Super Fourier Analysis and Localization in Disordered Wires

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The problem of calculating the zero-temperature mean conductance $\langle c\rangle$ of a disordered thick metallic wire coupled at both ends to ideal leads is formulated as a diffusion problem on a Riemannian symmetric superspace **G/K** (Efetov). The problem is solved exactly by Fourier transforming the diffusion kernel. Although the solution agrees with known results for the case of orthogonal and unitary symmetry, it has the surprising feature that $\langle c\rangle$ never falls below the minimum value of $e^2/2h$ for long wires with symplectic symmetry, in leading order of the expansion around the metallic and thick limit.

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It is a celebrated result of mathematical physics that random Schrödinger operators (RSO) in one dimension have, with probability one, a pure point spectrum and exponentially decreasing eigenfunctions. This result was first proved by Kunz and Souillard [1] and subsequently improved by many authors.

A major motivation for the study of RSO has been their use in the modeling of disordered solids in the single-particle approximation. It should be noted, however, that while mathematicians have studied RSO as self-adjoint operators on $l^2(\mathbb{Z})$ [or $L^2(\mathbb{R})$], present experiments on mesoscopic disordered solids [2] emphasize the role of finite-size effects and of the leads attached for the purpose of making voltage and current measurements. Thus, the challenge from modern solid-state physics is to study RSO in a *finite volume* and to take into account the *escape of flux* into external leads. To formulate the problem in concrete terms, let $H-i\Gamma$ be the non-self-adjoint operator defined on the interval $[0,L]$ by

$$((H-i\Gamma)\psi)(x) = -\frac{d^2\psi}{dx^2}(x)+\{U(x)-i\gamma[\delta(x)+\delta(x-L)]\}\psi(x)\,, \quad (1)$$

where U is a random potential and $\gamma>0$ a measure of the loss of probability due to escaping flux. Then an interesting quantity to calculate is (γ^2 times) the absolute square of the end-to-end Green's function,

$$c(E)\stackrel{\text{def}}{=}\gamma^2|\langle 0|(E-H+i\Gamma)^{-1}|L\rangle|^2\,,$$

for real energies E, this being *conductance* by a formula of the Landauer type [3].

The results described below do not concern (1), but the more realistic case of a "thick metallic wire," meaning a quasi-one-dimensional disordered system satisfying the inequalities $1\ll E_F\tau\ll k_F^2S$ with E_F the Fermi energy, τ the elastic mean free time, k_F the Fermi wave number, and S the cross-sectional area of the wire. Localization in such systems has previously been discussed in Ref. [4], on which the present Letter is based. It is well known that disordered conductors in the metallic limit come in three different symmetry classes, called orthogonal (potential scattering), unitary (magnetic field and/or magnetic impurities), and symplectic (strong spin-orbit scattering). The models I study correspond to these symmetry classes and differ from those of Ref. [4] only by the choice of boundary conditions which are adapted to the scattering-theoretic formalism of Iida, Weidenmüller, and Zuk [5] and Altland [6], who have shown how to treat disordered wires of finite length, coupled semirealistically at both ends to a number of continuum states supported by two ideal leads.

In Refs. [5] and [6] the mean, $\langle c\rangle$, and mean square, $\langle c^2\rangle$, of the conductance were calculated by means of an asymptotic expansion valid for short wires. Here I will present the *exact* result for $\langle c\rangle$ for wires of arbitrary length. I am, of course, aware that the focus in mesoscopic physics has shifted from averages to fluctuations and, ultimately, to the entire distribution function. While appearing feasible, the exact calculation of *higher* cumulants of the conductance requires further research and is left for the future. In any case, the computation of $\langle c\rangle$ is interesting in its own right, as it generalizes, for the first time, the Fourier transform to a class of homogeneous superspaces of rank greater than one.

The technique used is a mapping pioneered by Schäfer and Wegner [7] and Efetov [8], which is valid for the systems under consideration and associates with each symmetry class a nonlinear σ model defined over a Riemannian symmetric superspace **G/K**. I call these spaces "Efetov's spaces" and denote by I (II,III) the space corresponding to systems with orthogonal (unitary, symplectic) symmetry. (More precisely, II corresponds to model IIa of Ref. [4].) The derivation of the nonlinear σ model will not be repeated here. Instead, I will formulate directly the problem that emerges, when the quantity to be calculated is $\langle c\rangle$.

