SYMMETRY CLASSES OF DISORDERED FERMIONS AND TOPOLOGICAL INSULATORS

by

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Abstract. — Universal properties of disordered and chaotic quantum Hamiltonian systems can often be described by random matrix models. A key question in this context is that of symmetries and universality classes. In a 1962 paper known as the 'Threefold Way' Freeman J. Dyson proved that, given an arbitrary group of unitary and anti-unitary symmetries, every set of irreducible Hamiltonians commuting with these symmetries must be a set of Hermitian matrices with matrix elements that are either real numbers, or complex numbers, or quaternions. I will explain how to refine Dyson's threefold classification scheme by the so-called 'tenfold way' handling the case of disordered fermions. Developed in Köln in the mid-90's, the refined scheme encompasses noninteracting quasi-particles in disordered metals and superconductors as well as relativistic fermions in random gauge field backgrounds. The same scheme underlies a recent classification of topological insulators by Kitaev and others.

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1. Introduction

The plan of this talk is to (i) begin with an extensive introduction (of at least half an hour) of ideas, motivation and background, then (ii) move to the subject of symmetry classes, and (iii) finish with a brief look at related developments of great current interest.

2. The uncanny power of prediction by random matrix theory

The story I will start with tonight is a story about quantum mechanical systems far from integrability, with dynamics which is essentially unpredictable due to chaos and disorder; it's a story about the fluctuations in the energy spectra and scattering cross sections of such systems; and it's a story about random matrices.

2.1. Neutron resonances. — The story begins in nuclear physics in the 1950's, a time of accumulating experimental data about the scattering of slow neutrons from heavy target nuclei such as Thorium 232 or Uranium 238. The energy scale here is electron volts – down by a factor of 10^6 from the typical MeV energy scale of nuclear physics. Thus the sharp structures you see in this scattering cross section reflect the formation of quasi-bound states which live a million times longer than you would expect based on typical nuclear time scales; they are called compound nucleus resonances.

2.2. Wigner. — One of the first to think about them – in the middle of the 1950's – was Eugene Paul Wigner, a theoretical physicist of Jewish Hungarian origin. He is best known to the public at large for his pivotal role in initiating the Manhattan project. Hired by Princeton University [as a Professor of Mathematical Physics] in 1937, Wigner received the Nobel Prize in 1963 for the discovery and application of fundamental symmetry principles in nuclear and particle physics.

2.3. Bohr's picture of the compound nucleus. — In his attempt to devise a theory of compound nucleus scattering, Wigner was strongly influenced by a picture of the compound nucleus which Niels Bohr had proposed in 1936. Bohr imagined the incident neutron as a ball hitting a billiard table of other balls (protons and neutrons, of course) and sharing its kinetic energy with many if not all of the target particles for a long time — as opposed to making a quick strike and departing from the scene immediately, as would be the case in a direct reaction.

2.3.1. The idea of random matrix modeling is born. — Bohr's picture suggested to Wigner that the narrow resonances observed in neutron scattering were the fingerprint of a strongly interacting and complicated many-body problem, one he could not hope to compute and predict in detail. So he conceived of the idea that a statistical theory might be called for, akin to what is done in statistical mechanics and thermodynamics. And he proposed to model the resonance energies by the eigenvalues of a random matrix.

This proposal turned out to yield an excellent description, as shown in this figure. Here, the distribution of spacings between neighboring resonances [or levels] is compared to what one would expect from a certain random matrix model, the so-called Gaussian Orthogonal Ensemble – which is an ensemble of real symmetric matrices with matrix elements distributed according to a Gaussian. Note that the good agreement between GOE statistics and the Nuclear Data Ensemble (consisting of about 1700 levels) is not restricted to spacings but is seen in numerous other statistics, e.g., in the Dyson-Mehta statistic.

So much for the origin of the use of random matrices in the statistical modeling of complex quantum many-body systems such as atomic nuclei.

2.4. Quantum chaos. — For more than two decades, the use of random matrix models remained a stronghold of statistical nuclear theory. This began to change at the beginning of the 1980's, when computer power had grown to a point where it became possible to calculate, say, the first ten thousand energy eigenvalues of a single-particle quantum system such as the Sinai billiard shown here.

