\leftrightarrow} W = A \oplus A^{\mathrm{c}}$

where $A = \mathrm{span}_{\mathbb{C}}\{\widetilde{c}_1, \ldots, \widetilde{c}_n\} \subset W$ and the $\widetilde{c}_\alpha$ are

quasi-particle annihilation ops: $\widetilde{c}_\alpha |\mathrm{g.s.}\rangle = 0 \quad (\alpha = 1, \ldots, n)$

Fermi constraint: $\{A, A\} = 0$

Symmetries in quantum mechanics

Q: What's a symmetry in quantum mechanics?

A: An operator $T : \mathscr{R}\psi_1 \mapsto \mathscr{R}\psi_2$ on Hilbert rays that preserves all transition probabilities: $\left|\langle T\mathscr{R}\psi_2 , T\mathscr{R}\psi_1\rangle\right|^2 = \left|\langle \mathscr{R}\psi_2 , \mathscr{R}\psi_1\rangle\right|^2.$

Wigner's Theorem:

A symmetry T in quantum mechanics can always be represented on Hilbert space by an operator $\hat{T}$ which is either unitary or anti-unitary.

$$\langle \hat{T}\psi_2|\hat{T}\psi_1\rangle = \overline{\langle\psi_2|\psi_1\rangle}$$

Eugene P. Wigner

Remark 1: The symmetries form a group, G.

Remark 2: Symmetries commute with the Hamiltonian ($\hat{T}H = H\hat{T}$). Thus "chiral symmetry" ($\gamma_5 D \gamma_5 = -D$) is not a symmetry.

J. Math. Phys. 3 (1962) 1199

The Threefold Way.
Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
(Received June 22, 1962)

Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

Tenfold Way

Communications in Mathematical Physics

Symmetry Classes of Disordered Fermions

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Abstract: Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

The primary data of the classification are a Nambu space of fermionic field operators which carry a representation of some symmetry group. Our approach is to eliminate all

In this paper, it is proved that the symmetry classes of disordered fermions are in one-to-one correspondence with the 10 large families of symmetric spaces.

Symmetry operations of the Tenfold Way

- **Anti-unitary** symmetries:

1. Time reversal $T: U \oplus V \to U \oplus V, \qquad T^2 = (-1)^{2S}.$

2. Particle-hole conjugation $C: U \oplus V \to V \oplus U, \quad C^2 = \mathrm{Id}_W.$

- **Unitary** symmetries include

$\mathrm{U}(1)_A$ symmetry (charge operator Q),

$$\mathrm{e}^{\mathrm{i}\theta Q}: \; u + v \mapsto \mathrm{e}^{-\mathrm{i}\theta} u + \mathrm{e}^{+\mathrm{i}\theta} v \, .$$

$\mathrm{SU}(2) = \mathrm{Spin}(3)$ spin rotations generated by S_1, S_2, S_3 .

$$\hat{R}: \; u + v \mapsto \; Ru + (R^{-1})^t v.$$

Universal Model (including symmetries)

Clifford algebra (Kitaev) of pseudo-symmetries $J_1, \ldots, J_s$:

$$J_l J_m + J_m J_l = -2\delta_{lm} \mathrm{Id}_W, \qquad J_l \in \mathrm{U}(W) \cap \mathrm{O}(W).$$

Definition. A free-fermion ground state of symmetry class s is a polarization $W = A \oplus A^{\mathrm{c}}$ where the complex vector space A is subject to

- Fermi constraint: $\{A, A\} = 0$
- pseudo-symmetries: $J_1 A = \ldots = J_s A = A^{\mathrm{c}}$.

Example. Time-reversal symmetry (spin ½): $TA = A$

$$\curvearrowright J_1 = \gamma \circ T \quad (\gamma : \; c \leftrightarrow c^\dagger \text{ Hermitian conj.})$$

Indeed, $J_1 A = \gamma(TA) = \gamma A = A^{\mathrm{c}}$ and $J_1^2 = T^2 = -\mathrm{Id}_W$.

Kitaev Sequence ("real" and "complex" classes)

class	symmetries	s	pseudo-syms
D	none	0	Fermi constraint
DIII	T (time reversal)	1	$J_1 = \gamma T$
AII	T, Q (charge)	2	$J_2 = \mathrm{i}\gamma T Q$
CII	T, Q, C (ph-conj)	3	$J_3 = \mathrm{i}\gamma C Q$
C	S_1, S_2, S_3 (spin rot)	4	see below
CI	S_1, S_2, S_3, T	5	
AI	S_1, S_2, S_3, T, Q	6	
BDI	S_1, S_2, S_3, T, Q, C	7	

class	symmetries	s	pseudo-syms
A	Q	0	Fermi constraint
AIII	Q, C	1	$J_1 = \mathrm{i}\gamma C$

Diagonal Map

Bott-Kitaev Periodic Table

Question: does there exist a ``Diagonal Map''?

from Hasan & Kane, Rev. Mod. Phys. (2011):

	Symmetry				d							
	AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
U/U × U	A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
U	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
O/O × O	AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
O	BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
O/U	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
U/Sp	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
Sp/Sp × Sp	AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
Sp	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
Sp/U	C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
U/O	CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

?

Quantum Hall Effect

He-3 (B phase)

QSHI: HgTe

Majorana

$Bi_2 Se_3$

?