Each of Efetov's spaces is a supermanifold with $2q$ commuting and $2q$ anticommuting degrees of freedom, where $q=4$ (2,4) for space I (II,III). In addition, each of these spaces has the structure of a coset space **G/K**, **G** being a Lie group [the definition of which can be inferred from Eqs. (3.55)–(3.59) of Ref. [8]], and **K** a compact subgroup. **G** consists of 8×8 supermatrices in each case.

With Λ denoting the matrix $\mathrm{diag}(1_4,-1_4)$, **K** can be characterized as the subgroup of elements $k\in\mathbf{G}$ that satisfy $k\Lambda k^{-1}=\Lambda$. There is a one-to-one correspondence between the elements of **G/K**, which are cosets $g\mathbf{K}$ ($g\in\mathbf{G}$), and supermatrices Q of the form $Q=g\Lambda g^{-1}$. A Riemannian structure on each of the spaces **G/K** is defined by the **G**-invariant metric tensor $\mathrm{str}(dQdQ)$, where "str" denotes the supertrace.

In the metallic regime, and for zero temperature, information about the propagation of a single electron along a thick disordered wire of length $L\gg\sqrt{S}$, is encoded [4] in a function W on $\mathbf{G/K}\times\mathbf{G/K}\times\mathbb{R}^+$ defined by

$$\left(\frac{\partial}{\partial t}-\Delta\right)W(Q,\cdot\,;t)=0,\quad \lim_{t\to 0+}W(Q,Q';t)=\delta(Q,Q'),\tag{2}$$

where Δ is the Laplace-Beltrami operator on **G/K** and δ the unit kernel (Dirac's delta distribution). The information about the coupling of the disordered wire to the ideal leads, on the other hand, is encoded in a function f on **G/K**. The precise form of f is rather complicated in general; it can be found in Refs. [5] and [6] and need not be reproduced here. Simplifications occur for an "open system" with a large number γ of scattering channels (at fixed energy E_F) in the leads. In this case, which is the one that I will consider, f is approximated with sufficient accuracy by

$$f(Q)=(\gamma/d)Q_{13}^{12}\exp[-(\gamma/4d)\,\mathrm{str}\Lambda Q],$$

where I have used the notation of Ref. [4] for the matrix elements of Q, and $d=2$ (2,1) for model I (II,III). Now, if DQ denotes the **G**-invariant integration measure on **G/K** normalized by $\int_{\mathbf{G/K}}DQ=1$, the average conductance $\langle c\rangle$, measured in units of e^2/h, has the expression

$$\langle c\rangle=d\int_{\mathbf{G/K}}\int_{\mathbf{G/K}}\bar{f}(Q)W(Q,Q';dL/4\xi)f(Q')DQ'DQ\,.\tag{3}$$

Here ξ equals $4\pi\rho D$, with D the diffusion constant and ρ the density of states per unit of energy and unit of length of the wire.

Clearly, the problem of calculating $\langle c\rangle$ exactly as a function of length L amounts to the problem of expanding W in a complete system of eigenfunctions of Δ. The theory of such eigenfunction expansions, known also as the theory of the "Fourier transform," is completely understood [9] for the case of Riemannian symmetric spaces **G/K** of the classical (i.e., nonsuper) type. It turns out that the classical notion of Fourier transform can be generalized to a wide class of Riemannian symmetric superspaces **G/K**. For the case of a rank-one space, the "hyperbolic superplane," a detailed description of this generalization and a rigorous proof of the Fourier inversion theorem have been given in Ref. [10]. Relegating all mathematical details to a future publication, I here wish to announce the nontrivial result that the method of Ref. [10] can be adapted to Efetov's spaces.

