BCGS Intensive Week — Day 4: Morse Theory & Bott Periodicity (M. Zirnbauer)

PART 0: Some explanation of the words in the title. Motivation. Background.

(i) Morse Theory (developed in 1930 — 1965; Morse, Bott, Smale, Milnor ...)

One outcome of Morse Theory are the Morse inequalities:

for a compact manifold M with Morse function f, the number $m_q(f)$ of critical points of f with index q is no less than the Betti number $b_q(M)$.

- Fruitful new proof by E. Witten (1982) in "Supersymmetry and Morse Theory".

Witten's IDEA: de Rham complex $\Omega^* M$ = supersymmetric quantum mechanics & deformation to a harmonic-oscillator problem.

See, e.g., http://www.thp.uni-koeln.de/zirn ⤳ "Topology for Physicists" (pp. 46–64).

(ii) Bott Periodicity Theorem

'Modern' proofs available (cf. Day 5); Bott's original approach (1958) uses Morse theory.

Complex case.

$$\mathbb{Z} = \pi_1(U_n) = \pi_2(U_{2n}/U_n \times U_n) = \pi_3(U_{2n}) = \pi_4(U_{4n}/U_{2n}\times U_{2n}) = \dots$$
$$0 = \pi_1(U_{2n}/U_n\times U_n) = \pi_2(U_{2n}) = \pi_3(U_{4n}/U_{2n}\times U_{2n}) = \pi_4(U_{4n}) = \dots$$

Real case.

$$\mathbb{Z}_2 = \pi_1(O/O\times O) = \pi_2(U/O) = \pi_3(Sp/U) = \pi_4(Sp) =$$
$$= \pi_5(Sp/Sp\times Sp) = \pi_6(U/Sp) = \pi_7(O/U) = \pi_8(O) = \pi_9(O/O\times O)$$

$$\mathbb{Z} = \pi_1(U/O) = \pi_2(Sp/U) = \pi_3(Sp) = \pi_4(Sp/Sp\times Sp) =$$
$$= \pi_5(U/Sp) = \pi_6(O/U) = \pi_7(O) = \pi_8(O/O\times O) = \pi_9(U/O)$$

and 6 more such chains of identities (8-fold periodicity).

$\geq$ 2008: enter the physics of topological insulators and superconductors

(classification of gapped free-fermion ground states with symmetries).

Periodic Table of topological insulators/superconductors

from Hasan & Kane, Rev. Mod. Phys. (2011):

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Quantum Hall Effect

He-3 (B phase)

QSHI: HgTe

Majorana

$Bi_2 Se_3$

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

Diagonal map:

1d		2d		3d
Majorana chain	$\longrightarrow$	T-invt spinless superconductor	$\longrightarrow$	$\mathbb{Z}_2$ topological insulator
Kitaev chain	$\longrightarrow$	p+ip superconductor	$\longrightarrow$	^{3}He B-phase

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TODAY (modest goal): main ideas behind complex Bott periodicity.

CBP follows from the combination of two sub-results:

1. $\pi_d(U) = \pi_{d+1}(U/U\times U)$,
2. $\pi_{d+1}(U/U\times U) = \pi_{d+2}(U)$.

Both can be derived by using Morse theory, following Bott (1958). Roughly speaking, one shows that the unitary group furnishes a good approximation (→ homotopy type) to the loop space of a Grassmannian: $\Omega(U/U\times U) \approx U$, and vice versa.

Comment. The first identity, say in the form of $\pi_d(U_n) = \pi_{d+1}(U_N/U_{N-n}\times U_n)$, can also be obtained from the long exact homotopy sequence of a fiber bundle,

$$U_n \hookrightarrow U_N/U_{N-n}\times\cdot \longrightarrow U_N/U_{N-n}\times U_n ,$$

whose total space (Stiefel manifold) is weakly contractible.

HERE: "tools" ($\sim$ focus on aspects that are computationally powerful).

PART 1: A basic tale of CW complexes

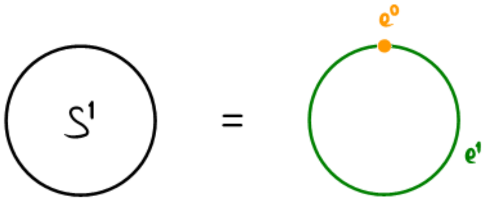
Cell decomposition.

