1-5 What are the fields?

We now make a few remarks on our way of looking at this subject. You may be saying: "All this business of fluxes and circulations is pretty abstract... ...

Also, the ideas of the field lines do not contain the deepest principle of electrodynamics, which is the superposition principle. Even though we know how the field lines look for one set of charges and what the field lines look like for another set of charges, we don't get any idea about what the field line patterns will look like when both sets are present together... ...

The best way is to use the abstract field idea. That it is abstract is unfortunate, but necessary. The attempts to try to represent the electric field as the motion of some kind of gear wheels, or in terms of lines, or of stresses in some kind of material have used up more effort of physicists than it would have taken simply to get the right answers about electrodynamics... ...

In the case of the magnetic field we can make the following point: Suppose that you finally succeeded in making up a picture of the magnetic field in terms of some kind of lines or of gear wheels running through space. Then you try to explain what happens to two charges moving in space, both at the same speed and parallel to each other. Because they are moving, they will behave like two currents and will have a magnetic field associated with them (like the currents in the wires of Fig. 1-8). An observer who was riding along with the two charges, however, would see both charges as stationary, and would say that there is *no* magnetic field. The "gear wheels" or "lines" disappear when you ride along with the object! All we have done is to invent a *new* problem. How can the gear wheels disappear?! The people who draw field lines are in a similar difficulty. Not only is it not possible to say whether the field lines move or do not move with charges - they may disappear completely in certain coordinate frames.

Maxwell in Chains

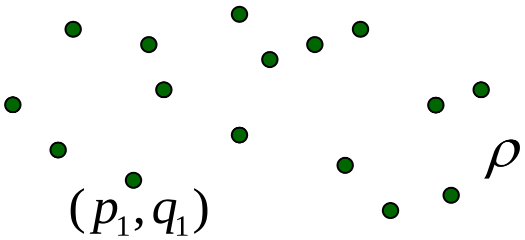
Colloquium, Ulm (July 14, 2009)

- ▹ Atomistic Picture: Chains; Boundary Operator
- ▹ Homogeneous and Inhomogeneous Maxwell Equations
- ▹ Metric and Star Operator; Constitutive Laws
- ▹ Aharonov-Casher Effect (Coil in Motion)
- ▹ Equivalence between Problems in Electrostatics and Magnetostatics
- ▹ Signal Emitted by the Discharge of a Capacitor
- ▹ Feynman Revisited

Atomistic Picture: Chains (I)

Approximate the continuous by the discrete.

Points p_i and point charges q_i



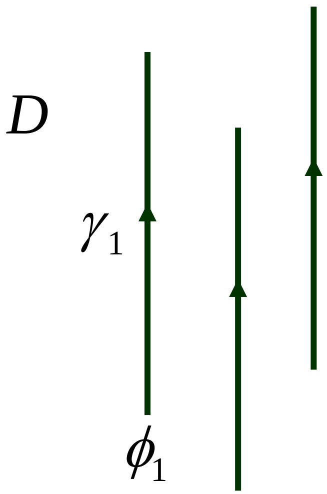
Charge density ρ is a 0-chain:

$$\rho = \sum_i q_i p_i = \sum_i q_i \, \delta(\mathbf{x} - \mathbf{x}_i) \, d^3x$$

Atomistic Picture: Chains (II)

Electric excitation D

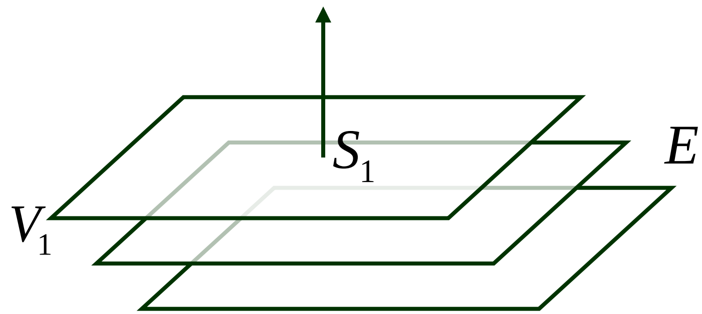
is a 1-chain (with inner orientation):



$$D = \sum_i \phi_i \gamma_i$$

Atomistic Picture: Chains (III)

Surface elements S_i and voltage drops V_i :



Electric field strength E

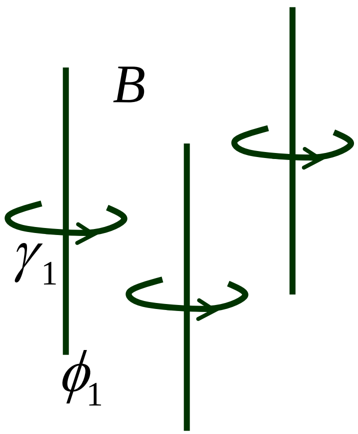
is a 2-chain (with outer orientation):

$$E = \sum_i V_i S_i$$

In the static limit, the surface elements of E combine to closed surfaces, the so-called equipotential surfaces.

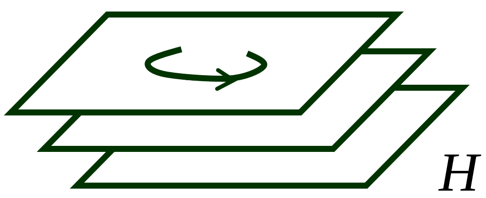
Atomistic Picture: Chains (IV)

Lines γ_i and magnetic fluxes ϕ_i :



Magnetic field strength B
is a 1-chain (with outer orientation): $B = \sum_i \phi_i \gamma_i$

Magnetic excitation H
is a 2-chain (with inner orientation)



Boundary Operator ∂ (I)

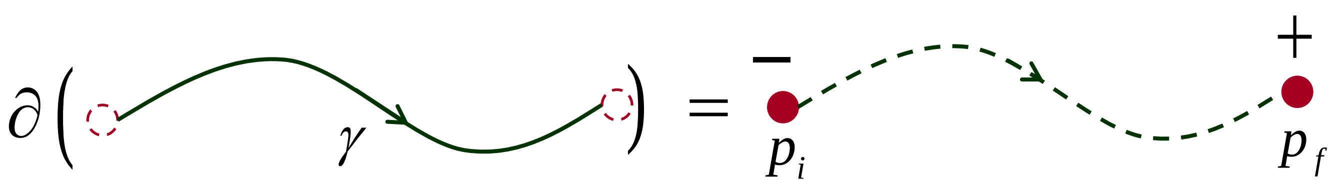
Introduce a linear operator ∂ ,

$$\partial : k-\text{chains} \rightarrow (k-1)-\text{chains} \text{ ,}$$

which extracts the boundary (while preserving the type of orientation).