Let ϕ_ν^m be an eigenfunction of $-\Delta$ with eigenvalue $\tilde{\Delta}(\nu)$, m being a set of quantum numbers that account for degeneracy. The main series of eigenfunctions appearing in the Fourier expansion of W have the eigenvalues

$$\text{(I)}\quad \tilde{\Delta}(l,\lambda_1,\lambda_2)=l^2+\lambda_1^2+\lambda_2^2+1\quad (l\in 2\mathbb{N}+1,\ \lambda_1\in\mathbb{R}^+,\ \lambda_2\in\mathbb{R}^+),$$

$$\text{(II)}\quad \tilde{\Delta}(l,\lambda)=l^2+\lambda^2\quad (l\in 2\mathbb{N}-1,\ \lambda\in\mathbb{R}^+),$$

$$\text{(III)}\quad \tilde{\Delta}(l_1,l_2,\lambda)=l_1^2+l_2^2+\lambda^2-1\quad (l_1\in 2\mathbb{N}-1,\ l_2\in 2\mathbb{N}-1,\ \lambda\in\mathbb{R}^+).$$

For spaces I and III there appear, in addition, the subsidiary series

$$\text{(I)}\quad \tilde{\Delta}(1,\lambda,\lambda)=2(\lambda^2+1)\quad (\lambda\in\mathbb{R}^+),$$

$$\text{(III)}\quad \tilde{\Delta}(i,l,l-2)=2(l-1)^2\quad (l\in 2\mathbb{N}+1)\ \text{and}\ \tilde{\Delta}(i,1,1)=0.$$

Finally, an eigenvalue zero, corresponding to the constant function, appears in all cases.

The Fourier expansion of W has the general form

$$W(Q,Q';t)=1+\int e^{-t\tilde{\Delta}(\nu)}\sum_m \phi_\nu^m(Q)\bar{\phi}_\nu^m(Q')d\mu(\nu),\tag{4}$$

where the integral sign actually stands for both integrations (over λ's) and summations (over l's), and the measure $d\mu(\nu)$ depends on how the functions ϕ_ν^m are normalized [11]. Taking ϕ_ν^0 to denote a function with the invariance property $\phi_\nu^0(kQk^{-1})=\phi_\nu^0(Q)$ for $k\in\mathbf{K}$, I fix the normalization by requiring that ϕ_ν^0 has the expansion

$$\phi_\nu^0(Q)=-2^{-4}\tilde{\Delta}(\nu)\,\mathrm{str}\Lambda Q+\cdots$$

around $Q=\Lambda$. Upon insertion of (4) into (3), the double integral factors into a sum of decoupled integrals over Q and Q'. Since I have only a generating function for ϕ_ν^m, but no closed-form expression in terms of elementary functions, I cannot evaluate these integrals analytically in general. However, in the limit under consideration ($\gamma\gg 1$) knowledge of the "Q-close-to-Λ" expansion is sufficient, and I am able to show that

$$\langle c\rangle=\tfrac{1}{2}d\int\tilde{\Delta}(\nu)\exp[-d(L/\xi+1/\gamma)\tilde{\Delta}(\nu)/4]d\mu(\nu)\,.\tag{5}$$

Finally, I calculate the measure $d\mu(v)$ by studying the asymptotic behavior of ϕ_v^0 in the "noncompact" directions of **G/K**. To do this, I write down a suitable integral representation for ϕ_v^0 ("Harish-Chandra's formula") and pass to an integral over a certain nilpotent group **N**, as explained in Refs. [9,10]. With the definition $T=\exp[-d(L/\xi+1/\gamma)\tilde{\Delta}/4]$, I then obtain expressions for the average conductance of the following form:

$$\text{(I)}\ \langle c\rangle=\pi\int_0^\infty d\lambda\tanh^2(\pi\lambda/2)T(1,\lambda,\lambda)+2^4\sum_{l\in 2\mathbf{N}+1}\int_0^\infty d\lambda_1\int_0^\infty d\lambda_2\, l(l^2-1)\lambda_1\tanh(\pi\lambda_1/2)\lambda_2\tanh(\pi\lambda_2/2)(l^2+\lambda_1^2+\lambda_2^2+1)\times\prod_{\sigma,\sigma_1,\sigma_2=\pm1}(-1+\sigma l+i\sigma_1\lambda_1+i\sigma_2\lambda_2)^{-1}T(l,\lambda_1,\lambda_2)\,,$$

$$\text{(II)}\ \langle c\rangle=2^2\sum_{l\in 2\mathbf{N}-1}\int_0^\infty d\lambda\,\lambda\tanh(\pi\lambda/2)l(\lambda^2+l^2)^{-1}T(\lambda,l)\,,$$

$$\text{(III)}\ \langle c\rangle=T(i,1,1)/2+\sum_{l\in 2\mathbf{N}+1}[T(i,l,l-2)+T(i,l-2,l)]/2+2^4\sum_{l_1,l_2\in 2\mathbf{N}-1}\int_0^\infty d\lambda\,\lambda(\lambda^2+1)\tanh(\pi\lambda/2)l_1l_2(\lambda^2+l_1^2+l_2^2-1)\times\prod_{\sigma,\sigma_1,\sigma_2=\pm1}(-1+i\sigma\lambda+\sigma_1l_1+\sigma_2l_2)^{-1}T(\lambda,l_1,l_2)\,.$$