Clean & Disordered Systems

Clean limit: translations are symmetries

Conserved momentum $k \in M \curvearrowright$ decomposition $A = \bigoplus_{k \in M} A_k$

Gapped system (insulator) $\curvearrowright \{A_k\}_{k \in M}$ vector bundle

Disorder: Non-Commutative Geometry approach, developed for IQHE by Bellissard et al. (following A. Connes)

C^*-algebra $\mathscr{A}$ of bounded operators

Pairing between cyclic cohomology and $K_0(\mathscr{A})$

$\curvearrowright$ Hall conductance = non-commutative Chern number

Prodan, Schulz-Baldes (2013), Thiang (2014), Kellendonk; Boersema, Loring (2015), Carey et al. (2016)

Diagonal Map: heuristic

Under the decomposition $A = \bigoplus_{k \in M} A_k$ (conserved momentum) the Fermi constraint $\{A, A\} = 0$ refines to

$$\{A_k, A_{-k}\} = 0 \quad \text{(for all } k \in M \text{)}.$$

Thus our free-fermion ground states are vector bundles $\pi : \mathscr{A} \to M$ subject to a $\mathbb{Z}_2$-equivariance condition

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\tau_{\mathscr{A}}} & \mathscr{A} \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\tau_M} & M \end{array}$$

with non-trivial involution $\tau_M : M \to M, \quad k \mapsto -k.$

Diagonal Map $(d,s) \to (d+1,s+1)$

Starting point: $\widetilde{J}_1, \ldots, \widetilde{J}_s$ and $\{\widetilde{A}_k\}_{k \in M_d}$

Preparatory step: jack up by (1,1) periodicity

$\curvearrowright$ New starting point: $J_1 , \ldots, J_s ; I, K$ and $\{A_k\}_{k \in M_d}$

Define: $\boxed{A_{k,t} := \mathrm{e}^{(t/2)KJ(A_k)} \cdot A_k}$ $\quad J(A) = \mathrm{i}(\Pi_A - \Pi_{A^c})$

Note:

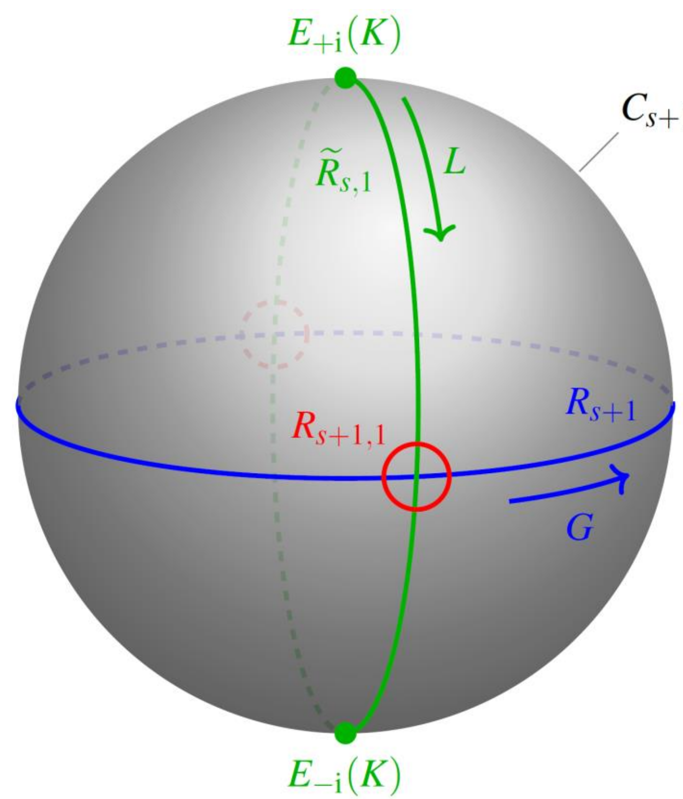
1. Fermi constraint: $\{A_{k,t} \, , A_{-k,-t}\} = 0$ ✓
2. Pseudo-syms: $J_1 A_{k,t} = \ldots = J_s A_{k,t} = A^{\mathrm{c}}_{k,t} = I A_{k,t}$ ✓
3. Degeneration: $A_{k,t=\pm\pi/2} = E_{\mp\mathrm{i}}(K)$ ✓

Outcome: V.B. $\{A_{k,t}\}_{(k,t) \in M_{d+1}}$ in class $s+1$ on $M_{d+1} = \widetilde{S}(M_d)$

Diagonal Map^2 = Bott Map

$$\begin{tikzcd} R_s(n) \simeq R_{s+1,1}(2n) \arrow[r, "\text{Bott Map}"] \arrow[d, "\text{Diagonal Map}"] & \Omega^\nu R_{s-1}(n) \simeq \Omega^\nu R_{s,1}(2n) \\ \left(\Omega^\nu C_{s+1}(2n)\right)^{\mathbb{Z}_2} \arrow[ur, "p"'] & \end{tikzcd}$$

$R_{s+1}(2n)$



The fibration p exists only for $s = 2, 6$.

Summary of colloquium

- Motivation: topological insulators and superconductors
- Bott periodicity theorem from Morse theory
- Free-fermion ground states with symmetries (imposed in a prescribed order) realize the complex/real Bott sequences.
- For Kitaev's Periodic Table replace Bott Map by Diagonal Map
- **Reference:** R. Kennedy and M.R. Zirnbauer, arXiv:1509.2537, Commun. Math. Phys. 342 (2016) 909—963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Thank you!

(The End)