It came as a surprise – to many, at least – that the same statistical laws which had been discovered in statistical nuclear theory, also applied to the situation of a Hamiltonian system like this billiard, with two degrees of freedom but only one constant of the motion (the energy). This finding led to a change of perspective: it was realized that in order for random matrix modeling to be applicable, it is not necessary for the quantum system to be 'many-body' and very complicated; instead, what is needed is chaotic dynamics making the quantum motion quickly explore all of the constant energy shell.

Emboldened by computer experiments, Bohigas, Giannoni, and Schmit made a conjecture (in 1984), which goes roughly like this: take a Hamiltonian system with chaotic classical dynamics and quantize it; then the spectrum of the quantum system at high energies is expected to exhibit universal fluctuations – those of a random matrix of the appropriate symmetry class.

While this conjecture is still lacking a precise mathematical formulation – and much more so a proof – it is corroborated by semiclassical arguments and a very substantial amount of numerical evidence.

Now, the success of random matrix theory did not end there. After the pioneering phase in statistical many-body theory and the extension to quantum chaotic systems at high energy, the story moved on to (some ground states of) elementary particle physics.

2.5. QCD Dirac spectra. — To keep this part of the story short: on spin bundles (and related objects) one has a first-order differential operator called the Dirac operator, and by twisting the bundle one can couple it to a gauge field. Again, the object of interest is the spectrum.

In the setting of quantum chromodynamics (or QCD) the infinite-volume limit of the Dirac spectrum carries important information about symmetry breaking phenomena in the QCD ground state.

Now in 1993, Verbaarschot and Zahed at Stony Brook made a proposal which turned out to have a lasting influence, giving rise to a substantial amount of activity in a new research field called *chiral random matrix theory*. V & Z suggested that the statistical properties of the Dirac eigenvalues (in the massless case assumed here) were those of a so-called chiral random matrix model. The characteristic feature of a chiral random matrix is that it has zeroes on the diagonal blocks, to account for the fact that the Dirac operator for massless fermions anti-commutes with the 'chirality' operator γ_5.

Chiral random matrix theory made predictions for the distribution of the low-lying eigenvalues and the spectral density in sectors of zero or non-zero topological charge, and all of these were borne out by the results from large-scale lattice gauge simulations, as shown in these view graphs.

2.6. Riemann zeta function. — Let me finish this motivational section with a striking example taken from mathematics: the statistical behavior of the Riemann zeroes.

The Riemann zeta function is the analytic function of a complex variable, say s, which is defined in the half-plane $\mathfrak{Re}(s) > 1$ by this power series [the so-called Dirichlet series] and extended to the rest of the complex plane by analytic continuation. According to the famous Riemann hypothesis – one of the great open problems in mathematics – its non-trivial zeroes all lie on the line parallel to the imaginary axis with $\mathfrak{Re}(s) = 1/2$.

There are infinitely many of these Riemann zeroes, and one may explore the idea of doing statistical analysis on them. Since the early 1970's, it had been known from a theorem and a conjecture by Montgomery that there is a connection with random matrices. Motivated by this, Andrew Odlyzko of AT&T Bell Laboratories carried out extensive numerical calculations (in 1987) to compute a large number of very high zeroes. Here you see the results of a study where he started at the zero of order 10^{20} and computed a stretch of 72 million consecutive zeroes. And lo and behold – the distribution of spacings between neighboring Riemann zeroes follows *perfectly* the distribution of spacings between neighboring eigenvalues of the GUE – a Hermitian random matrix with Gaussian distributed entries.

2.7. Universality of spectral fluctuations. — To summarize, there is a fascinating phenomenon referred to as the *universality of spectral fluctuations*: take any of the linear equations of wave mechanics, be it the Schrödinger equation, the Dirac equation or (for that matter) the wave equation of Maxwell electrodynamics, and look at the energy spectrum. If there is enough disorder or chaos, then you will see fluctuations which obey the laws predicted by random matrix theory for the appropriate *symmetry class* [GOE for nuclear spectra, GUE for the Riemann zeroes]. This finding has been made in numerous examples, including the neutron resonances of atomic nuclei, disordered metallic grains, chaotic billiards, microwaves in a cavity, acoustic modes of a vibrating amorphous solid, quarks in the ground-state configuration of a non-abelian gauge field, and (last but not least) the Riemann zeroes.