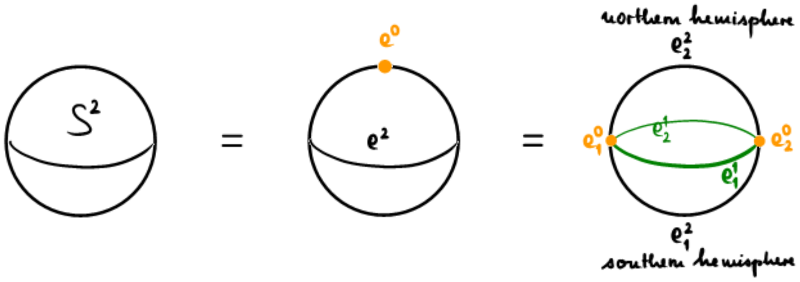
(closed) n-disk $D^n := \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$, $\partial D^n = S^{n-1}$ (n-1)-sphere.

An n-cell is a space homeomorphic to $\mathrm{int}(D^n)$ (open n-disk).

Definition. A cell decomposition of a topological space X is a collection $\{e_\alpha\}_{\alpha \in I}$ of cells $e_\alpha \in X$ such that $X = \bigsqcup_{\alpha \in I} e_\alpha$ (disjoint union). The n-skeleton of the cell decomposition is $X^n := \bigsqcup_{\dim(e_\alpha) \leq n} e_\alpha$.

Examples.





$\mathbb{R}P^2 \cong S^2/\mathbb{Z}_2 = e_1^0 \sqcup e_1^1 \sqcup e_1^2$.

Definition. A Hausdorff space X with a finite cell decomposition $X = \bigsqcup_{\alpha \in I} e_\alpha$ is called a CW complex if for each n-cell e there exists a map $\varphi_e : D^n \to X$ such that

1. The restriction $\varphi_e : \mathrm{int}(D^n) \to e$ is a homeomorphism;
2. The image of $\varphi_e : \partial D^n \to X$ is contained in X^{n-1}.

If there are infinitely many cells, one poses the additional requirements of

- "closure finiteness": for each cell e the closure $\bar{e}$ intersects only finitely many other cells.
- "weak topology": a subset $A \subseteq X$ is closed iff $A \cap \bar{e}$ is closed for every cell e.

Remarks. $\bar{e} = \varphi_e(D^n)$ (here the Hausdorff property is needed).

X^n $(n = 0, 1, 2, \ldots)$ is always closed (in fact, X^n is a CW complex).

Exercise: find a CW complex for S^3, T^2, $\mathbb{R}P^n$, Klein bottle.

PART 1 (continued).

Def. Topological space X, set Y. A surjective map $p: X \to Y$ induces "identification topology" on Y: $C \subset Y$ open $\Leftrightarrow$ $p^{-1}(C) \subset X$ open.

Example: principal bundle $p: X \to X/G$ (a.k.a. quotient topology)

e.g., $X = \mathbb{R}^2 \setminus \{0\}$, $G = SO(2)$, $X/G = \mathbb{R}_+$.

Note: identification map $p: X \to Y$ amounts to the same as

an equivalence relation $x_1 \sim x_2 \Leftrightarrow p(x_1) \sim p(x_2)$.

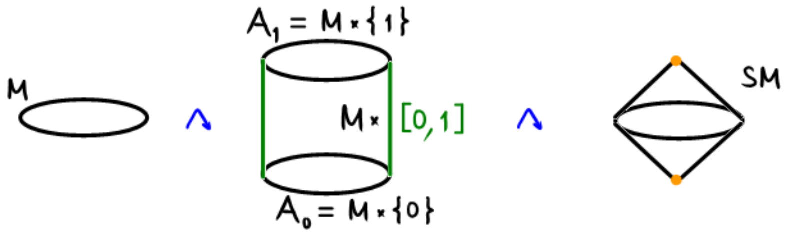
$$\begin{tikzcd} X \arrow[d] \arrow[dr, "p"] & \\ X/\sim \arrow[r, "\text{homeomorphism}"'] & Y \end{tikzcd}$$

We now highlight two operations that will be of use below.

1. Collapsing a subspace.

Let $A \subset X$ be a subspace of the topological space X. Then one defines $X/A := X/\sim$ where the equivalence classes are A and $\{x\}$ for all points $x \in X \setminus A$.

