Aspects of Symmetry and Topology for Free-Fermion Ground States with Disorder

Martin R. Zirnbauer

@ Bad Honnef (CRC 183) Summer School

June 22-24, 2017

Periodic Table of topological insulators/superconductors

from Hasan & Kane, Rev. Mod. Phys. (2011):

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

QSHI = Quantum Spin Hall Insulator

Quantum Hall Effect

He-3 (B phase)

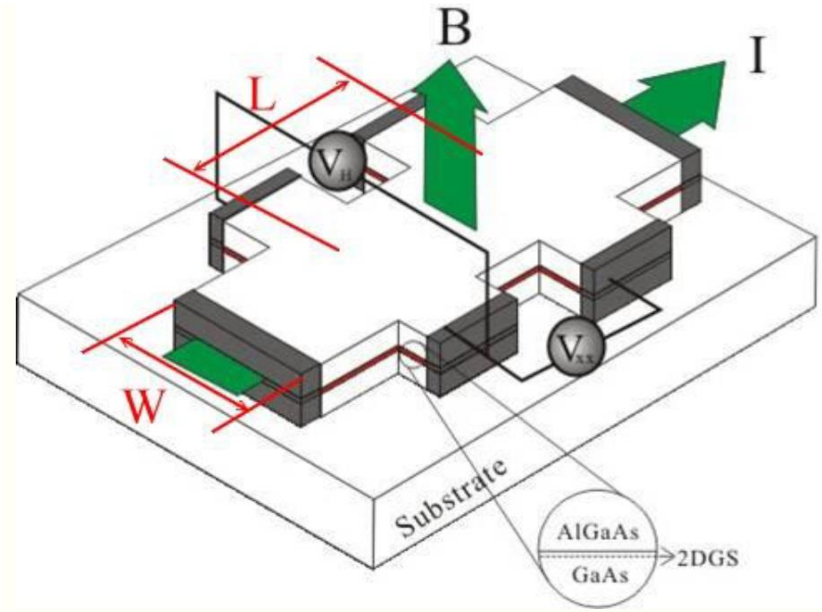
QSHI: HgTe

Majorana

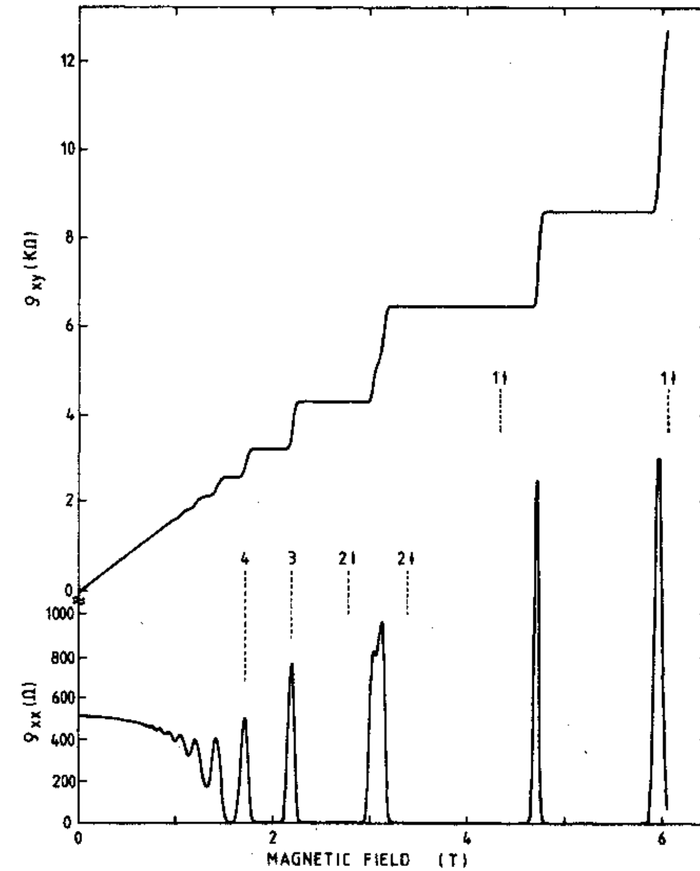
$Bi_2 Se_3$

Schnyder, Ryu, Furusaki, Ludwig (2008); Kitaev (2009); Teo & Kane (2010); Stone, Chiu, Roy (2011); Freedman, Hastings, Nayak, Qi, Walker, Wang (2011); Abramovici & Kalugin (2012); Freed & Moore (2013)

Integer Quantum Hall Effect



2D disordered electrons; high *B*, low *T*

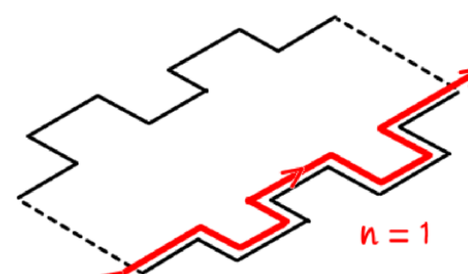


D.J. Thouless (1982-5; Nobel Prize Physics 2016):

Ground state = complex line bundle $\mathscr{A}$ over T^2

Hall conductance $= I/V_{\rm H} = n\,e^2/h$

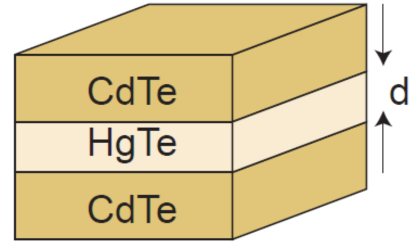
n = (integral of) first Chern class of $\mathscr{A}$



bulk-boundary correspondence

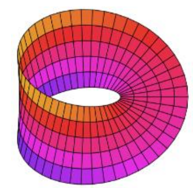
Quantum Spin Hall Insulator

(Kane & Mele, 2005)



Strong spin-orbit scattering (preserves time-reversal invariance) causes band inversion ↷ the bundle of Fermi projections is twisted.

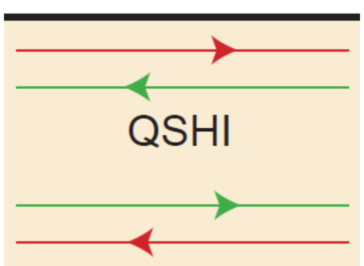
This twisting is detected by the Kane-Mele index.



