Literature

Belavin, Polyakov & Zamolodchikov (Nucl. Phys. B, 1984)

G. Segal (ICMP 1988)

Dijkgraaf (Les Houches 1995)

Polchinski ("String Theory", vol. 1, 1998)

1. Conformal invariance of critical systems

Setting of statistical mechanics: discrete space(time) $\Sigma \xrightarrow{\varphi}$ target space M, energy function $H[\varphi]$; partition sum $Z = \int \mathcal{D}\varphi \, e^{-\beta H[\varphi]}$.

Examples. Ising model: Σ = vertices in $\Lambda \subset \mathbb{Z}^d$, $M = \mathbb{Z}_2$;

Lattice gauge theory: Σ = edges in $\Lambda \subset \mathbb{Z}^d$, M = gauge group G.

Notation: field components $\varphi_i = \xi_i \circ \varphi$ (ξ_i coordinates on M);
n-point functions $\langle \varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n) \rangle = Z^{-1} \int \mathcal{D}\varphi \; \varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n) \, e^{-\beta H[\varphi]}$.

FACT/CREED: at a critical point $\beta = \beta_c$ (second-order phase transition, diverging correlation length, massless excitations) some n-point functions have a continuum limit, which is universal (i.e. independent of the microscopic details) and depends (within a given universality class) only on the conformal structure of Σ.
This principle (of conformal invariance) is most powerful in two (1+1) dimensions.

Example ($d=2$): holomorphic conserved current j (e.g. free boson: $j = \partial\varphi$)

$$\langle j(z_1)\, j(z_2) \rangle_\Sigma =: K^{(\Sigma)}(z_1, z_2)$$

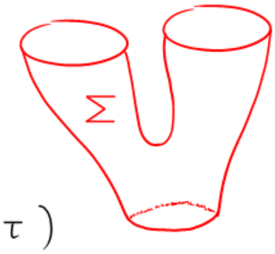
transforms under a conformal transformation $f: \Sigma \to \Sigma'$ by pullback:

$$K^{(\Sigma)}(z_1, z_2) = f'(z_1) f'(z_2) K^{(\Sigma')}(f(z_1), f(z_2)).$$

In particular, for $\Sigma = \Sigma' = S^2$ one has $K(z_1, z_2) = \dfrac{k}{(z_1 - z_2)^2}$,

and this is invariant under Möbius transformations $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc = 1$.

2. Free boson on a Riemann surface



Riemann surface Σ: a complex manifold with $\dim_{\mathbb{C}} \Sigma = 1$.

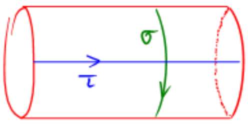
No metric tensor is given (!), just a complex structure:

$* d\sigma = d\tau\,, \quad * d\tau = -d\sigma$ (star operator in local coordinates σ, τ)

Cartan derivative decomposes as $d = \partial + \bar\partial\,, \quad \partial f = \frac{\partial f}{\partial z} dz\,, \quad \bar\partial f = \frac{\partial f}{\partial \bar z} d\bar z\,,$

$\partial, \bar\partial$ coordinate-independent: $\quad *d = i^{-1}(\partial - \bar\partial)$. $\qquad (z = \sigma + i\tau\,, \ \bar z = \sigma - i\tau)$.

Example: Σ = cylinder



Free boson: $\quad H[\varphi] = \int_\Sigma d\varphi \wedge * d\varphi = \int_\Sigma d\sigma \wedge d\tau \left(\left(\frac{\partial \varphi}{\partial \sigma}\right)^2 + \left(\frac{\partial \varphi}{\partial \tau}\right)^2 \right) = -2i \int_\Sigma \varphi\, \partial \bar\partial \varphi\,.$

$0 = \int \mathcal{D}\varphi \, \frac{\delta}{\delta \varphi(x)} e^{-\beta H[\varphi]} \implies$ equation of motion $\partial \bar\partial \varphi = 0$.

Language:

$f \in \mathrm{Hol}(\Sigma) \iff \bar\partial f = 0$

$f \in \overline{\mathrm{Hol}(\Sigma)} \iff \partial f = 0$

$f \in \mathrm{Harm}(\Sigma) \iff \partial \bar\partial f = 0$.