Example 1:

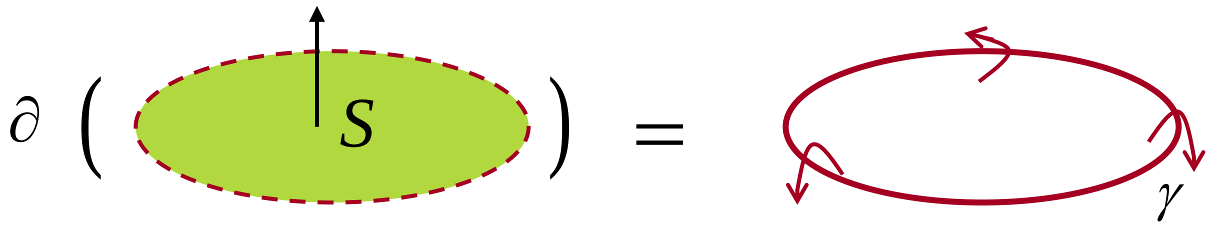
The boundary of a line element (with inner orientation) is the final point minus the initial point: $\partial\gamma = p_f - p_i$



Boundary Operator ∂ (II)

Example 2:

The boundary of a surface element with outer orientation is a closed line, still with outer orientation: $\partial S = \gamma$

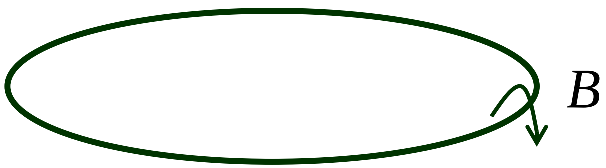


Homogeneous Maxwell Equations (I)

▹ The 1-chain of B is closed:

$$\boxed{\partial B = 0}$$ (no magnetic monopoles)

„Magnetic flux lines have no beginning and no end.“

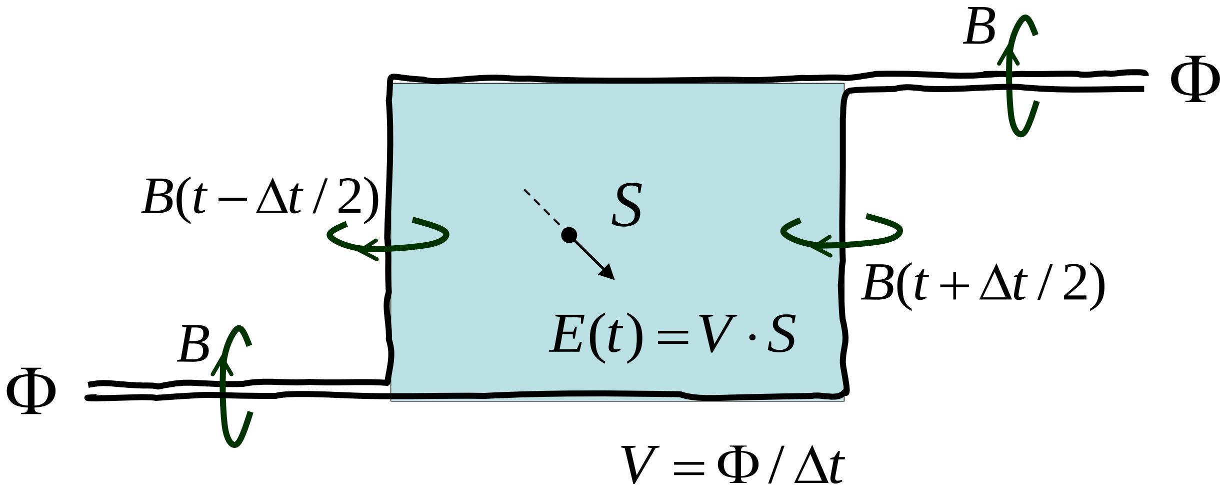


Homogeneous Maxwell Equations (II)

- (1-chain of $\dot{B}$) + (boundary of E) = 0

$$\boxed{\dot{B} + \partial E = 0}$$ (Faraday´s law of induction)

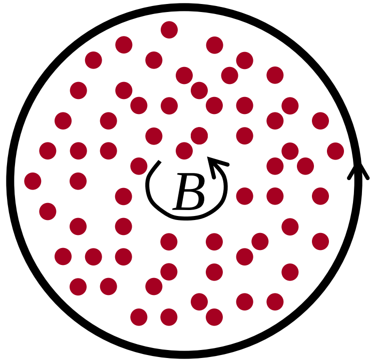
Pictorially: $\Delta t \cdot \partial E(t) = B(t - \Delta t/2) - B(t + \Delta t/2)$

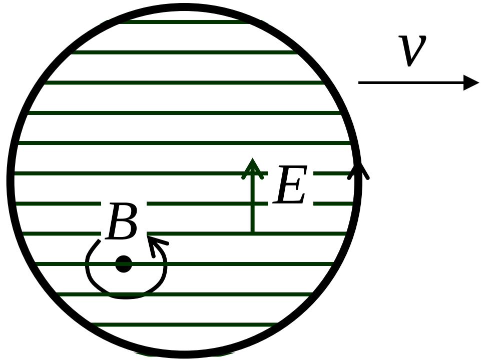


A Simple Application

Current carrying coil at rest (cross section)

Coil in motion (velocity v)





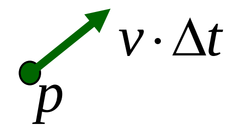
In formulas: $E = \iota(v)B$ (+ closed surfaces).

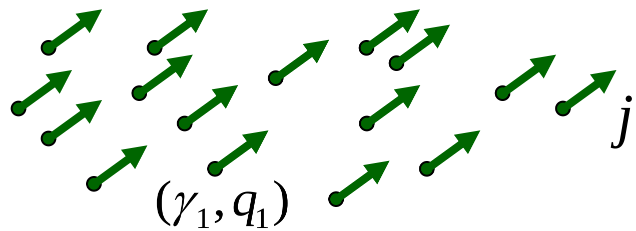
$\iota(v)B$ means this: attach vector $v\Delta t$ to the lines of B ; multiply resulting surface elements by Δt^{-1} ; take limit $\Delta t \to 0$.

Atomistic Picture: Chains (V)

Define line element γ by attaching velocity vector v to point p:

$$\gamma = \iota(v)\,p := \lim_{\Delta t \to 0} \Delta t^{-1} \times$$





Current density j is a 1-chain:

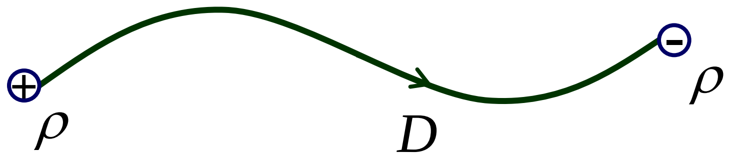
$$j = \sum_i q_i \gamma_i = \iota(v)\rho$$

Inhomogeneous Maxwell Equations (I)

▹ (0-chain of ρ) + (boundary of D) = 0

$$\boxed{\rho + \partial D = 0}$$ (Gauss's law)

„Charges are sources of electric excitation".