Bulk-boundary correspondence:

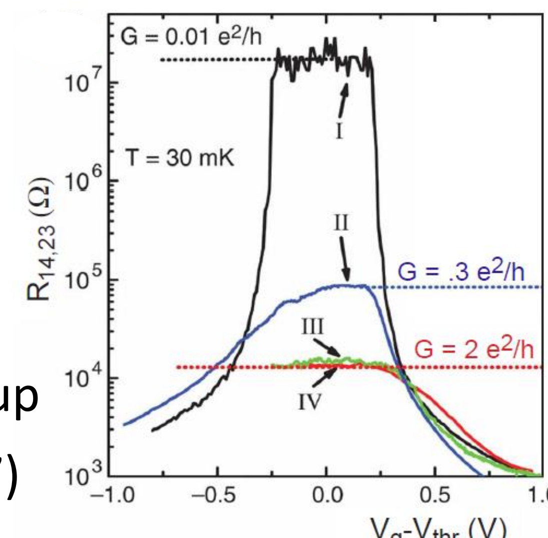
non-trivial bulk topological invariant

→ perfectly conducting surface mode



Molenkamp group

(Würzburg, 2007)



Majorana chain (gapped 1d superconductor)

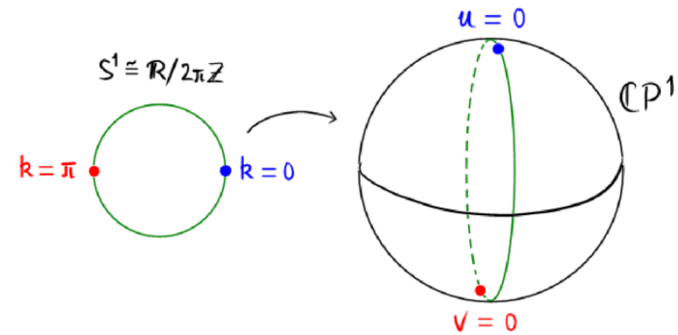
No symmetries, “spinless fermions”, single band; momentum k

Bogoliubov transformation: $\gamma_k = u(k)\, c_k + v(k)\, c^\dagger_{-k} \quad (k \in \mathbb{R}/2\pi\mathbb{Z})$

Fermi constraint: $\{\gamma_k, \gamma_{-k}\} = 0$

$\curvearrowright \quad u(k)\, v(-k) + v(k)\, u(-k) = 0\,.$

Weak pairing: $u(0) = v(\pi) = 0$ (topol. invt.)



Weak pairing: bulk-boundary correspondence → gapless edge state

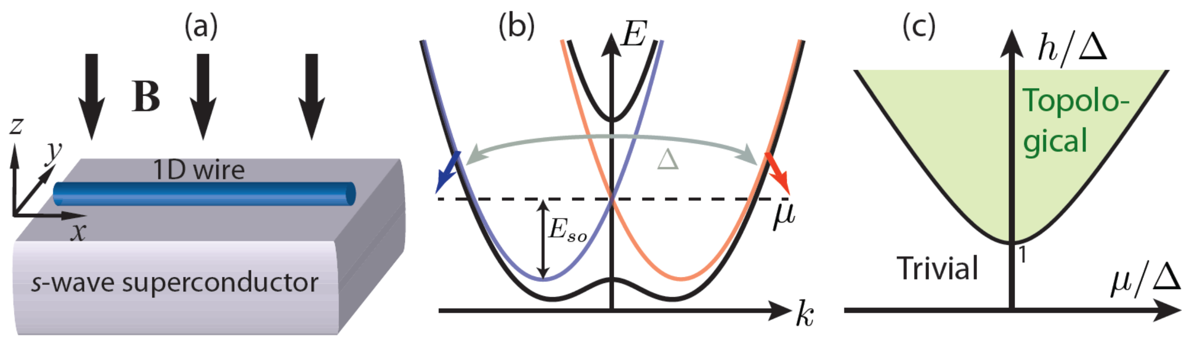


FIG. 6. (a) Basic architecture required to stabilize a topological superconducting state in a 1D spin-orbit-coupled wire. (b) Band structure for the wire when time-reversal symmetry is present (red and blue curves) and broken by a magnetic field (black curves). When the chemical potential lies within the field-induced gap at $k = 0$, the wire appears ‘spinless’. Incorporating the pairing induced by the proximate superconductor leads to the phase diagram in (c). The endpoints of topological (green) segments of the wire host localized, zero-energy Majorana

Review of Second Quantization: Free-Fermion Ground States as Complex Structures

Complex vector space V ket-vectors

Dual vector space V^* bra-vectors

Exterior algebra $\Lambda(V)$ fermionic Fock space

Exterior multiplication $V \ni v \mapsto \varepsilon(v) : \Lambda^q(V) \to \Lambda^{q+1}(V)$

$$\Psi \mapsto v \wedge \Psi$$

particle creation operators $|j\rangle \mapsto a_j^\dagger$

Inner product $V^* \ni \varphi \mapsto \iota(\varphi) : \Lambda^q(V) \to \Lambda^{q-1}(V)$

$$v_{j_1} \wedge \ldots \wedge v_{j_q} \mapsto \sum_{l=1}^{q} (\varphi \cdot v_{j_l}) \, (-1)^{l-1} \, v_{j_1} \wedge \ldots \wedge \hat{v}_{j_l} \wedge \ldots \wedge v_{j_q}$$

particle annihilation operators $\langle j| \mapsto a_j$

Space of (quantum) fields $W = V \oplus V^*$ Nambu space

Symmetric bilinear form (non-degenerate)

$$\{\cdot,\cdot\} : W \otimes W \to \mathbb{C}, \quad (v_1 \oplus \varphi_1) \otimes (v_2 \oplus \varphi_2) \mapsto (\varphi_1 \cdot v_2) + (\varphi_2 \cdot v_1)$$

anti-commutator bracket of fermion fields

Clifford algebra $Cl(W)$ (algebra of) polynomials in the fields

Representation of Clifford algebra $Cl(W)$ on $\Lambda(V)$:

$$\iota(\varphi)\,\varepsilon(v) + \varepsilon(v)\,\iota(\varphi) = (\varphi \cdot v)\, 1_{\Lambda(V)}$$

$$\varepsilon(v_1)\,\varepsilon(v_2) + \varepsilon(v_2)\,\varepsilon(v_1) = 0 = \iota(\varphi_1)\,\iota(\varphi_2) + \iota(\varphi_2)\,\iota(\varphi_1)$$

canonical anti-commutation relations (CAR)

Assume V Hilbert space.