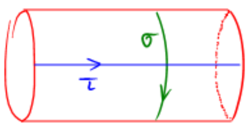
Chiral factorization

The cohomology of Σ poses an obstruction to harmonic functions being sums of holomorphic and anti-holomorphic parts.

Exact sequence:

$$0 \longrightarrow H^0(\Sigma) \longrightarrow \mathrm{Hol}(\Sigma) \oplus \overline{\mathrm{Hol}(\Sigma)} \longrightarrow \mathrm{Harm}(\Sigma) \longrightarrow H^1(\Sigma) \longrightarrow 0\,.$$

Example (Σ = cylinder): harmonic function $\quad 2i\tau = \underbrace{(\sigma + i\tau)}_{\notin \mathrm{Hol}(\Sigma)} - (\sigma - i\tau)$.



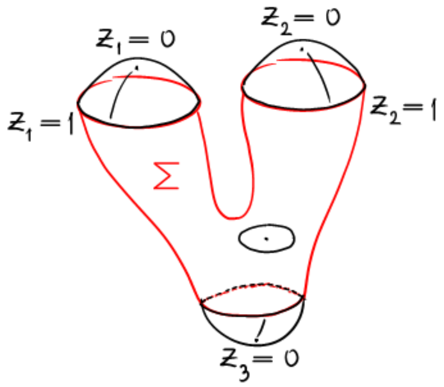
3. CFT: axioms ($d = 1+1$)

A (unitary) CFT is a **functor** Φ from the **category** of Riemann surfaces Σ with punctures (= parametrized boundaries) to the category of Hilbert spaces $\mathcal{H}$,

$$\Phi : \text{Riem} \longrightarrow \text{Hilb} \,,$$

$$(\text{TFT}: \quad \text{Man} \longrightarrow \text{Vect})$$

with some additional properties (see below).



punctures $i = 1, \dots, n$ at $z_i = 0$
(local coordinate z_i; disk $|z_i| \leq 1$)

REMARKS.

i) **Sewing** of punctured Riemann surfaces:

ii) "Functor" means there is an assignment $X \mapsto \mathcal{H}_X \equiv \mathcal{H}$ and

$$(\Sigma : X \to Y) \longrightarrow (\Phi_\Sigma : \mathcal{H}_X \to \mathcal{H}_Y)$$

which satisfies the **semigroup law** $\Phi_{\Sigma_2 \circ \Sigma_1} = \lambda\, \Phi_{\Sigma_2} \circ \Phi_{\Sigma_1} \quad (\lambda \in \mathbb{C})$.

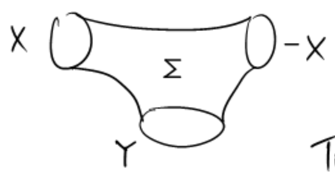
iii) Physics interpretation: think of Φ_Σ as the normalized partition sum for Σ with prescribed boundary data on $\partial\Sigma = (-X) \cup Y$.

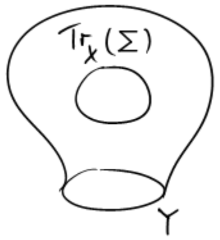
Additional properties:

$$\mathcal{H}_{X \cup Y} = \mathcal{H}_X \otimes \mathcal{H}_Y \,, \qquad \mathcal{H}_{-\Sigma} = \mathcal{H}_X^* \,, \qquad \mathcal{H}_\emptyset = \mathbb{C} \,,$$

$$\Phi_{\Sigma_2 \cup \Sigma_1} = \Phi_{\Sigma_2} \otimes \Phi_{\Sigma_1} \,, \qquad \Phi_{-\Sigma} = \Phi_\Sigma^\dagger \,, \qquad \Phi_{\text{disk}} = 1 \ (\text{ground state}).$$

Partial trace $\quad \text{Tr}_X : \Sigma \to \text{Tr}_X(\Sigma)$





$$\text{Tr}_{\mathcal{H}_X} \Phi_\Sigma = \Phi_{\text{Tr}_X(\Sigma)}$$

Further remarks: i) See G. Segal (1988) for a concise system of axioms using the notion of "modular functor".

ii) CFT n-point functions are obtained by inserting vectors $\varphi_{i_1}, \dots, \varphi_{i_n}$ into the n slots of the linear form $\Phi_\Sigma : \mathcal{H}^{\otimes n} \to \mathbb{C}$.