3. Dyson's Threefold Way

Now, you might be inclined to ask: *why* are the fluctuations universal? The reason is understood in some cases, but still mysterious in many others. From my viewpoint (looking at the generic model situation) the most fitting theoretical explanation uses the robust tools of field theory: supersymmetric non-linear sigma models, Wilson's idea of the renormalization group, and the notion of universality at RG-fixed points.

Unfortunately, this is too technical a story to be suitable for a colloquium-style talk. There exists, however, a less technical and equally important aspect: the role of symmetries. To this I turn next.

3.1. Dyson. — This is Freeman Dyson, who came to early fame for his work on quantum electrodynamics (in 1949), proving the equivalence of Feynman's diagrammatic approach with the operator theory of Schwinger and Tomonaga. In 1953 he took a post at the Institute for Advanced Study in Princeton, where he still lives and works to this day.

By Wigner's presence in Princeton, Dyson witnessed the emergence of random matrices and statistical nuclear theory from their very inception. He got interested in the subject and chose to work on it for about two years (or so) at the beginning of the 1960's, introducing such lasting and important concepts as what's nowadays called Dyson's Coulomb gas and Dyson's Brownian motion.

3.2. Dyson's abstract. — He concluded his spurt of five substantial papers on random matrix theory with an article 'The Threefold Way. Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics'.

[As an aside, let me point out that there is a play of words here. In the late 1950's, Murray Gell-Mann had coined the word 'eightfold way' for the particle physics of flavor SU_3 symmetry with its 8 generators. Gell-Mann, famous for his extraordinary range of general knowledge, had taken the cue from the 'Noble Eightfold Path' of Buddhism. Dyson mimicks him with the choice of title 'The Threefold Way'.]

In previous work, he [Dyson] had introduced and studied ensembles of matrices of three different types, which we will get to presently. In this final paper of the series he makes the point that his earlier work is conceptually complete in that it covers all cases of interest. He writes: "In particular, it is proved that the most general matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types."

Let us now see what that means.

3.3. Unitary and anti-unitary symmetries. — The setting is in quantum mechanics, where one has a Hilbert space V with a Hermitian scalar product, and the Hamiltonian is a Hermitian linear operator on V. Now, a central role in Dyson's scheme is played by symmetries, and these may be unitary (e.g., space rotations) or anti-unitary (e.g., time reversal). In the former case, the Hermitian scalar product is preserved, while in the latter case it is preserved but for an extra complex conjugation. A unitary symmetry $[U]$ commutes with the time evolution operator, an anti-unitary symmetry $[T]$ interchanges with it while reversing the direction of time.

3.4. The setting. — Dyson now considers a general Hilbert space V with a group G acting on it by unitary and anti-unitary operators. The physical meaning of the group G is that of the group of symmetries of the particular quantum system which is to be treated statistically by a random matrix model. The symmetry group G is meant to be quite arbitrary; in Dyson's words, it "may be a rotation group, or an isotopic-spin rotation group, or a time-inversion group, or all of these in combination."

Needless to say, the 'good' Hamiltonians are those that commute (in the sense just described) with all of the symmetry operations from G. The question (pointedly asked by Dyson) then is: what can be said about the structure of the set of all good Hamiltonians, which are compatible with these symmetry constraints?

3.5. Double commutant theorem. — Dyson's answer comes from a technical discussion using theorems due to Hermann Weyl, John von Neumann, and Eugene Wigner (all of whom were contemporaries of his at Princeton). I will explain only the outcome, sparing you the technicalities. The argument is made in two steps.

First, Dyson utilizes the unitary symmetries. If they act reductively (an assumption implicit in Dyson's paper), then the Hilbert space decomposes as an orthogonal sum of subspaces, each of which is invariant and irreducible under the joint action of the symmetries and the Hamiltonians.

Here is a picture of the situation: you may think of each subspace as a 'rectangle' of quantum states, where the symmetries act (say) vertically, and the Hamiltonians act horizontally. (Of course, in order for the two actions to commute, the Hamiltonian matrix elements have to be the same in each row, and those of the symmetries have to be the same in each column.) As an example, consider the case of the rotation group $G = \mathrm{SO}_3$. In this case the rectangles are labeled by total angular momentum, L, and the rows are labeled by the projection of angular momentum, M, on some axis – conventionally, the z-axis. The x- and y-components of angular momentum then act vertically as raising and lowering operators.