$(\lambda \in \mathbb{C})$

Hilbert isomorphism $h : V \to V^*, \quad \lambda v \mapsto \bar{\lambda} \langle v, \cdot \rangle$

ket-to-bra correspondence $|j\rangle \mapsto \langle j|$

Real structure $\gamma : W \to W$ by $v \oplus \varphi \mapsto h^{-1}\varphi \oplus h v$

Hermitian conjugation of field ops.

Note: γ complex anti-linear and $\gamma^2 = 1_W$.

Remark. Hermitian scalar product induced on $\Lambda(V)$

$\rightsquigarrow \varepsilon(v)$ adjoint to $\iota(hv)$.

Subspace of real fields $W_{\mathbb{R}} = \mathrm{Fix}_W(\gamma)$ Majorana

CAR bracket restricts to $\{\cdot,\cdot\}: W_{\mathbb{R}} \otimes W_{\mathbb{R}} \to \mathbb{R}$.

Orthogonal group $O(W_{\mathbb{R}})$ Bogolinbov transformations

$U(V) \hookrightarrow O(W_{\mathbb{R}})$ by $W_{\mathbb{R}} \ni v \oplus hv \mapsto (gv) \oplus h(gv)$, $g \in U(V)$.

Fact / Definition. A free-fermion ground state is (the same as) a CAR-bracket preserving complex structure J on $W_{\mathbb{R}}$, i.e.

$$J \in \mathrm{End}(W_{\mathbb{R}}), \quad J^2 = -1, \quad \{J\psi_1, J\psi_2\} = \{\psi_1, \psi_2\} \qquad (\psi_1, \psi_2 \in W_{\mathbb{R}}).$$

Remarks. $J \rightsquigarrow$ polarization $W = E_{+i}(J) \oplus E_{-i}(J) \equiv A \oplus A^c$.

quasi-particle annihilation ops. creation ops.

$$\{A, A\} = 0 = \{A^c, A^c\}$$

$$\hat{J} = -\frac{i}{2} \sum_\alpha (\gamma_\alpha^\dagger \gamma_\alpha - \gamma_\alpha \gamma_\alpha^\dagger), \quad J \cdot \gamma_\alpha \equiv [\hat{J}, \gamma_\alpha] = i\gamma_\alpha$$

Remarks (cont'd).

iJ is a (spectrally) "flattened" Hamiltonian.

The space of CAR-preserving complex structures on $W_{\mathbb{R}}$ is a compact complex manifold, $\mathcal{C}_{CAR}(W_{\mathbb{R}})$.

Fact. $\mathcal{C}_{CAR}(W_{\mathbb{R}}) \cong O(W_{\mathbb{R}}) / \mathcal{U}(V)$.

Proof. The group $O(W_{\mathbb{R}})$ of Bogoliubov transformations acts transitively on ground states, i.e. $\mathcal{C}_{CAR}(W_{\mathbb{R}}) = O(W_{\mathbb{R}}) \cdot J_{vac}$.

The vacuum-J is left invariant by the unitaries: $\mathcal{U}(V) \cdot J_{vac} = J_{vac}$.

Remark. $\dim_{\mathbb{C}} \mathcal{C}_{CAR}(W_{\mathbb{R}}) = n(n-1)/2$, $\dim W_{\mathbb{R}} = 2n$.

$\mathcal{C}_{CAR}(W_{\mathbb{R}})$ has 2 connected components

ground states with even/odd fermion parity.

Example: $\mathcal{C}_{CAR}(\mathbb{R}^2) = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

Interlude. Similar story for bosons:

Replace $\Lambda(V) \longrightarrow S(V)$ symmetric algebra bosonic Fock space

$\varepsilon(v) \longrightarrow \mu(v)$ symm. multiplication boson creation ops.

$\iota(\varphi) \longrightarrow \delta(\varphi)$ derivation boson annihilation operators

$Cl(W) \longrightarrow \mathcal{W}(W)$ Weyl algebra.

CCR bracket $[\cdot,\cdot] : W \otimes W \to \mathbb{C}$,

$$(v_1 \oplus \varphi_1) \otimes (v_2 \oplus \varphi_2) \mapsto (\varphi_1 \cdot v_2) - (\varphi_2 \cdot v_1)$$

restricts to $\sqrt{-1} \times$ symplectic form on $W_{\mathbb{R}}$.

Fact. A free-boson ground state without BEC (i.e. $\langle \psi \rangle = 0$)

is a CCR-bracket preserving complex structure J on $W_{\mathbb{R}}$.

$\mathcal{C}_{CCR}(W_{\mathbb{R}}) \cong Sp(W_{\mathbb{R}}) / U(V)$ non-compact complex $n(n+1)/2$ manifold. ∎

Comment: non-linear sigma model.

NLσM for disordered superconductors with broken time-reversal and broken spin-rotation symmetry (class D) has for its target space a Riemannian symmetric superspace based on

$$(Sp/U) \times (O/U)$$

boson-boson fermion-fermion sector; see MZ, JMP (1996).

$\leftrightarrow$ complex structures preserving super-bracket on $\mathbb{R}^{2n|2n}$.

Summary. The space W of Fermi field operators comes with

— CAR bracket $\{\cdot,\cdot\}$ ($\mathbb{C}$-bilinear, symmetric)

— real structure $W_{\mathbb{R}} = \mathrm{Fix}_W(\gamma)$

Free-fermion ground states $\xleftrightarrow{1:1}$ $\mathcal{C}_{\mathrm{CAR}}(W_{\mathbb{R}})$.

Comment on Fermi projections as ground states.

Free-fermion ground states: symmetries

Symmetries in quantum mechanics

Q: What's a symmetry in quantum mechanics?