4. Energy – momentum (stress – energy) tensor

n – point function $\langle A(x_1, \dots, x_n)\rangle$ where $A(x_1,\dots,x_n) \equiv \varphi_{i_1}(x_1)\cdots\varphi_{i_n}(x_n)$.

Consider diffeomorphism generated by vector field ξ.

$$0 = \mathcal{L}_\xi \langle A\rangle + \frac{1}{2\pi}\left\langle A \cdot \int T_{\mu\nu}\, d\xi^\mu \wedge * dx^\nu \right\rangle \qquad (\text{this defines } T_{\mu\nu}).$$

Invariance under rotations $(d\xi^\mu = * dx^\mu) \Rightarrow T_{\mu\nu} = T_{\nu\mu}$ (T is symmetric),

Scale invariance at critical point $(d\xi^\mu = dx^\mu) \Rightarrow T_{\mu\nu}\, dx^\mu \wedge * dx^\nu = 0$ (T is traceless),

Take $\xi = 0$ at support of n-pt fctn $\Rightarrow dT_{\mu\nu} \wedge * dx^\nu = 0$ (T is conserved).

In complex coordinates: $T_{\mu\nu}$ traceless $\Rightarrow T_{z\bar z} + T_{\bar z z} = 0$,

$T_{\mu\nu}$ symmetric $\Rightarrow T_{z\bar z} = T_{\bar z z} = 0$.

$T_{\mu\nu}$ conserved $\Rightarrow \dfrac{\partial}{\partial \bar z} T_{zz} = 0$ and $\dfrac{\partial}{\partial z} T_{\bar z\bar z} = 0$.

Abbreviated notation $T_{zz} = T(z)$, $T_{\bar z\bar z} = \tilde T(\bar z)$.

Ward identity as a contour integral.

Let C be a contour enclosing the points of support of A.

Take $\xi^z = \varepsilon(z)$ (holomorphic) inside C, ε vanishes on $\operatorname{supp}(A)$, $\xi^z \to 0$ at ∞, $\xi^{\bar z} \equiv 0$.

Then
$$0 = \mathcal{L}_\varepsilon \langle A\rangle - \frac{1}{2\pi i}\oint_C dz\, \varepsilon(z)\, \langle T(z)\, A\rangle.$$

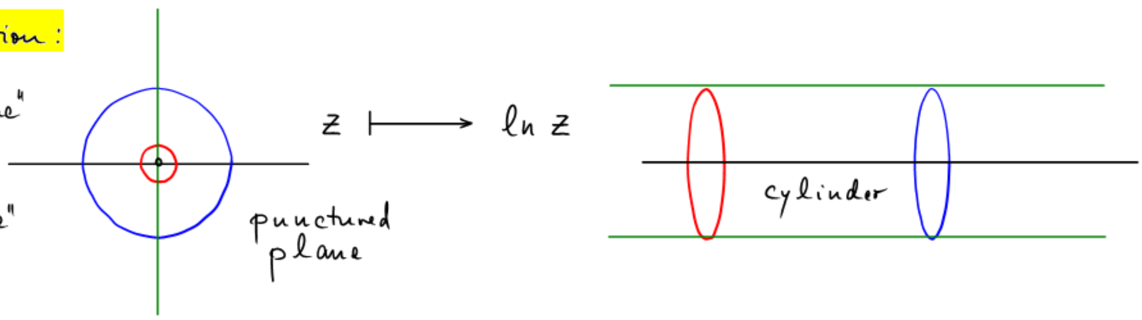
Similar with $T(z) \leftrightarrow \tilde T(\bar z)$.