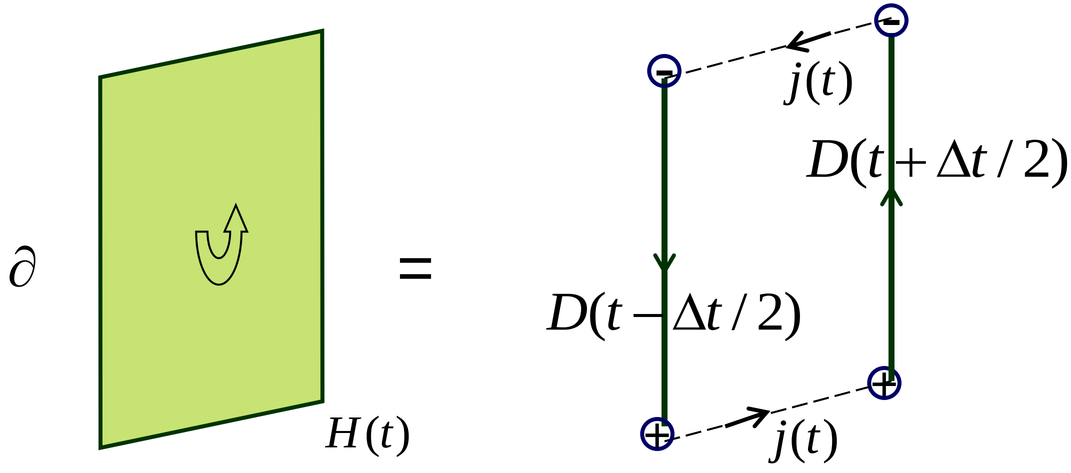


Inhomogeneous Maxwell Equations (II)

- The boundary of the 2-chain of H equals the 1-chain of $(j+\dot{D})$

$$\boxed{\partial H = j + \dot{D}}$$ (Ampere-Maxwell law)

Pictorially: $\Delta t \cdot \partial H(t) = \Delta t \cdot j(t) + D(t+\Delta t/2) - D(t-\Delta t/2)$



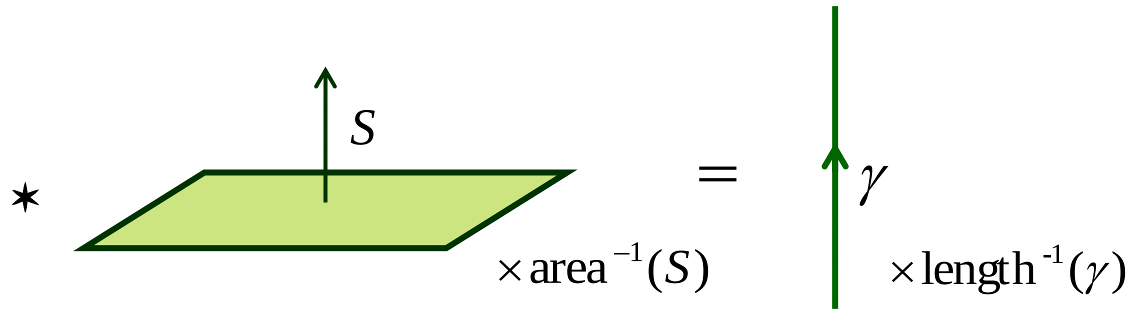
Metric: Star Operator (Hodge)

Introduce a linear operator $\star$,

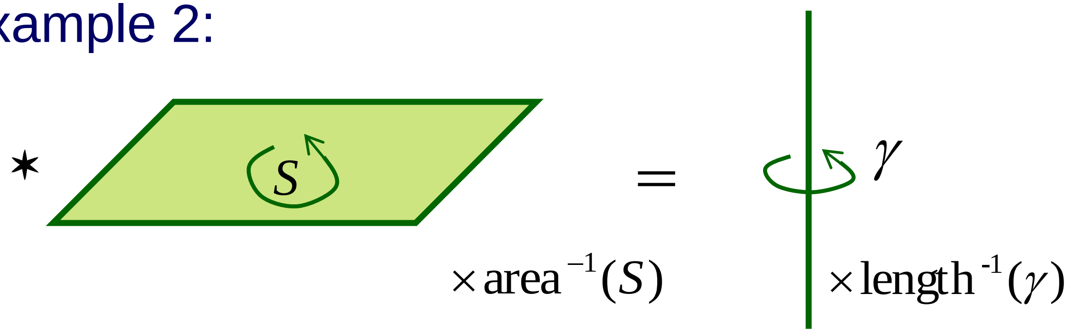
$\star : k-\text{chains} \rightarrow (3-k)-\text{chains}$,

which turns surface elements into line elements,
(in the perpendicular direction), and vice versa.

Example 1:

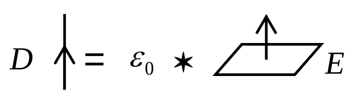


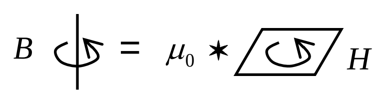
Example 2:



Constitutive Laws

$$\boxed{D = \varepsilon_0 \star E}$$ and $$\boxed{B = \mu_0 \star H}$$



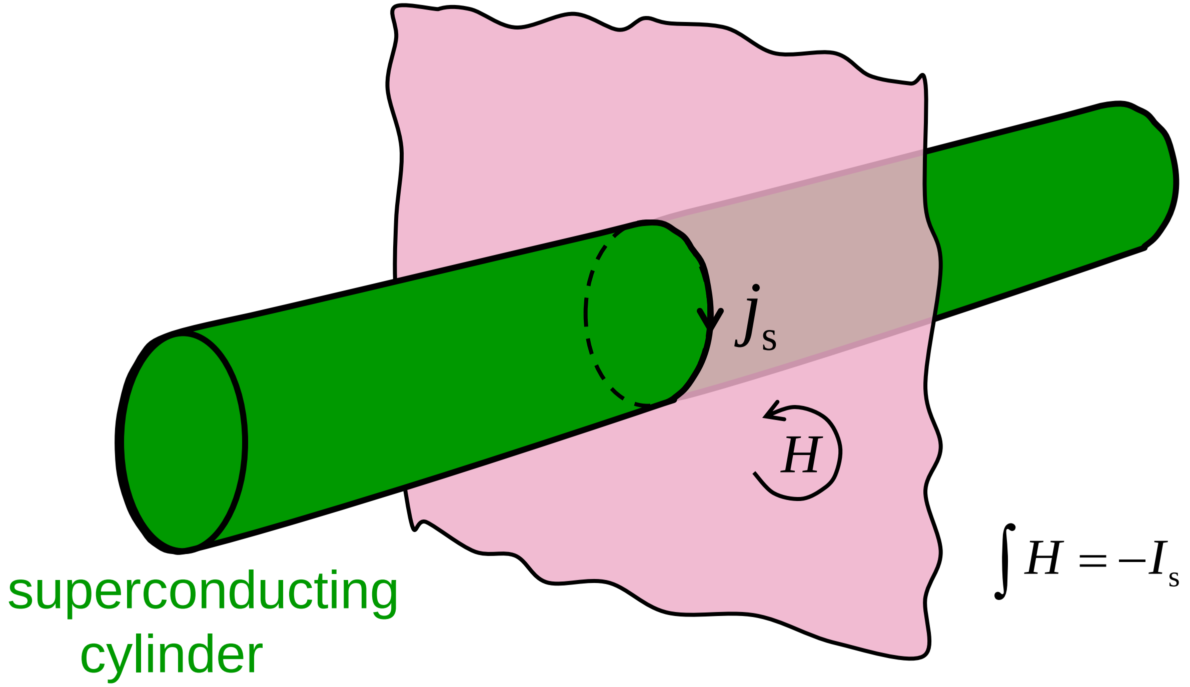


Discussion (I)

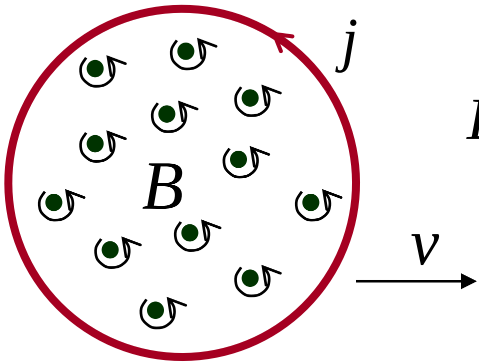
- The notion of chains is intuitive, and the operators ∂ , $\star$ and $\iota(v)$ are easily grasped by students.
- The right-hand rule isn´t ever used, so parity invariance of Maxwell´s theory is manifest!
- In the form presented, Maxwell´s equations are not modified by general relativity (the metric appears only in the constitutive laws).
- The nature of the chains E, B, D, H precisely matches the way they´re measured.

Discussion (II)

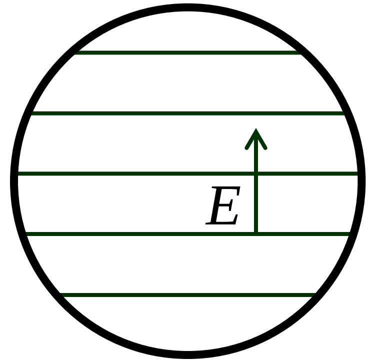
- The electric excitation D is measured by Maxwell´s double plates.
- Measurement of the magnetic excitation H :



Aharonov-Casher Effect

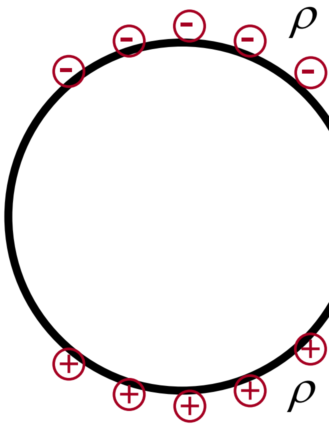


$$E = \iota(v)B$$

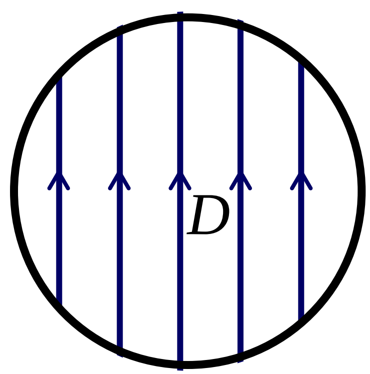


$$B = \Phi\gamma = \mu_0 v_{\mathrm{m}}\gamma \quad \Rightarrow \quad v_{\mathrm{e}} = v_{\mathrm{m}} \times v / c^2$$

$$D = \varepsilon_0 \star E$$



$$\rho = -\partial D$$

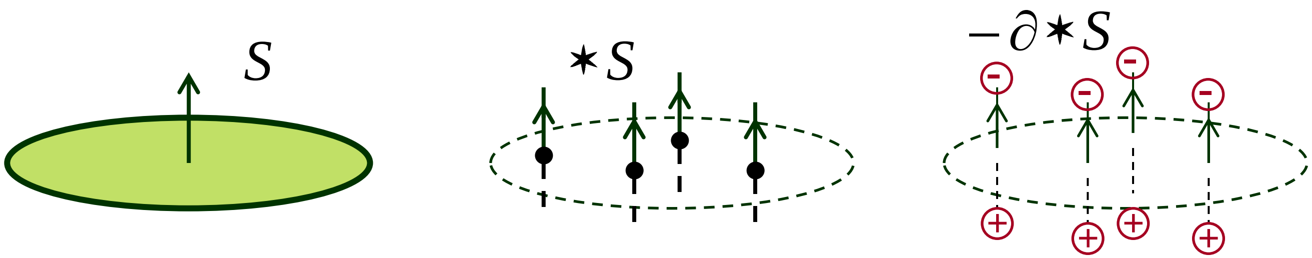


ElectroMagnetostatics (I)

Charge density of a dipole layer S

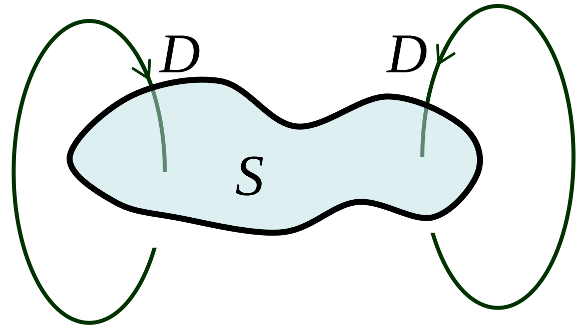
with ν = dipole moment / unit area: $\rho = -\nu \partial \star S$

Verification:

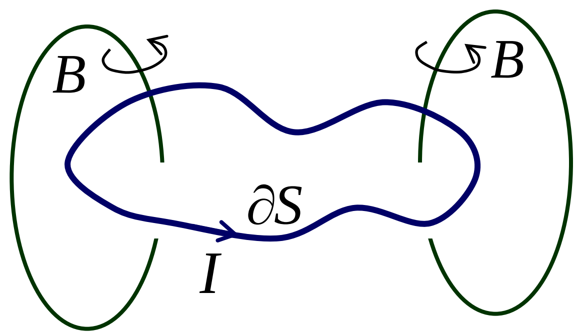


ElectroMagnetostatics (II)

Electric excitation of a dipole layer S :



Magnetic field strength of a current loop ∂S :



ElectroMagnetostatics (II)

Theorem (Ampere): $$D_{\text{dipole layer } S} = -\frac{\nu}{\mu_0 I} B_{\text{current loop } \partial S}$$

Proof. Ansatz: $$D = \nu \star S + D_1$$

$$0 = \rho + \partial D = \partial(-\nu \star S + \nu \star S + D_1) \quad \Rightarrow \quad \partial D_1 = 0$$

$$\partial E = 0 = \partial \star D = \partial(\nu S + \star D_1) \quad \Rightarrow \quad \partial \star D_1 = -\nu \partial S$$

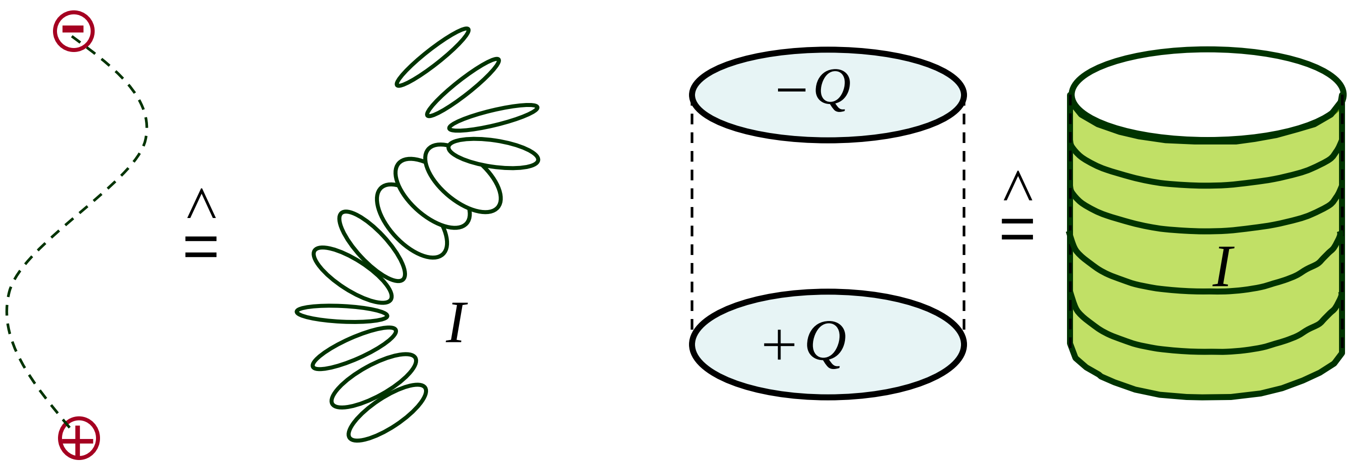
Compare this with

$$I\partial S = j = \partial H = \mu_0^{-1} \partial \star B \quad \Rightarrow \quad \boxed{\begin{aligned} \partial B &= 0 \\ \partial \star B &= \mu_0 I \partial S \end{aligned}}$$

The same equations have the same solutions.

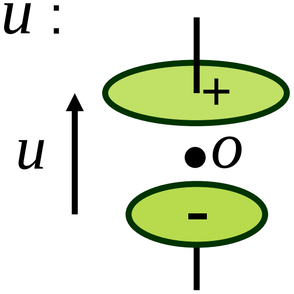
ElectroMagnetostatics (III)

Further examples of the equivalence
electrostatics $\Leftrightarrow$ magnetostatics:



Discharge of a Capacitor (I)

Capacitor with electric dipole moment u :



is discharged instantaneously at time zero.

What´s the electromagnetic signal emitted?

Charge density in dipole approximation: $\rho = \partial \iota(u) o$

Total (time integrated) current pulse: $\int_{-\varepsilon}^{\varepsilon} jdt = -\iota(u) o$

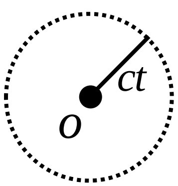
Discharge of a Capacitor (II)

Wave equation for B : $\square B = \mu_0 \partial \star j$

Initial value problem: $\square B = 0 \quad (t > 0)$, and

$$B\big|_{t=0+} = 0, \quad \dot{B}\Big|_{t=0+} = \mu_0 c^2 \partial \star \int_{-\varepsilon}^{\varepsilon} j dt = -\varepsilon_0^{-1} \partial \star \iota(u) o$$

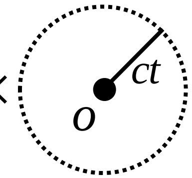
Green's function of wave equation:

$\Sigma_t = t \times$ 

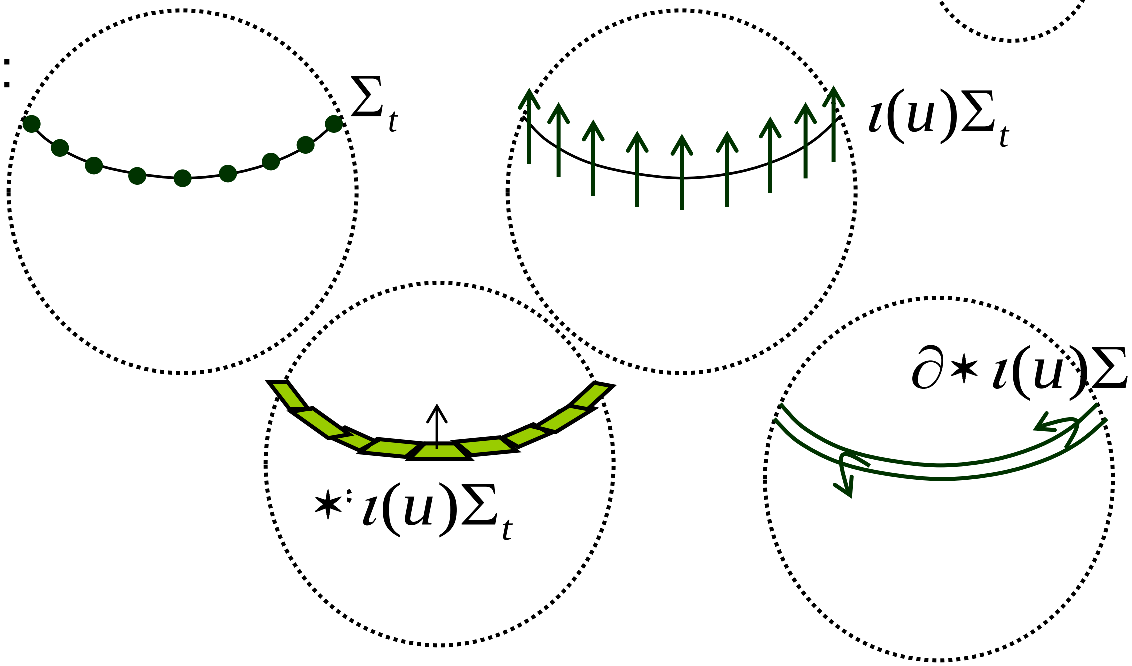
(0-chain with total mass t)