The upshot of a theorem (due to Weyl) is that the situation remains qualitatively the same for *any* symmetry group acting reductively [on V] by complex linear operators.

Our attention now focuses on the matrix elements of H between the quantum states (in any row) of a fixed rectangle (or irreducible block).

3.6. Enter the anti-unitaries. — This takes us to the second step, which is to consider the effect of one anti-unitary symmetry, say T. (By the way, if there are two of these, say T_1 and T_2, we need only consider one of them, as the product $T_1 T_2$ is a unitary symmetry and we may assume that this has already been taken care of in step one of the discussion .) So, pick one anti-unitary symmmetry and note that its square is a unitary operator.

Motivated by physical examples (such as time reversal and charge conjugation, all of which are involutions) Dyson requires that T^2 be a projective realization of unity (on the subspace under consideration, if not on the Hilbert space as a whole). A simple application of the associative law then leads to the conclusion that there exist but two possibilities: T^2 is either plus the identity or minus the identity.

Recalling the orthogonal decomposition $V = \oplus_\lambda V_\lambda$, one is led to a trichotomy.

1. The anti-unitary symmetry T may be absent, or it may exchange pairs of distinct subspaces, $V_\lambda \overset{T}{\leftrightarrow} V_{\bar{\lambda}}$. In either case there is no further condition on the symmetry-compatible Hamiltonians; hence we are in the general class of Hermitian matrices.
2. T stabilizes a given block and squares to $+1$. In this case there exists a real basis $Te_i = e_i$, and with respect to it, the matrix of the Hamiltonian is real symmetric.
3. $T^2 = -1$. Here the dimension of the subspace must be even, and we can find a basis made from pairs of vectors $e_i, e_{\bar{i}}$ satisfying [these relations]. One then immediately sees that the Hamiltonian consists of quaternions.

These are all the possibilities, and there aren't any more.

3.7. Example. — To illustrate this abstract classification business, let me give an example, elaborating on Case 3. Although Dyson was not able to think of a good physical example back in 1962 [as you can see from these quotes], Case 3 did rise to prominence (beginning in the late 1970's) in the theory of disordered metals: it turned out to be the symmetry class of disordered electrons with spin-orbit scattering and no interactions breaking time-reversal symmetry. The Hamiltonian looks like this:

$$H = \frac{p^2}{2m} + eU(x) + V_{\text{SO}} \cdot (\sigma \times p) \, .$$

Note that the spin-orbit term (like the other terms) is T-invariant, since both spin σ and momentum p are T-odd while the spin-orbit field, a vector, is T-even.

As an aside, let me note that the appearance of the spin-orbit interaction is a relativistic effect: any moving particle with a spin-magnetic moment [and a nonzero velocity tangential to the plane of the spin] acquires an electric dipole moment, which gives rise to a velocity- and spin-dependent coupling with the electric field. In a metal with a Fermi velocity of the order of 10^{-3} times the speed of light, this is not a negligible effect.

Here is how the Hamiltonian matrix looks:

$$(k, \uparrow) \xrightarrow{a} (k', \uparrow) \, ,$$

$$(-k,\downarrow) \xrightarrow{\bar{a}} (-k',\downarrow) \ .$$

In words: if the matrix element between states with momenta k,k' and spin up is a, then the matrix element between states with momenta $-k,-k'$ and spin down is the complex conjugate, $\bar{a}$. And if the matrix element from $|k,\uparrow\rangle$ to $|-k',\downarrow\rangle$ is b, then the matrix element from $|-k,\downarrow\rangle$ to $|k',\uparrow\rangle$ must be $-\bar{b}$. Thus the Hamiltonian matrix consists of 2×2 blocks of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} .$$

Such 2×2 blocks are called *quaternions*.

This symmetry class acquired some notoriety in condensed matter physics, as the consequences of spin-orbit interactions turn out to be quite exotic. The basic mechanism is shown here in a semiclassical picture: an electron makes a loop on its way through a disordered metal; the amplitude for this process interferes with another one where the loop is traversed in the opposite sense, with the direction of momentum *and* spin reversed.