Numerical evaluation of these expressions as a function of $s=L/\xi+1/\gamma$ yields the curves displayed in Fig. 1. They show $\langle c\rangle$ always to be less than (greater than) the Ohmic value $1/s$ for models I and II (model III).

The limits $s\to 0$ and $s\to\infty$ lend themselves to further analytic evaluation [12]. By making a heat kernel expansion in ordinary space (rather than Fourier space), one obtains for s small:

$$\text{(I)}\ \langle c\rangle=1/s-2/3+4s/45+O(s^2)\,,$$

$$\text{(II)}\ \langle c\rangle=1/s-4s/45+O(s^2)\,,$$

$$\text{(III)}\ \langle c\rangle=1/s+1/3+s/45+O(s^2)\,.$$

It turns out that these asymptotic expansions approximate the exact result very well, the deviations in relative magnitude being smaller than 0.4 (1.7,1.2) percent in the range $s<0.5$ (1.0,2.0) for I (II,III). In the opposite limit, $s\to\infty$, one easily finds by keeping only the dominant terms in the Fourier series:

$$\text{(I)}\ \langle c\rangle\simeq 2^{-4}\pi^{7/2}s^{-3/2}\exp(-s)\,,$$

$$\text{(II)}\ \langle c\rangle\simeq 2^{1/2}\pi^{3/2}s^{-3/2}\exp(-s/2)\,,$$

$$\text{(III)}\ \langle c\rangle\simeq 1/2+2^5 3^{-2}\pi^{3/2}s^{-3/2}\exp(-s/4)\,.$$

Note especially the appearance of a constant, $1/2=T(i,1,1)/2$, for model III. The Fourier coefficient $T(i,1,1)$ is associated with a function ϕ which satisfies $\Delta\phi=0$ and has finite integral $\int_{\mathbf{G/K}}\bar{\phi}(Q)\phi(Q)DQ=1$ [13]. Its presence in the Fourier series prevents $\langle c\rangle$ from going to zero when $L\to\infty$. In other words, while the localization length is finite and is given by $\xi=4\pi\rho D$ ($2\xi=8\pi\rho D$) for model I (II), it is infinite for model III, in leading order of the expansion around the limit $k_F^2S\gg E_F\tau\gg 1$.

Surely, the result for model III comes as a tremendous surprise and calls for further research. To investigate the stability with respect to changes in the boundary conditions, I have reconsidered the calculation of

$$K(x,\omega)=\langle G^R(0,x;E_F)G^A(x,0;E_F+\omega)\rangle$$

for a wire of infinite length *without leads.* This correlator has been computed by Efetov and Larkin [4] in the limit $\omega\to 0$. We now ask whether their result is modified by the discovery of the zero mode ϕ for model III. Evaluation of K by the mapping onto the nonlinear σ model leads again to Eqs. (2), except that $-\Delta$ is now replaced by $-\Delta+V$ where V operates by multiplication with $V(Q)=-2\pi i d^{-2}\rho\xi\omega\,\mathrm{str}\Lambda Q$. The deformation of Δ by V lowers the group symmetry of (2), and the calculation can therefore no longer be done analytically in general. For a small frequency ω, however, one may attempt to use perturbation theory in ω, applied to the eigenfunctions and eigenvalues of $-\Delta$. The function ϕ behaves asymptotically on **G/K** as $\phi\sim\lambda^{-1}$ (with λ as defined in Ref. [4]), which entails that the integral