A: An operator $T : \mathscr{R}\psi_1 \mapsto \mathscr{R}\psi_2$ on Hilbert rays that preserves all transition probabilities: $\left|\langle T\mathscr{R}\psi_2 , T\mathscr{R}\psi_1\rangle\right|^2 = \left|\langle \mathscr{R}\psi_2 , \mathscr{R}\psi_1\rangle\right|^2.$

Wigner's Theorem: cf. D. Freed, arXiv: 1112.2133

A symmetry T in quantum mechanics can always be represented on Hilbert space by an operator $\hat{T}$ which is either unitary or anti-unitary.

$$\langle \hat{T}\psi_2 | \hat{T}\psi_1 \rangle = \overline{\langle \psi_2 | \psi_1 \rangle}$$

Eugene P. Wigner

Remark 1: The symmetries form a group, G.

Remark 2: Symmetries commute with the Hamiltonian ($\hat{T}H = H\hat{T}$).
Thus "chiral symmetry" ($\gamma_5 D \gamma_5 = -D$) is not a symmetry.

Recall $W = V \oplus V^*$, real subspace $W_{\mathbb{R}} = \mathrm{Fix}_W(\gamma)$, CAR bracket $\{\cdot,\cdot\}$.

Notation. Let $G(V)$ be the group of (anti-)unitary operators on V.

Let $\widehat{G}(W)$ be the group of complex (anti-)linear operators $\hat{g} : W \to W$ which stabilize $W_{\mathbb{R}}$ (i.e. $\hat{g}\, W_{\mathbb{R}} = W_{\mathbb{R}}$) and preserve the CAR bracket :

$$\{\hat{g}\psi_1, \hat{g}\psi_2\} = \overline{\{\psi_1, \psi_2\}}.$$

Lemma. $G(V) \hookrightarrow \widehat{G}(W)$ by Hilbert isomorphism:

$$\hat{g}(v \oplus \varphi) = (gv) \oplus (h g h^{-1} \varphi) \qquad \text{(proof as exercise)}.$$

Definition. A transformation $\hat{g} \in \widehat{G}(W)$ is called a *symmetry* of the free-fermion ground state J if it commutes with iJ.

Remark. $\hat{g}\, iJ\, \hat{g}^{-1} = iJ \iff \hat{g}\, E_{+i}(J) = E_{+i}(J)$.

Unitary symmetries may include

— $U(1)_Q$ symmetry (charge operator Q)

$$e^{i\theta Q} : \quad v \oplus \varphi \mapsto e^{i\theta} v \oplus e^{-i\theta} \varphi$$

— $SU(2) \equiv Spin(3)$ spin rotations generated by S_1, S_2, S_3

$$\hat{R} : \quad v \oplus \varphi \mapsto R v \oplus R^{-1T} \varphi$$

— Subgroups of the group of Euclidean motions:

translations, reflections, crystallographic groups

Basic anti-unitary operators

1. Time reversal

$T : V \to V, \quad T^2 = -1_V$ on fundamental fermions (Dirac)

$\wedge T : \Lambda^n(V) \to \Lambda^n(V)$

On field space $W = V \oplus V^*$: $\quad v \oplus \varphi \mapsto Tv \oplus hTh^{-1}\varphi$

2. Particle-hole conjugation

is defined on field space, $\quad C : W \to W, \quad C^2 = 1_W$

and lifts to an operation $C : \Lambda^{\mathrm{half}+q}(V) \to \Lambda^{\mathrm{half}-q}(V)$ on Fock space

or $C\,Q\,C^{-1} = -Q$ (Q charge operator, normal-ordered)

Recall T (C) symmetry of ground state $J \Leftrightarrow TA = A \quad (CA = A)$

for $A = E_{+i}(J)$

WARNING.

1. Particle physics

Time reversal anti-unitary and $E, B \overset{T}{\longmapsto} E, -B$ on gauge fields.

Charge conjugation C unitary on fermions but $E, B \overset{C}{\longmapsto} -E, -B$

Remark. Consistency condition $\boxed{C \circ e^{i\theta Q} = e^{i\theta Q} \circ C}$

2. Condensed matter physics

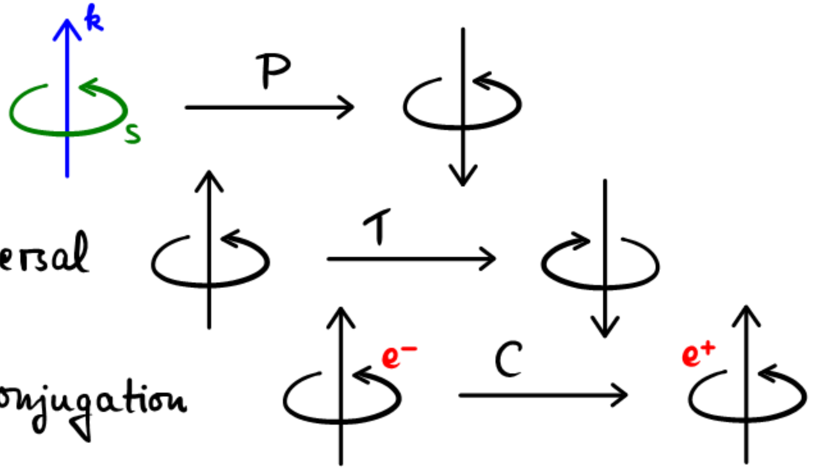
e.g. $H = \frac{1}{2m}(p - eA)^2$

Both T, C anti-unitary and $E, B \overset{T,C}{\longmapsto} E, B$ (background fields)