6. CFT basics

Radial quantization:

$\ln|z| =$ "time"

$\arg(z) =$ "space"



Language: 'operator' $\equiv$ local field $A_i \in \{\varphi_i, \partial\varphi_i, \dots\}$

Operator – state correspondence:

insert operator $A_i(0)$ at $z = 0$ $\Longleftrightarrow$ impose state $|A_i\rangle \in \mathcal{H}$ for boundary circle at $\ln|z| = -\infty$

Main principle: associativity of the algebra of local fields

Operator product expansion (OPE): $A_i(x)\, A_j(y) = \sum_k C_{ij}^k(x,y)\, A_k(y)$

Justification from operator-state correspondence.

Primary fields: conformal dimension

$$T(z)\, \varphi_i(0) = \frac{\Delta_i}{z^2}\varphi_i(0) + \frac{1}{z}\frac{\partial \varphi_i}{\partial z}(0) + \dots \qquad \text{\& similar for } \tilde{T}(\bar{z})$$

(Algebraic meaning: φ_i highest-weight vector of Virasoro representation)

OPE of energy-momentum tensor with itself:

$$T(z)\, T(0) = \frac{c/2}{z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\frac{\partial T}{\partial z}(0) + \dots \qquad (T \text{ not a primary field})$$

c = **central charge**, conformal charge.

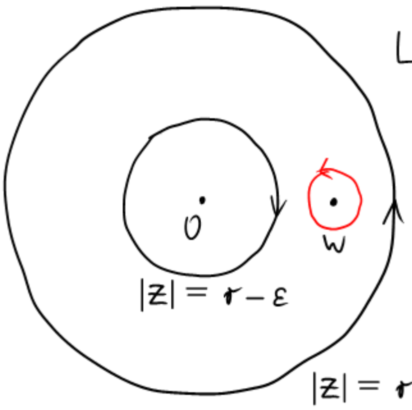
Interpretation of c. The integrated law of T for a conformal mapping $f: M \to N$ is

$$T^{(M)}(z) = f'(z)^2\, T^{(N)}(f(z)) + \frac{c}{12}(\mathcal{S}f)(z)$$ Schwarzian derivative (vanishes for f Möbius).

For $f(z) = e^{2\pi i z/L}$ one gets $\langle T\rangle_{cyl} = (\dots)\underbrace{\langle T\rangle_{plane}}_{=0} + \frac{\pi^2 c}{6L^2}$ (→ Casimir energy density)

counts no. of d.o.f.

OPEs & **commutation relations** (use radial quantization):



$$L_n := \frac{1}{2\pi i}\oint T(z)\, z^{n+1}\, dz \qquad \text{(Virasoro generators)}$$

Example (primary field φ_i):

$$[L_{-1}, \varphi_i] = L_{-1}\varphi_i - \varphi_i L_{-1}$$

$$\mathrel{\hat{=}} \frac{1}{2\pi i}\oint_{\odot} T(z)\, \varphi_i(\cdot)\, dz = \frac{\partial \varphi_i}{\partial z}$$

from simple pole in OPE

2-point function of primary fields on $\Sigma = \mathbb{C}$:

$$\langle \varphi_{i_1}(z_1,\bar{z}_1)\, \varphi_{i_2}(z_2,\bar{z}_2)\rangle = \delta_{i_1 \bar{i}_2}\, (z_1 - z_2)^{-\Delta_i}(\bar{z}_1 - \bar{z}_2)^{-\bar{\Delta}_i}$$

is invariant under fractional linear transformations (= Möbius transformations)

3-pt fctn of primary fields (spinless case, $\Delta_i = \bar{\Delta}_i$)

$$\langle \varphi_{i_1}(z_1,\bar{z}_1)\, \varphi_{i_2}(z_2,\bar{z}_2)\, \varphi_{i_3}(z_3,\bar{z}_3)\rangle = \frac{C_{i_1 i_2 i_3}}{|z_1 - z_2|^{\Delta_1+\Delta_2-\Delta_3}\, |z_3 - z_1|^{\Delta_3+\Delta_1-\Delta_2}\, |z_2 - z_3|^{\Delta_2+\Delta_3-\Delta_1}}.$$