satisfies $\square \Sigma_t = 0 \ (t > 0), \quad \Sigma_{t=0+} = 0, \quad \dot{\Sigma}_{t=0+} = o$

Discharge of a Capacitor (III)

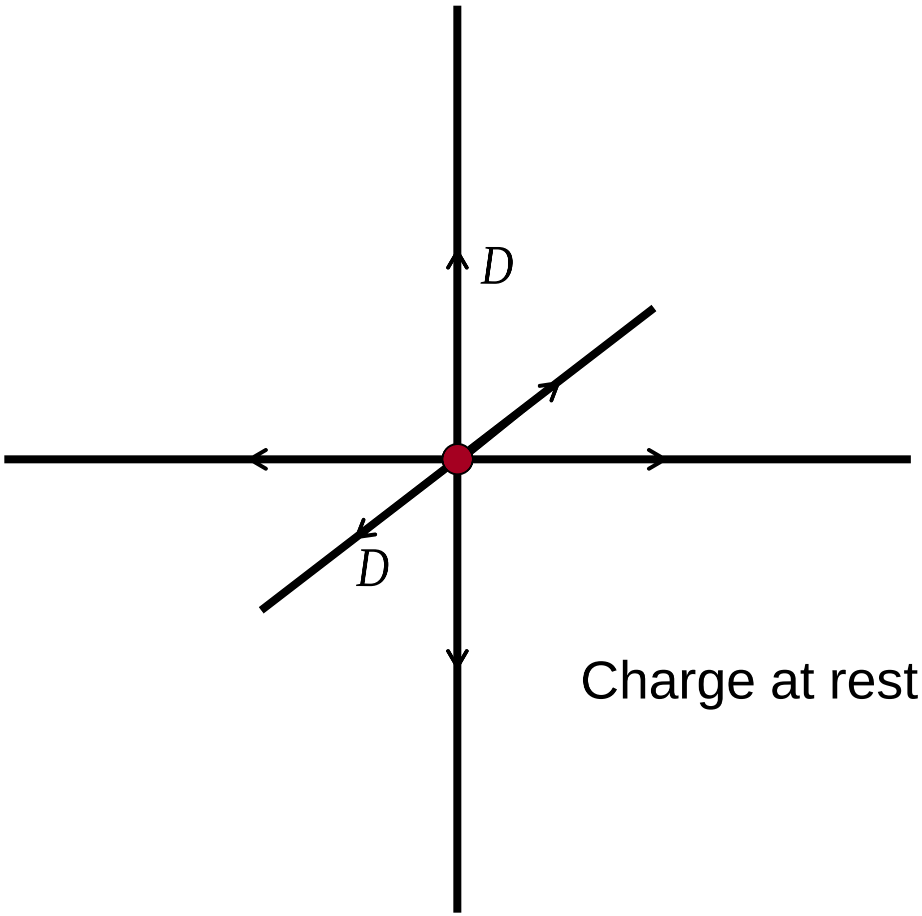
Solution: $B\big|_t = -\varepsilon_0^{-1}\partial \star \imath(u)\Sigma_t$ with $\Sigma_t = t \times$ 

Pictorially:



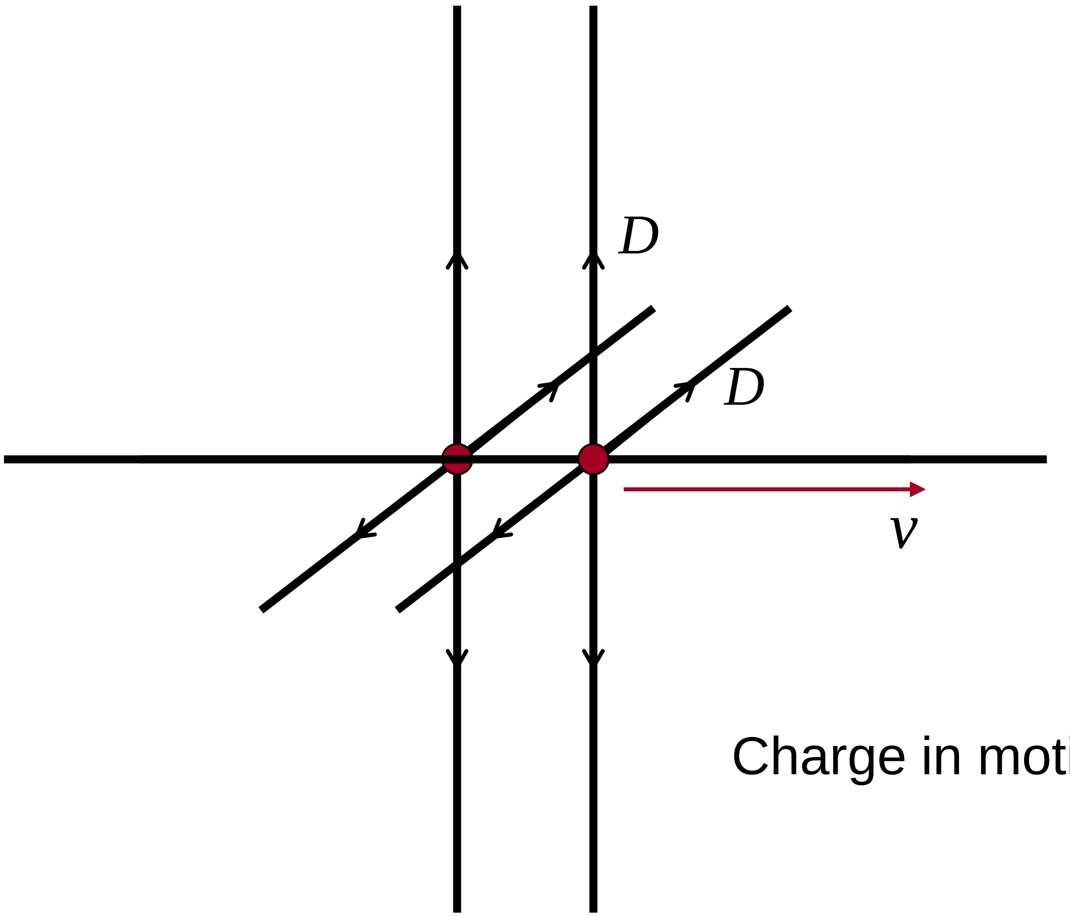
In formulas: $$B_{\text{azimut}} = \frac{|\mu|\sin\theta}{4\pi\varepsilon_0 c}\frac{\partial}{\partial r}\left(\frac{\delta(r-ct)}{r}\right)$$