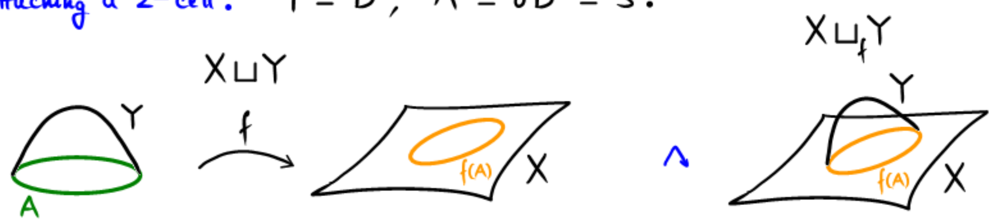
Example: Suspension SM. Let $X = M \times [0,1]$ and define the equivalence relation $\sim$ on X to have for its equivalence classes the subspaces $A_0 = M \times \{0\}$, $A_1 = M \times \{1\}$ and $\{x\}$ for all points $x \in X$ outside of $A_0 \cup A_1$. Then $SM := X/\sim$. In a rather sloppy notation this is sometimes written as $SM := M \times [0,1] \,/\, (M \times \{0\}) \cup (M \times \{1\})$.



2. Attaching one space to another by a map. For two topological spaces X, Y and a closed subspace $A \subset Y$, let there be a map $f: A \to X$.

Consider the disjoint union $X \sqcup Y$ and define $X \sqcup_f Y := (X \sqcup Y)/\sim$ where the equivalence classes are $\{y\}$ for $y \in Y \setminus A$, $\{x\}$ for $x \in X \setminus f(A)$, $\{y, f(y)\}$ for $y \in A$.

Example: attaching a 2-cell. $Y = D^2$, $A = \partial D^2 = S^1$.



Note. For a CW complex X with n-skeleton X^n ($n = 0, 1, 2, \dots$) the space X^n is obtained from the preceding space by attaching all the n-cells.

Exercises. 1. $\pi_d(S^n) = 0$ (trivial group) for $1 \leq d \leq n$. **Hint:** $S^n = \{p\} \sqcup_f D^n$ with $f(\partial D^n) = p$.

2. $\pi_d(X) = \pi_d(X \sqcup_f D^n)$ $(f: \partial D^n \to X)$ for $d < n-1$.

PART 2. Morse Theory – fundamental theorems

Manifold M, function $f : M \to \mathbb{R}$, $M^a := \{ x \in M \mid f(x) \leq a \}$.
Assume f smooth and proper.

Thm 1. If f has no critical values in the interval $[a, b]$, then
M^a is homotopy equivalent to M^b (in fact, a deformation retract of M^b).

Thm 2. Let $x \in M$ be a non-degenerate critical point of f of index n.
If x is the only critical point in $f^{-1}[f(x)-\varepsilon, f(x)+\varepsilon]$, then
$M^{f(x)+\varepsilon}$ is homotopy equivalent to $M^{f(x)-\varepsilon}$ with an n-cell attached.

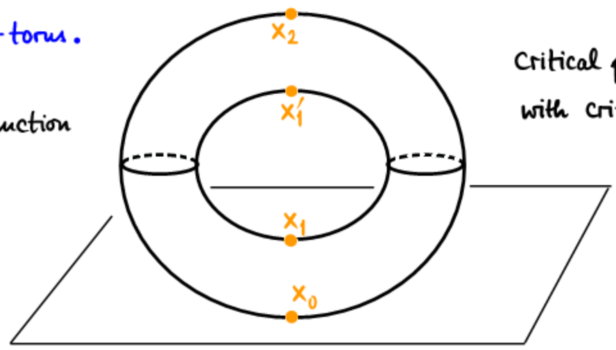
The proof uses the Morse lemma: in some neighborhood $\mathcal{U}$ of a non-degenerate critical point x with index n there exists a chart $\{x_1, \dots, x_m\}$ such that
$f = f(x) - x_1^2 \dots - x_n^2 + x_{n+1}^2 + \dots + x_m^2$ holds on $\mathcal{U}$.

Fact. For any manifold M, there exist functions (so-called Morse functions) with
no degenerate critical points and with no two critical values the same.

Corollary. Every manifold is a CW complex with one n-cell for each critical point (of a Morse function) of index n.