Remark. parity

time reversal

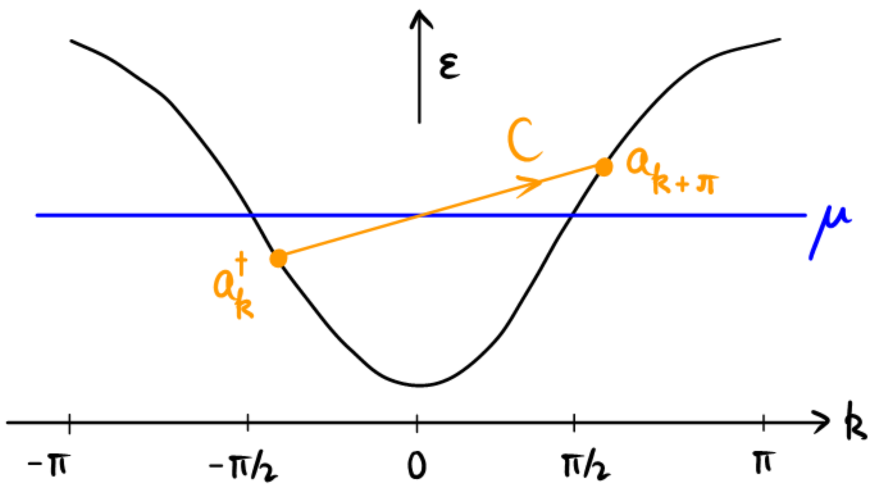
charge conjugation



Example: cosine band

$$H - \mu N = \sum_k \varepsilon(k)\, a_k^\dagger a_k \,, \qquad \varepsilon(k+\pi) = -\varepsilon(k),$$

$$\int \varepsilon(k)\, dk = 0 .$$



$$a_k^\dagger \xmapsto{\;C\;} a_{k+\pi}$$

e.g. $\varepsilon(k) = -\cos k$

Some further examples.

Hubbard model at half filling

($\rightsquigarrow$ antiferromagnets)

Half-filled lowest Landau level (D.T. Son)

The Tenfold Way
of free-fermion ground states

Note. Given a free-fermion ground state J, we can ask what is its group of symmetries $G_J \subset \widehat{G}(W)$. Conversely, we can fix a choice of symmetry group $G \subset \widehat{G}(W)$ and ask what is the set of symmetry-allowed ground states J.

Setting. G_0 unitaries, cosets, $G/G_0 = \mathbb{Z}_2^T \times \mathbb{Z}_2^C$

Symmetry group $G = G_0 \cup \underbrace{G_T \cup G_C \cup G_{CT}}_{\text{cosets}} \subset \widehat{G}(W)$, G_0 reductive.

Block decomposition $iJ \overset{\forall g \in G}{=} g\, iJ\, g^{-1} = \bigoplus_\lambda iJ_\lambda$

Theorem ("Tenfold Way"). All blocks J_λ that occur in the decomposition of J are of one of 10 types, and these are in one-to-one correspondence with the large families of classical symmetric spaces.

Proof. This is an easy corollary of a result by HHZ (CMP, 2005).

Tenfold Way

Communications in
Mathematical Physics

Symmetry Classes of Disordered Fermions

P. Heinzner[1], A. Huckleberry[1], M.R. Zirnbauer[2]

[1] Fakultät für Mathematik, Ruhr-Universität Bochum, Germany.
E-mail: heinzner@cplx.ruhr-uni-bochum.de; ahuck@cplx.ruhr-uni-bochum.de
[2] Institut für Theoretische Physik, Universität zu Köln, Germany.
E-mail: zirn@thp.uni-koeln.de

Received: 10 June 2004 / Accepted: 9 December 2004
Published online: ■■■ –

Abstract: Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

The primary data of the classification are a Nambu space of fermionic field operators which carry a representation of some symmetry group. Our approach is to eliminate all

In this paper, it is proved that the symmetry classes of disordered fermions are in one-to-one correspondence with the 10 large families of symmetric spaces.

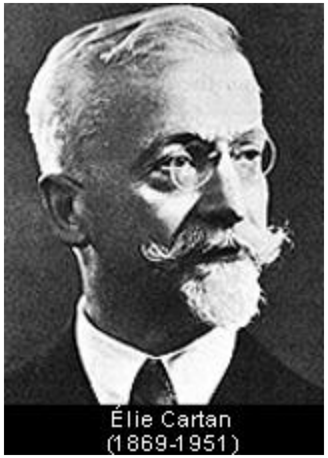
What's a symmetric space?

Riemann tensor: $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm}$

Def.: A (locally) symmetric space is a Riemannian manifold $X = U/K$ with covariantly constant curvature: $\nabla R = 0$.

Ex. 1: the round two-sphere $X = \mathrm{S}^2$, $\quad ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

Ex. 2: the set $X = \mathrm{Gr}_n(\mathbb{C}^N) = \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$ of all subspaces $\mathbb{C}^n \simeq V \subset \mathbb{C}^N$



Classification:

Globally symmetric spaces classified by E. Cartan (1926)

10 large families: *A*, *A*I, *A*II, *A*III, *BD*, *BD*I, *C*, *C*I, *C*II, *D*III

U, U/O, U/Sp, U/U×U, O, O/O×O, Sp, Sp/U, Sp/Sp×Sp, O/U

Tenfold Way: basic realizations ($T^2 = -1$, $C^2 = +1$)

J	G_0	T	C	comment
O/U	$\{e\}$	no	no	superconductor
U/Sp	$\{e\}$	yes	no	superconductor
$Sp/Sp\times Sp$	$U(1)_Q$	yes	no	Wigner-Dyson (GSE)
Sp	$U(1)_Q$	yes	yes	"chiral"
$U/U\times U$	$U(1)_Q$	no	no	Wigner-Dyson (GUE)
U	$U(1)_Q$	no	yes	"chiral"
Sp/U	$SU(2)_{spin}$	no	no	superconductor
U/O	$SU(2)_{spin}$	yes	no	superconductor
$O/O\times O$	$SU(2)_{spin}\times U(1)_Q$	yes	no	Wigner-Dyson (GOE)
O	$SU(2)_{spin}\times U(1)_Q$	yes	yes	"chiral"

cf. MZ, "Symmetry Classes" arXiv: 1001.0722

Tenfold Way: basic realizations ($T^2 = -1$, $C^2 = +1$)

	J	G_0	T	C	comment
O	O/U	$\{e\}$	no	no	superconductor
			no	yes	
$CT = +TC$: U	U/Sp	$\{e\}$	yes	no	superconductor
			yes	yes	
$CT = -TC$: O/U	$Sp/Sp\times Sp$	$U(1)_Q$	yes	no	Wigner-Dyson (GSE)
	Sp	$U(1)_Q$	yes	yes	"chiral"
	$U/U\times U$	$U(1)_Q$	no	no	Wigner-Dyson (GUE)
	U	$U(1)_Q$	no	yes	"chiral"
	Sp/U	$SU(2)_{spin}$	no	no	superconductor
	U/O	$SU(2)_{spin}$	yes	no	superconductor
	$O/O\times O$	$SU(2)_{spin} \times U(1)_Q$	yes	no	Wigner-Dyson (GOE)
	O	$SU(2)_{spin} \times U(1)_Q$	yes	yes	"chiral"

Idea of proof.