$$a=\int_{\mathbf{G/K}}\bar{\phi}(Q)\,\mathrm{str}(\Lambda Q)\phi(Q)DQ$$

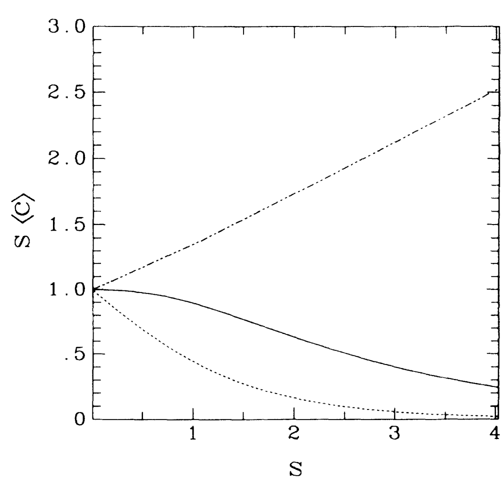

FIG. 1. The product $s/\langle c\rangle$ as a function of $s=L/\xi+1/\gamma$ for the case of orthogonal symmetry (dotted line), unitary symmetry (solid line), and symplectic symmetry (dash-dotted line).

converges. Therefore, standard first-order perturbation theory is applicable to ϕ and gives an additional contribution to $K(x,\omega)$ of the form

$$(2\pi\rho/S)^2|b|^2\exp(i\pi a\rho\omega|x|/2)\,, \qquad (6)$$

where

$$b=\int_{\mathbf{G/K}} Q_{31}^{21}\phi(Q)DQ$$

converges, too.

Having introduced the term (6), I will now argue that it must vanish after all. By conservation of probability, K is subject to the sum rule

$$-i\omega S^2\int_{\mathbf{R}} K(x,\omega)dx=4\pi\rho\,. \qquad (7)$$

Note further that (6) has integral $16\pi\rho|b|^2/(-ia\omega S^2)$. The result of Ref. [4] for K exhausts the sum rule (7) and, therefore, the term (6) would violate conservation of probability, if b were nonzero. Since all computations made in Ref. [4] are verifiably correct (normalization included), I am forced to conclude that $b=0$. This conclusion is supported by group theory, as $Q_{13}^{12}\,[\phi(Q)]$ is the matrix element of a finite-dimensional (infinite-dimensional) representation of $\mathbf{G}$. Hence, the zero mode ϕ does *not* contribute to the low-frequency limit of K, and the result of Ref. [4] is complete as it stands.

We are thus faced with a seemingly paradoxical situation. The results of Ref. [4], on the one hand, prove the localization of all states of the system. On the other hand, the above result for model III cannot but mean that there exist states whose support extends throughout the disordered sample, in this case. To reconcile these facts, recall that the first statement applies to an *isolated system*, while the second has been established for an *open system* coupled to perfect leads. It has been known since the work of Hikami, Larkin, and Nagaoka [14] that systems with symplectic symmetry differ from those with orthogonal and unitary symmetry in that the leading quantum corrections to the conductivity act *against* localization and lead to the existence of extended states and an Anderson transition in two dimensions. The message of the present Letter is that this qualitative difference extends to quasi-one-dimensional systems: The intrinsic tendency toward localization at large L is not strong enough in the case of symplectic symmetry, *and in this case only*, to resist the broadening of energy levels caused by the escape of flux into the leads.

In summary, I have used a super generalization of the Fourier transform to diagonalize the Laplace-Beltrami operator for Efetov's spaces and thereby derive exact expressions for the average conductance of a disordered thick metallic wire of arbitrary length L. While obtaining complete agreement with known results for wires with orthogonal and unitary symmetry, I find the conductance of an open wire with symplectic symmetry never to fall below the minimum value $e^2/2h$. This is so in spite of the fact that all states of the isolated system are localized, and it highlights the need for proper theoretical treatment of the boundary conditions.

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[11] The system of functions $\{\bar{\phi}_v^m\}$ is an unnormalized dual system of $\{\phi_v^m\}$.

[12] Very short wires with transverse size $\sqrt{S}>L$ are not accessible within the approximations leading to the *one-dimensional* problem (2). Nevertheless, the limit $s\rightarrow 0$ is well defined and interesting as a mathematical limit.

[13] This term contributes to (5) in spite of the vanishing of $\tilde{\Delta}(v)$ at $v=(i,1,1)$, because the Plancherel measure $d\mu(v)$ is *singular* there.

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