This mode of quantum interference is called the *spin-singlet cooperon*. As a result of spin-orbit scattering, the interference between amplitudes tends to be negative, causing a suppression of the return probability and hence an enhancement of transmission. In the jargon of the field, one speaks of *weak anti-localization* – an effect observable in experiments [e.g., in disordered magnesium films with gold impurities] as an increase of the electrical resistivity with increasing magnetic field (at low enough temperature).

3.8. Summary. — Let's get back on track by repeating the quote from the abstract of Dyson's Threefold Way: "the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of ensembles each of which belongs to one of the three known types."

So far, we have just done an exercise in algebra (in the setting of quantum theory and symmetry groups), and it led to what are called symmetry classes, more precisely the Wigner-Dyson symmetry classes. But now, with the algebraic structure understood, one can get started with some statistical theory by putting probability measures on these matrix spaces (the simplest are the so-called Gaussian measures or Gaussian ensembles) and analyzing the correlations. One then discovers the universality classes which exhibit the kind of universal behavior discussed earlier.

For that reason, Dyson's threefold way is generally regarded as one of the corner stones of random matrix theory. Conceived in the context of the statistical theory of complex many-body systems, it has become influential in several areas of theoretical physics, including mesoscopic systems, disordered electrons, and quantum chaos.

4. 10-Way Classification

Having given you all this introductory material in the way of motivation and background, let me now move on to the heart of this talk: The Tenfold Way – starting again with some motivation.

Around the middle of the 1990's, there was an accumulation of evidence that an extension to Dyson's threefold way was called for:

- The QCD-motivated chiral random matrix ensembles paralleled Dyson's classification scheme [orthogonal symmetry for the $\mathrm{SU}(2)$ gauge group, unitary for $\mathrm{SU}(3)$, and symplectic for adjoint fermions] but they clearly transcended it.
- There was a growing level of activity in the mesoscopic physics of hybrid systems involving both metallic and superconducting components. A key process in such systems is the one shown in this figure: an electron incident on a superconductor is reflected as a hole with the same momentum and opposite velocity. [The missing charge is transferred as a Cooper pair to the superconducting condensate.] This process, called *Andreev reflection*, gives rise to a novel mechanism of quantum interference which is foreign to the Wigner-Dyson symmetry classes.
- Mathematicians (once again at Princeton, namely N. Katz and P. Sarnak) had introduced and studied ensembles of so-called L-functions, which are cousins of the Riemann zeta function. They found phenomena which did not fit into the known classes, and Sarnak [in particular] voiced the need for an extension of Dyson's classification.

These strands of thought (excluding the work on L-functions, which came to my attention only in the summer of 1997) were brought together in the fall of 1995, and an extended scheme was conjectured and submitted for publication in the spring of 1996.

Given the boundary conditions of this talk, I cannot tell this story true to history [which involved a number of technical arguments, including field-theoretic computations], so I will present a streamlined version.

4.1. The setting. — First we have to specify the rules of our game.

4.1.1. Our setting: Fock space. — This is an elaboration of the previous setting in that we replace Dyson's general Hilbert space by the more refined structure of a Fock space for fermions. Fock space is still a Hilbert space, but it comes with a grading by particle number. [F_0 is the vacuum. F_1 is the same as the single-particle Hilbert space V of dimension N. F_2 is the two-particle space, and so on, until we fill all the available states, and the sequence terminates by the Pauli principle]. The particle creation [c^*] and annihilation operators [c] acting on Fock space satisfy the canonical anti-commutation relations.

4.1.2. Our setting: symmetries. — Following Dyson, we adopt the setting of a symmetry group G acting on Fock space by unitary and anti-unitary operators. We require the group of unitary symmetries to be defined on the single-particle space V and extend it to Fock space in the natural way. [I should mention that this requirement excludes Yangian or quantum group symmetries, which arise at the many-particle level.] There is no further restriction; in Dyson's words: the group of unitary symmetries may be completely arbitrary.