G_0 reductive by assumption. Hence $W = \bigoplus_\lambda W_\lambda$ ← isotypic component (irreps of type λ)

Block form of complex structure:

$$J = g J g^{-1} = \begin{pmatrix} \ast & & & & \\ & \ast & & 0 & \\ & & J_\lambda & & \\ & 0 & & \ast & \\ & & & & \ddots \end{pmatrix} \begin{matrix} \\ \\ \}\, W_\lambda \\ \\ \\ \end{matrix}$$

(for all $g \in G_0$)

Reduction by G_0 (non-Abelian case):

$$W_\lambda = R_\lambda \otimes M_\lambda\,, \quad M_\lambda = \mathrm{Hom}_{G_0}(R_\lambda, W_\lambda) \cong \mathbb{C}^{m_\lambda}$$

standard irrep   multiplicity

Transfer $T, C, \{\cdot,\cdot\}$ from W_λ to M_λ

Key observation: $T, C, \{\cdot,\cdot\}$ may only change their parity

Example. Reduction by fund. rep. of $SU(2)$ causes $\{\cdot,\cdot\} \longrightarrow [\cdot,\cdot]$

$$T^2 = -1 \longrightarrow T^2 = +1$$

Special case $G_0 = \{\text{translations}\}$.

Momentum k conserved $\rightsquigarrow$ $W = \oplus_k W_k$, $\quad W_k = V_{-k} \oplus V_k^*$

CAR structure $\rightsquigarrow$ pairing $\{\cdot,\cdot\} : W_{-k} \otimes W_k \to \mathbb{C}$ (non-degen.)

Real structure $W_k \xrightarrow{\gamma} W_{-k}$ (Hermitian conjugation)

Hermitian scalar product on W_k: $\langle \psi_1, \psi_2 \rangle = \{\gamma\psi_1, \psi_2\}$

Conditions on $J = \oplus_k J_k$:

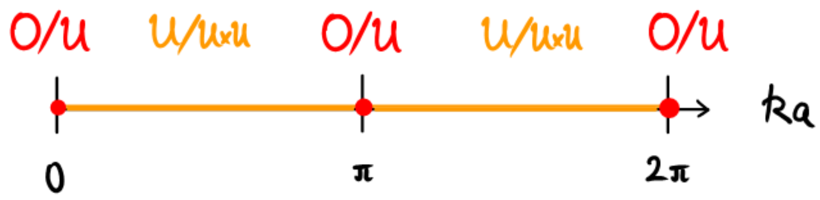
$2n = \dim W_k$

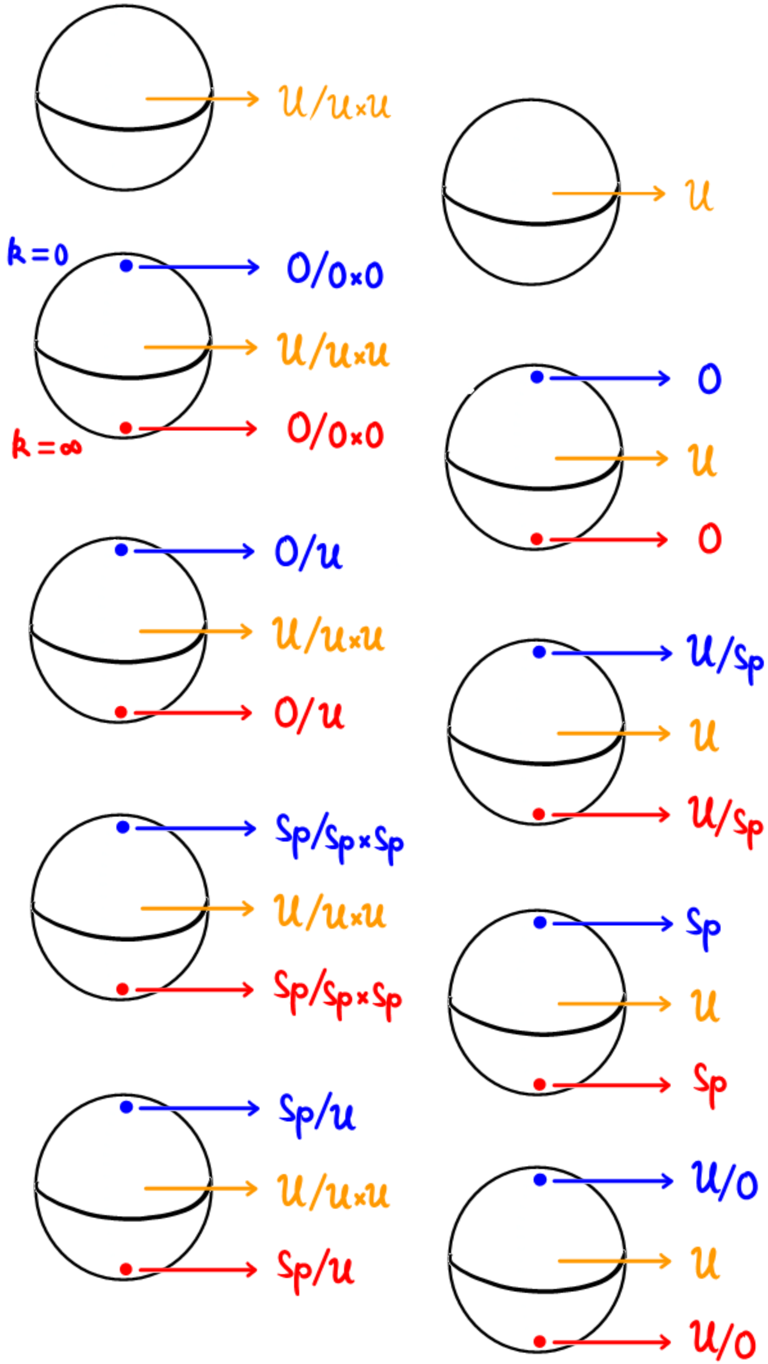
$J_k^T = J_{-k}^{-1}$ $\rightsquigarrow$ if $k = -k \equiv k_0$ then $J_{k_0} \in O(2n)/U(n)$

$J_k^2 = -1$ and $J_k = -J_k^*$ $\rightsquigarrow$ $J_k \in U(2n)/U(n) \times U(n)$

$$\langle J_k\psi_1, \psi_2\rangle_{W_k} = \{\gamma J_k \psi_1, \psi_2\} = \{J_{-k}\gamma\psi_1, \psi_2\} = \{\gamma\psi_1, J_k^{-1}\psi_2\} = -\langle \psi_1, J_k\psi_2\rangle_{W_k}$$

Example: Majorana chain.