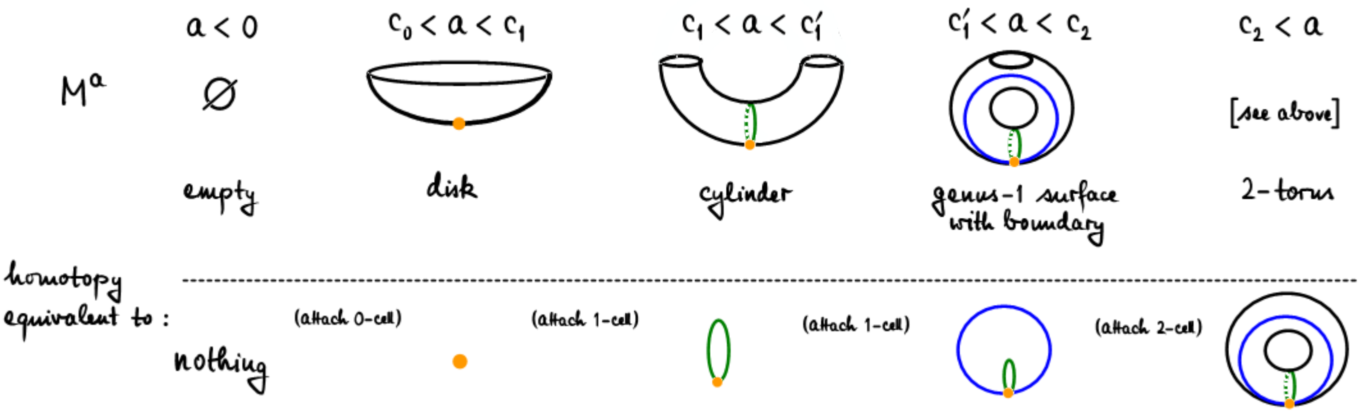
Example: 2-torus.

f = height function



Critical points x_k
with critical values $c_k = f(x_k)$

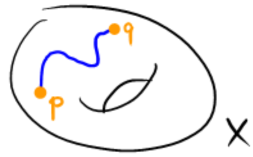
$c_2 > c_1' > c_1 > c_0$



Exercises. 1. Construct a CW complex for a Riemann surface of genus 2.
2. Construct a CW complex for SM (given that of M).

PART 3. Bott Periodicity Theorem (sketch of complex case).

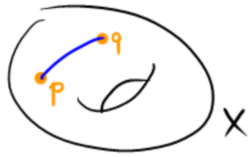
(said to be one of the most surprising theorems in topology)



Setting. Riemannian manifold X (compact, connected).

$\nu = (p, q; h)$ homotopy class h of curves in X joining p to q

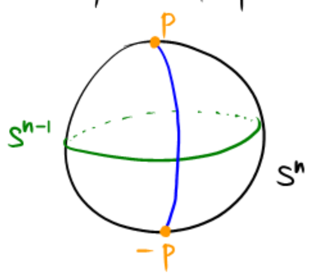
X^ν = space of minimal geodesics —— " ——



Thm (Bott). If X is a symmetric space, then so is X^ν.

Remark. The most interesting situation occurs when q is the antipode of p.

Example 1. $X = S^n$, $\nu = (p, -p; \cdot)$: $X^\nu = S^{n-1}$



Example 2.

Let $C_0(n) := \{ J \in \mathrm{End}(\mathbb{C}^{2n}) \mid J^\dagger = -J,\ J^2 = -1 \}$.

Subspace $X = \{ J \in C_0(n) \mid \dim E_{+i}(J) = n \} \simeq \mathrm{Gr}_n(\mathbb{C}^{2n}) \underset{\uparrow}{\simeq} U_{2n}/U_n \times U_n$.

Let $\nu = (J_1, -J_1; \cdot)$ with, say, $J_1 = i\sigma_3 \otimes 1_n$. $\qquad J = g(i\sigma_3 \otimes 1_n)\, g^\dagger$

CLAIM: $X^\nu \simeq U_n$. **Proof** of $U_n \hookrightarrow X^\nu$:

Let $C_1(n) = \{ J \in C_0(n) \mid J J_1 + J_1 J = 0 \}$.

For $J \in C_1(n)$ consider $\gamma(t) = e^{(\pi t/2) J_1 J}\, J_1\, e^{-(\pi t/2) J_1 J} = e^{\pi t J_1 J}\, J_1$.

Properties of γ: (i) $\gamma(0) = J_1$ and $\gamma(1) = -J_1$ | from $(J_1 J)^2 = -1$

(ii) $\gamma(\tfrac{1}{2}) = -J$ | from $e^{(\pi/2) J_1 J} = J_1 J$

(iii) γ is minimal geodesic in X.

$J \in C_1(n)$ is a pair of linear transformations $E_{+i}(J_1) \underset{v}{\overset{u}{\rightleftarrows}} E_{-i}(J_1)$.