As for the anti-unitaries, we again allow for the possible presence of time-reversal symmetry, which is defined on the single-particle Hilbert space and extends to Fock space in the usual way. Moreover, the structure of Fock space opens the possibility for another anti-unitary operation to be a symmetry; this is particle-hole conjugation, which transforms the particle vacuum into the fully occupied state, and in general, a state of n particles into a state of n holes.

4.2. Statement of problem. — We can now formulate the problem to be solved. Let F be any fermionic Fock space carrying a G-action, where G is an arbitrary symmetry group made from generators as described above; i.e., with the most general element $g \in G$ being any combination of unitary symmetry operations and/or time reversal and/or particle-hole conjugation.

Our object of interest is the set $\mathscr{H}$ of all Hamiltonians which are G-invariant one-body operators, i.e., operators which are *quadratic* in the particle creation and annihilation operators [$c^\dagger, c$] and *commute* with all operators from the symmetry group G. [More generally, and without causing any serious complications, one may consider any polynomial in such operators.]

The question then is: what can we say about the structure of the set $\mathscr{H}$? Can we classify the types of irreducible block which occur in this setting?

4.3. HHZ. — After toying with this question for a number of years, the final and definitive answer was given in a joint paper with Peter Heinzner and Alan Huckleberry [from the Bochum Mathematics Department]: every irreducible block which occurs in this setting is a classical irreducible *symmetric space*, and conversely, every classical irreducible symmetric space arises in this way.

4.4. What's a symmetric space?— Now to make this a meaningful statement, I must briefly explain the notion of symmetric space. This comes from differential geometry and can be defined in several equivalent ways. I will give you the infinitesimal (or algebraic) version, which is probably the best one to describe to a general audience.

Let $\mathfrak{g}$ be a Lie algebra, and assume that you are given an involution of $\mathfrak{g}$, i.e., a linear mapping $\theta : \mathfrak{g} \to \mathfrak{g}$ which squares to the identity map, $\theta^2 = \mathrm{Id}$ and preserves the bracket relations:

$$\theta([X,Y]) = [\theta(X), \theta(Y)] \; .$$

Then we look for the set of Lie algebra elements which are odd under θ. We call this set, $\mathfrak{p}$, the negative θ-eigenspace:

$$\mathfrak{p} = \{X \in \mathfrak{g} : \; \theta(X) = -X\} \; .$$

Simplifying a little bit: $\mathfrak{p}$ is an (infinitesimal version of) symmetric space.

[More precisely, a Riemannian symmetric space is a Riemannian manifold with invariant geometry modeled on the Euclidean tangent space $\mathfrak{p}$.]

4.4.1. Example. — As an elementary example, consider the Lie algebra $\mathfrak{so}_3$ of the group of rotations of three-dimensional Euclidean space. Define $\theta : \mathfrak{so}_3 \to \mathfrak{so}_3$ in terms of the standard basis J_x, J_y, J_z of rotation generators as

$$\theta(J_z) = J_z \; , \quad \theta(J_x) = -J_x \; , \quad \theta(J_y) = -J_y \; .$$

θ squares to the identity and leaves the bracket relations $[J_x, J_y] = J_z$, etc., unchanged; thus it is an involutory automorphism of $\mathfrak{so}_3$. The negative θ-eigenspace $\mathfrak{p}$ is the real plane spanned by J_x and J_y. We may view this plane as the tangent plane of a two-sphere, which (in its rotation-invariant geometry) is a symmetric space.

4.5. The 10-way table. — Let me now run through the 10-way table of symmetry classes, omitting the Wigner-Dyson classes, as these have already been the focus of our attention.

4.5.1. Class CI. — For each symmetry class there is a family name [due to Elie Cartan, the French mathematician, who did the pioneering work of classifying symmetric spaces]; here: CI.

Each class corresponds to an infinite family of classical symmetric spaces [typically quotients of compact Lie groups]; here: $\mathrm{Sp}(2N)/\mathrm{U}(N)$.

For each class we display the standard form of the elements of $\mathfrak{p}$ [the negative θ-eigenspace], which gives us the Hamiltonians. For the case at hand, the diagonal blocks vanish and the off-diagonal blocks are complex symmetric matrices, say Z.