10 "atoms" make ...

Symmetry				d					
AZ	Θ	Ξ	Π	1	2	3	4	5	6
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

... 10 simple "molecules"

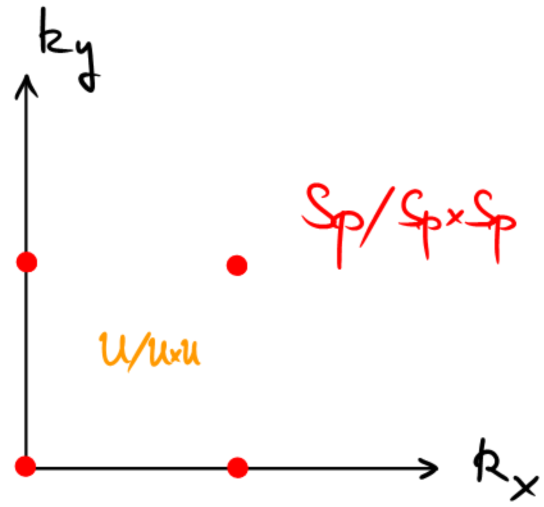
Example: quantum spin Hall insulator (with translation inv.)

Charge Q conserved

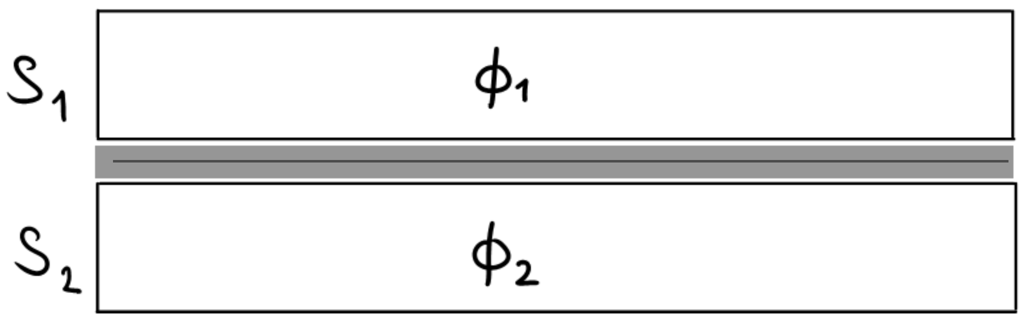
$$\rightsquigarrow \quad J_k \in U(n)/U(p)\times U(q)$$
$$p+q=n$$

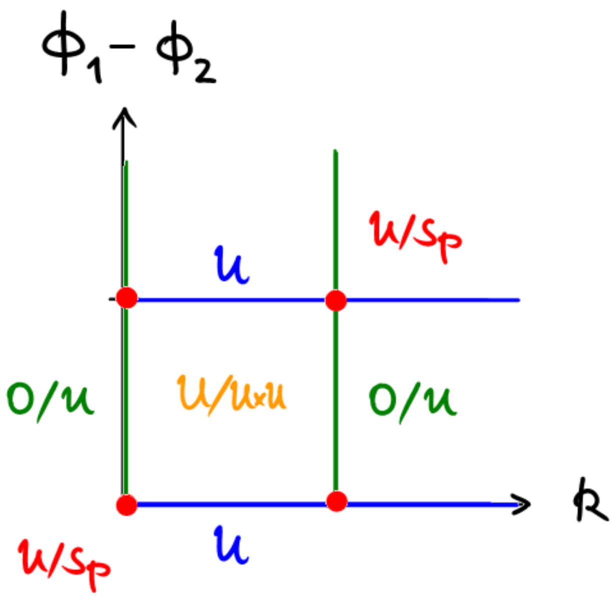
Time-reversal symmetry

$$\rightsquigarrow \quad J_{k=-k} \in Sp(n)/Sp(p)\times Sp(q)$$



Example: anomalous topological pump (Zhang & Kane 2014)





$S_{1,2}$ T-invariant superconductors with spin-orbit scattering

Recapitulate

- Insulator = system with energy gap for excitations (carrying charge/spin/energy)

- Free-fermion ground state = complex structure J on $W_{\mathbb{R}}$ preserving CAR

  = polarization $W = A \oplus A^c$;

  reduces to Fermi projection for charge-conserving systems.

- Tenfold Way: in the setting of HHZ (CMP, 2005)

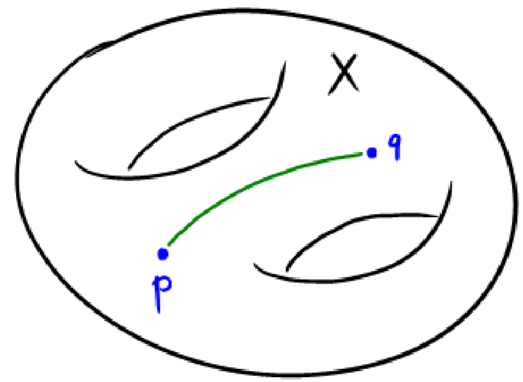
  every block in the decomposition $J = \bigoplus_{\lambda} J_\lambda$ by isotypic components is of one of 10 known types.

  Applies to topological crystalline insulators,
  statistical topological insulators, ...