$J^2 = -1 \rightsquigarrow v = -u^{-1}$; $J^\dagger = -J \rightsquigarrow u^{-1} = u^\dagger$.

Hence $C_1(n) \simeq U_n$ (as sets).

Exercise. For $X = C_1(n)$ and $\nu = (J_2, -J_2; \text{any})$ with, say, $J_2 = i\sigma_1 \otimes 1_n$, find X^ν.

Hint. Extend the construction of before.

(sketch cont'd)

$\Omega_\nu X :=$ space of all (piecewise differentiable) curves.

Take the length fctn $\ell(\cdot)$ as a Morse function (actually, Morse-Bott function) for $\Omega_\nu X$.

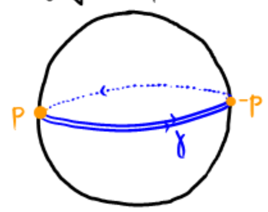
By generalizing the procedure for constructing a CW complex from a Morse function, Bott shows that X^ν is a 'good' approximation to $\Omega_\nu X$:

as the index-0 critical manifold of ℓ, the space X^ν of minimal geodesics captures the homotopy type of $\Omega_\nu X$ up to corrections (which take the form of cells attached to X^ν) due to positive-index critical manifolds (constituted by non-minimal geodesics):

$$\Omega_\nu X \simeq X^\nu \sqcup e^\mu \sqcup \ldots$$

where $\dim e^\mu \geqslant |\nu| :=$ smallest <u>non-zero</u> index of a geodesic; the index of a geodesic turns out to be the (properly counted) number of p-conjugate points in its interior.

Example. $X = S^2$, $\nu = (p, -p)$, $X^\nu = S^1$



index$(\gamma) = 2$

Thm (1, Bott): $\pi_d(X^\nu) = \pi_d(\Omega_\nu X)$ for $0 < d < |\nu| - 1$.

Remark. $\pi_d(\Omega_\nu X) = \pi_{d+1}(X)$.

Application (condensed matter physics).

<u>Recall</u>: a topological insulator in symmetry class A ($U(1)$ symmetry ~ charge is conserved) is modeled by

$$\left\{ \begin{array}{l} \text{a rank-}n\text{ complex vector (sub)bundle } V \xrightarrow{p} M,\ p^{-1}(k) \equiv V_k \text{ space of valence states at momentum } k \\ \simeq \\ \text{a classifying map } M \to Gr_n(\mathbb{C}^N),\ k \mapsto V_k. \end{array} \right.$$

In d dimensions and for $M = S^d$, such objects are classified by $\pi_d(Gr_n(\mathbb{C}^N))$.

Pertinent consequences of Bott's results.

1. $\pi_d(U_n) = \pi_{d+1}(U_{2n}/U_n \times U_n)$ for $d <$ [insert]
2. $\pi_d(U_{2n}/U_n \times U_n) = \pi_{d+1}(U_{2n})$ for $d < 2n+1$,

   more generally, $\pi_d(U_N/U_n \times U_{N-n}) = \pi_{d+1}(U_N)$ for [insert]
3. Periodicity Theorem: $\pi_d(U/U\times U) = \pi_{d+2}(U/U\times U)$; also $\pi_d(U) = \pi_{d+2}(U)$.
4. $\pi_{even}(U/U\times U) = \mathbb{Z}$ (QHE) | $\pi_{even}(U) = 0$ (symmetry class AIII:

   $\pi_{odd}(U/U\times U) = 0$ (no QHE) | $\pi_{odd}(U) = \mathbb{Z}$ Su-Schrieffer-Heeger model)

Appendix

Literature.

R. Bott, The stable homotopy of the classical groups, Ann. Math. 70 (1959) 313–337.

J. Milnor, Morse theory (Princeton University Press, 1963).

R. Bott, The periodicity theorem for the classical groups and some applications Adv. Math 4 (1970) 353–411.

Symmetric space. On a Riemannian manifold M one has for every point $p \in M$ an operation σ_p of geodesic inversion (the Riemannian analog of the Euclidean-geometry operation of reflection at a point) in some neighborhood of p. M is called a *locally symmetric space* if for all p the map σ_p is an isometry (on its domain of definition). M is called a *globally symmetric space* if the locally defined isometry σ_p extends for all p to an isometry $\sigma_p : M \to M$.

The Bott Periodicity Theorem makes a statement about the homotopy groups of globally symmetric spaces of "classical type" — these are the spaces mentioned in PART 0.