Symmetry class CI is realized by quasi-particle excitations (or Bogoliubov-deGennes quasiparticles) in disordered spin-singlet superconductors in the Meissner phase. [This class includes, as an important special case, unconventional superconductors with d-wave pairing symmetry, such as the cuprate high-T_c superconductors.]

4.5.2. Class C. — The Cartan label C stands for the family of symmetric spaces $\mathrm{Sp}(2N)$. The Hamiltonian matrix now has non-zero diagonal blocks occupied by a Hermitian matrix, say W. Coming from the previous class, CI, this class is realized by breaking time-reversal symmetry; which is achieved, for example, by going into the mixed phase where magnetic vortices are present.

4.5.3. Class DIII. — The most natural realization of the next class, DIII, is in superconductors with spin-triplet pairing. Going from spin-singlet to spin-triplet changes the form of Z from symmetric to skew, and the symmetric space becomes $\mathrm{SO}(2N)/\mathrm{U}(N)$. Another physical realization of this class is by the B-phase of superfluid ^{3}He.

4.5.4. Class D. — The fourth superconducting class, D, corresponding to the symmetric space $\mathrm{SO}(2N)$, is sometimes referred to as *Majorana fermions*. It is realized by spin-triplet superconductors with a p-wave symmetry that breaks time-reversal symmetry spontaneously. A promising candidate for this scenario is Strontium Ruthenate. Another realization is by the A-phase of superfluid ^{3}He.

4.5.5. Class AIII. — We now leave the 'superconducting' classes and move on to the so-called chiral classes. These have a natural realization by massless Dirac fermions. In the case of class AIII we want the gauge field to be $\mathrm{SU}(N)$, with the number of colors N no less 3. The off-diagonal blocks Z here are rectangular complex matrices, of size $p \times q$, and the difference $p - q$

has a physical interpretation as the topological charge of the gauge field background.

4.5.6. Class BDI. — Now if we make the complex rectangular matrices Z real, we go from class AIII to class BDI. Real Z is appropriate for gauge group $\mathrm{SU}(2)$ or $\mathrm{Sp}(2N)$.

4.5.7. Class CII. — Finally, the elements of the matrix Z may be quaternions. This will be the case if the Dirac fermions are placed in the adjoint representation of the gauge group, or if the gauge group is $\mathrm{SO}(N)$.

4.6. Example. — Now you may say: all this is very neat and elegant, but what is it good for? To illustrate the usefulness of our scheme, let's take a quick look at one particular example, which is on record in the literature.

Conventional superconductors have an excitation gap at low energy. However, in a superconductor with d-wave pairing symmetry (as found in the cuprates, materials known as high-temperature superconductors), the order parameter vanishes at several points in the Brillouin zone, so low-energy excitations do exist. Their dispersion relation in principal approximation is Dirac, giving a linearly increasing density of states [of the clean system].

In the first half of the 1990's there was quite some activity to work out the density of states for a system with disorder. And although all groups agreed that they were addressing exactly the same physical problem, there was a cacophony of different predictions for the density of states, varying from vanishing linearly to vanishing with a disorder-dependent exponent, to finite (at zero energy), to logarithmically divergent.

4.7. Resolution. — When the controversy did not abate, we decided to write a Physics Report with Alex Altland and Ben Simons, where we pointed out that all proposals but one were inappropriate because they derived from models which belonged to symmetry classes different from the particular symmetry class of the problem at hand, which is CI.

This case study once again underlined a point made by Wigner and Dyson: that it is crucial to understand what is the symmetry class of the problem you are looking at.

4.8. Current directions of research. — Here are a few directions of current research.

- There is ongoing work concerning the supersymmetric non-linear sigma models of the tenfold way, in order to further develop their mathematical foundations and find analytical solutions.
- With the classification and phenomenology of fermionic systems firmly in hand, it is natural to ask whether a similar (field-theoretic) scheme can be developed for bosons. This turns out to be more difficult (for various reasons) and results at present are still preliminary.
- The third [point] concerns another variation of the theme of fermionic symmetry classes: the classification of topological band insulators and superconductors.

4.9. Topological band insulators and superconductors. — With an average of more than five arXiv submissions per day, this is an extremely active field of current research. The word 'topological' here indicates that one is speaking about states of matter characterized by a topological invariant. An early precursor in this direction is the work of Thouless et al. defining a topological invariant for quantum Hall systems (i.e., two-dimensional electron systems in a strong magnetic field).