Magnetic excitation by point charge



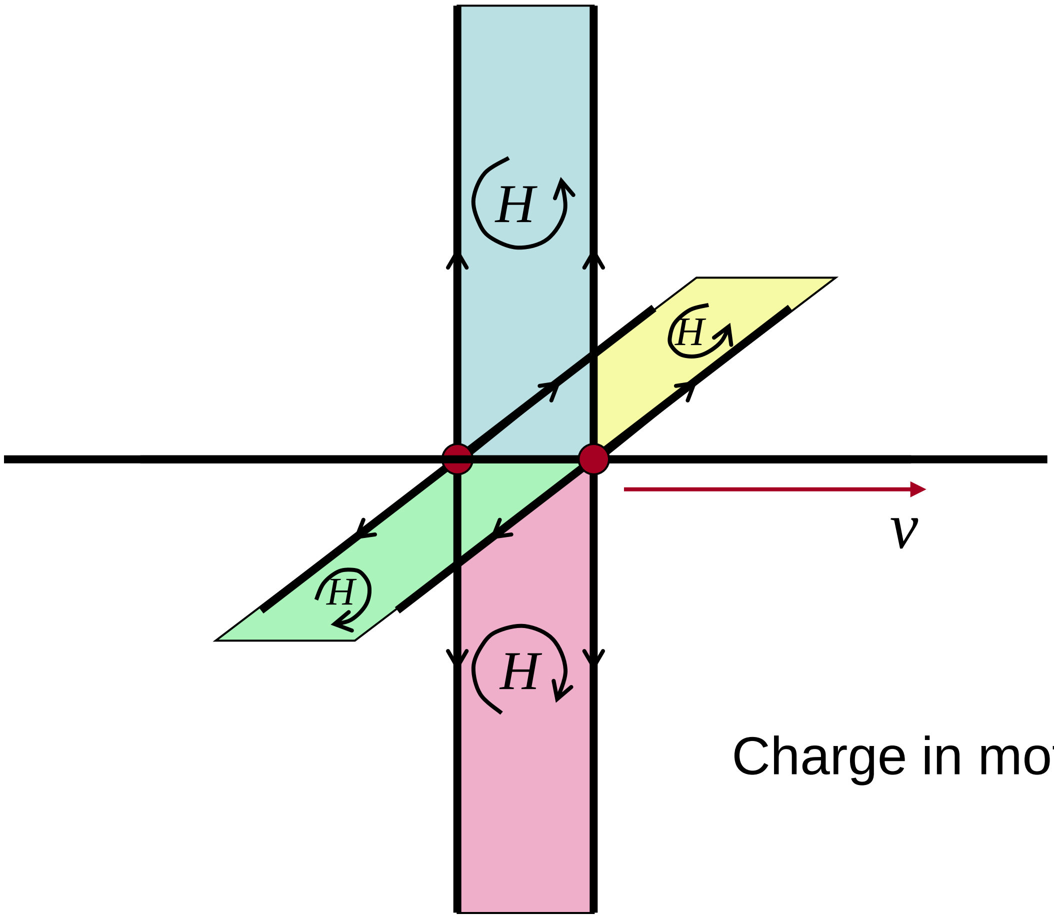
Charge at rest: $H = 0$

Magnetic excitation by point charge



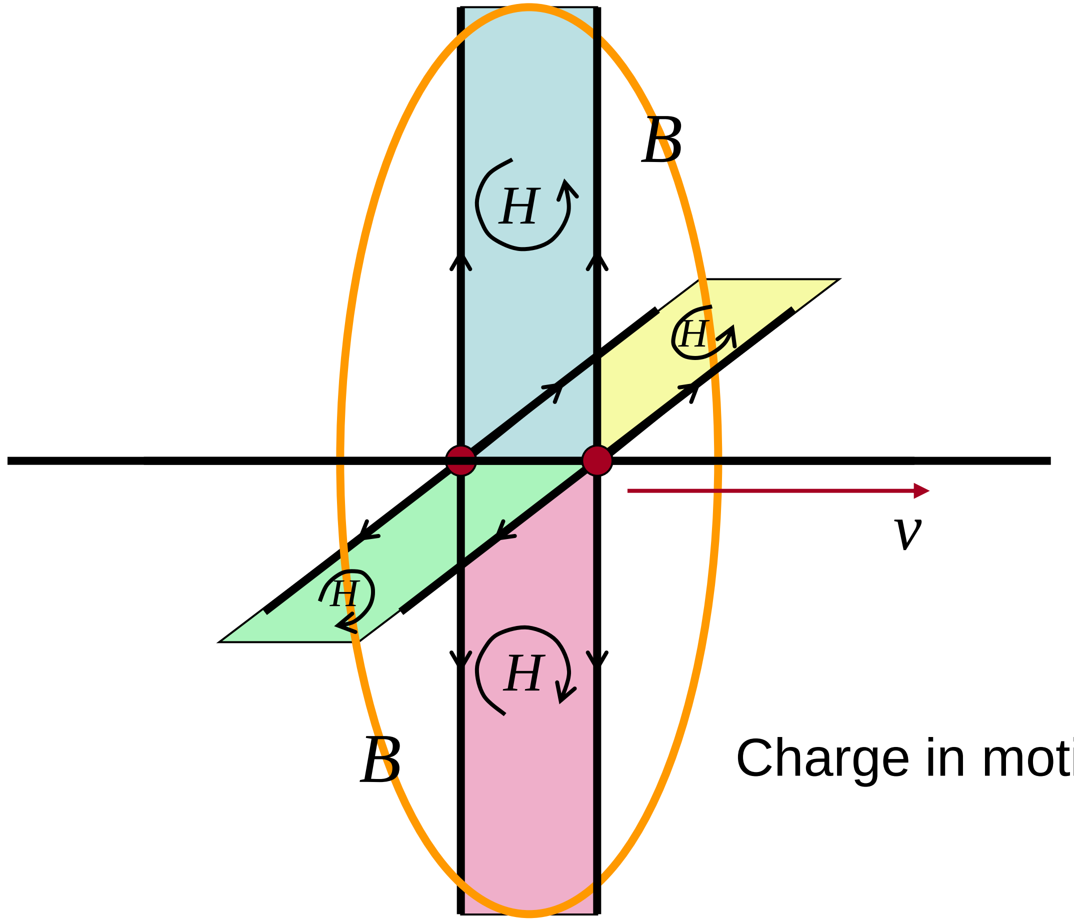
Charge in motion

Magnetic excitation by point charge



Charge in motion: $H \neq 0$

Magnetic excitation by point charge



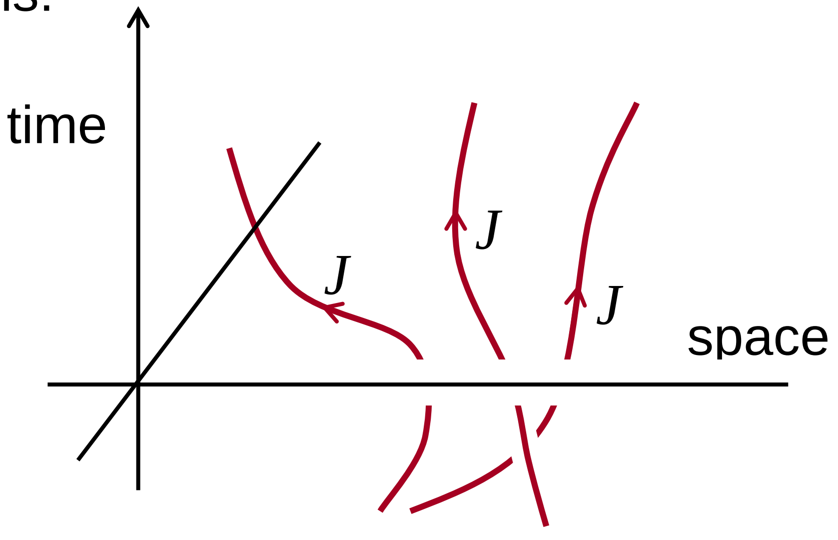
Charge in motion: $H \neq 0$

In the case of the magnetic field we can make the following point: Suppose that you finally succeeded in making up a picture of the magnetic field in terms of some kind of lines or of gear wheels running through space. Then you try to explain what happens to two charges moving in space, both at the same speed and parallel to each other. Because they are moving, they will behave like two currents and will have a magnetic field associated with them (like the currents in the wires of Fig. 1-8). An observer who was riding along with the two charges, however, would see both charges as stationary, and would say that there is *no* magnetic field. The "gear wheels" or "lines" disappear when you ride along with the object! All we have done is to invent a *new* problem. How can the gear wheels disappear?! The people who draw field lines are in a similar difficulty. Not only is it not possible to say whether the field lines move or do not move with charges - they may disappear completely in certain coordinate frames.

Quantum Hall Effect (I)

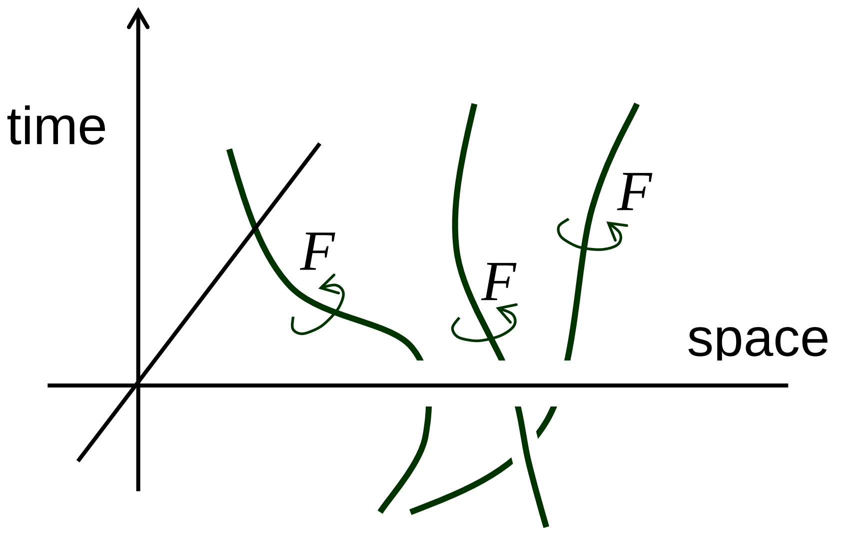
Two-dimensional electron gas in a strong magnetic field:
2+1 space-time dimensions

1-chain J (3-current) consists of the world lines of charged excitations:



Quantum Hall Effect (II)

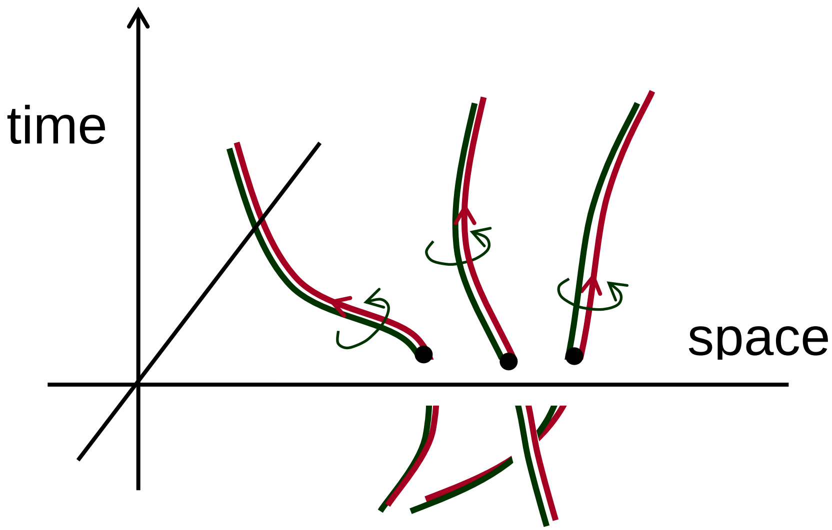
1-chain F (Faraday form) consists of the world lines of the points of intersection of the flux lines of B with the plane of the electron gas:



Quantum Hall Effect (III)

In every plateau regime (= quantum Hall state)
we have (Jürg Fröhlich): $\boxed{J = \sigma_{\mathrm{H}} F}$

σ_{H} = Hall conductivity (a pseudo scalar)



Every excitation with charge q is accompanied
by a magnetic flux $\Phi = q / \sigma_{\mathrm{H}}$