In short, a topological insulator is a material which does not conduct electric charge in its bulk but does show metallic behavior due to conducting states at the surface. Because of its topological origin, this behavior is robust with respect to perturbations by disorder.

The theoretical ideas initiating the field are due to Kane; a concrete proposal for good materials in which to look for the effect (which has come to be called the $\mathbb{Z}_2$ topological insulator) was made by Zhang and collaborators; and by now there exists a large number of experimental verifications, the first one being by a group at Würzburg.

Topological insulators exist as two-dimensional and three-dimensional materials. The first 2D material was a layered structure of mercury telluride spaced by cadmium telluride. As the thickness of the layers is varied, strong spin-orbit interactions cause an inversion of the electronic band structure. When this happens, experimental observations show a quantization of the electrical conductance (similar to what's seen in the famous quantum Hall effect) which is attributed to pairs of surface modes with opposite momentum and opposite spin.

4.10. Mathematical model. — To set up a mathematical description, we model the crystal or solid body by a lattice. Each lattice point is associated with an n-dimensional vector space, where n is the number of bands, or states per unit cell. The single-electron Hilbert space V then is the l^2-space of Γ, and the ground state for N electrons is constructed as a vector in the N-particle sector of Fock space.

There is some discrete group acting on the lattice by translations, and we can decompose that single-electron Hilbert space by the irreducible representations (labeled by momentum k) of this group. If the system has translational invariance (and electron-electron interactions are negligible) then the ground state (of a band insulator) is simply given by a mapping from k-space to the Grassmannian of m-planes in $V_k \simeq \mathbb{C}^n$. (In physics language, we have a decomposition into M valence/occupied bands and $n-m$ conduction/empty bands for every k-value.)

One can now associate with any such mapping (or ground state) its homotopy class. It turns out that the interesting physics occurs in those materials where the ground state lies in a homotopy class which is non-trivial.

4.11. Classification. — In work of Kitaev (and others) it was realized that the example of the $\mathbb{Z}_2$ topological insulator (of Kane, Zhang etc.) fits into a grand scheme the general mathematical setting of which is again that of the Tenfold Way. More precisely, ground states (of topological insulators and superconductors) are given by mappings from k-space (or first Brillouin zone) into the classical irreducible compact symmetric space determined by the symmetries of the physical situation at hand. If the single-electron spectrum has a gap (and the valence band is filled while the conduction band empty) then topology takes over and ground states are classified by the homotopy classes of these maps. Kitaev pointed out that if k-space is a d-dimensional sphere, then the relevant homotopy classification is known to topologists by a principle called (complex or real) Bott periodicity.

4.11.1. Bott periodicity. — More precisely, according to the Bott (periodicity) clock the stable homotopy groups of any symmetric space are either trivial, or $\mathbb{Z}_2$, or $\mathbb{Z}$, and if one arranges the symmetric spaces in a certain natural order, then the table of homotopy groups is periodic in the space dimension, with period 2 in the complex case and period 8 in the real case.

The situation is expressed in this table (due to Kitaev), which has caused quite an impression on the physics community and has been reprinted in a number of articles including a recent Review of Modern Physics by Kane and Hasan.

Actually, the story (in the real case) is a lot more complicated than it might look at first sight. Direct (and naive) application of real Bott periodicity would predict that the homotopy groups remain the same not in descending direction (as seen in the figure) but in the *ascending* direction. Roughly speaking, the discrepancy comes about because the maps to be classified are defined not on position space but on momentum (or Fourier) space and the presence of an anti-unitary operator (such as time reversal) gives a condition relating momentum to its negative. Miraculously, the effect of this symmetry constraint can be captured very simply by replacing the dimension d by $-d$.

This is my last slide. Let me finish with the following remark. There exist some recent papers complaining that the Kitaev table based on the Altland-Zirnbauer classification is incomplete. I take exception to these complaints. The apparent incompleteness of the published table is due to an abuse (or oversimplification) of the Tenfold Way, and disappears when the results of Heinzner, Huckleberry and myself are used. So, the Tenfold Way isn't a finished story and needs to be disseminated better, at least in the condensed matter community.

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