- "Altland-Zirnbauer" classification $\neq$ Tenfold Way

Bott Periodicity

Bott Periodicity (I)



X compact Riemannian manifold

ν triple $(p, q; h)$

X^ν space of minimal geodesics of class ν

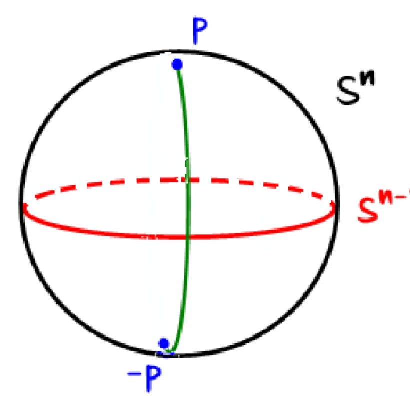
$|\nu|$ minimum over all non-minimal geodesics of class ν of no. of negative eigenvalues of Hessian of length function

Thm (Bott, 1959). If X is a symmetric space, then so is X^ν, and $\pi_d(X^\nu) = \pi_{d+1}(X) \quad \text{for} \quad 0 < d < |\nu| - 1\,.$

Example. $X = S^n, \quad X^{(p,-p)} = S^{n-1}, \quad |\nu| = 2(n-1)$

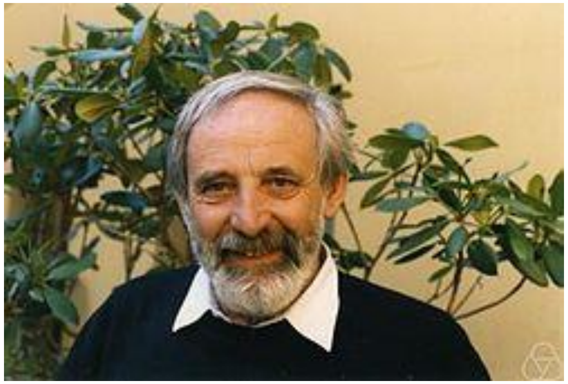
$\Longrightarrow \pi_d(S^{n-1}) = \pi_{d+1}(S^n) \quad \text{for} \quad 0 < d < 2n-3$

(Freudenthal suspension theorem).



Bott Periodicity (II)

ν–sequences: complex and real



Raoul Bott (1923-2005)

		π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
C_{even}	$\mathrm{U}/\mathrm{U}\times\mathrm{U}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C_{odd}	U	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R_0	O/U	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R_1	U/Sp	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
R_2	$\mathrm{Sp}/\mathrm{Sp}\times\mathrm{Sp}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
R_3	Sp	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
R_4	Sp/U	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
R_5	U/O	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
R_6	$\mathrm{O}/\mathrm{O}\times\mathrm{O}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R_7	O	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1. Bott Periodic Table of the stable homotopy groups of symmetric spaces

Bott Periodicity (III): Morse Theory

Manifold M, function $f: M \to \mathbb{R}$ (smooth and proper).

$$M^a := \{x \in M \mid f(x) \le a\}.$$

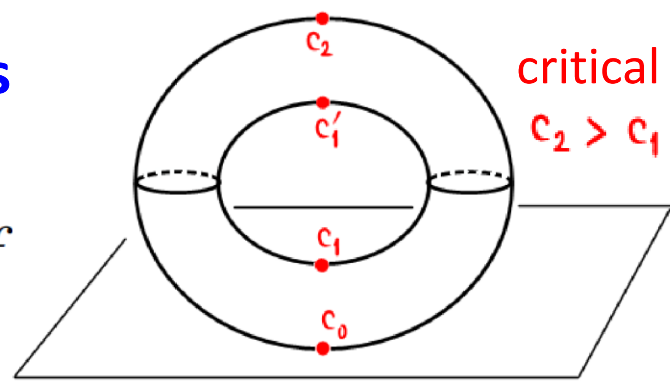
homotopy-equivalent

Thm 1. If f has no critical values in $[a,b]$, then $M^a \sim M^b$.

Thm 2. Let $x \in M$ be a non-degenerate critical point of f of index n. If x is the only critical point in $f^{-1}[f(x)-\varepsilon, f(x)+\varepsilon]$, then $M^{f(x)+\varepsilon} \sim M^{f(x)-\varepsilon} \cup e_n$ (n-cell e_n).

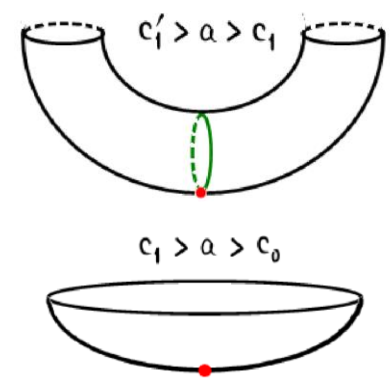
Example: 2-torus

height function f



critical values

$c_2 > c_1 > c_1' > c_0$



Fact. Functions f of the needed kind (Morse functions) do exist.

Bott Periodicity (IV): Idea of proof

$M = \Omega^\nu X$ (space of paths in X from p to q of homology class h),
Morse function $f \equiv$ length of path.

$M^a = \emptyset$ if $a < L \equiv$ geodesic distance from p to q,

$M^a = X^\nu$ if $L < a < L + \varepsilon$,

$\vdots$

$\Omega^\nu X = X^\nu \cup e_n \cup e'_n \cup \ldots$ where $\dim e_n \geq |\nu|$.

Hence $\pi_{d+1}(X) = \pi_d(\Omega^\nu X) = \pi_d(X^\nu)$ if $0 < d < |\nu| - 1$.

Bott Periodicity (V): Clifford algebra

$W \cong \mathbb{C}^{2n}$ Hermitian vector space with compatible symmetric bilinear form $\{\cdot\,,\cdot\}: W \otimes W \to \mathbb{C}$.

$J_1, \ldots, J_s$ generators of Clifford algebra on W:

$$J_l J_m + J_m J_l = -2\delta_{lm}\mathrm{Id}_W, \quad (J_l^* = J_l^{-1} = J_l^T).$$

Note: $W = E_{+\mathrm{i}}(J) \oplus E_{-\mathrm{i}}(J)$.

Let $C_0(n) := \mathrm{Gr}_n(W)$ Grassmann m'fld of complex n-planes in W.
Define

$$C_s(n) := \{A \in C_0(n) \mid J_1 A = \ldots = J_s A = A^{\mathrm{c}}\},$$
$$R_s(n) := \{A \in C_s(n) \mid \{A, A\} = 0\}.$$

Lemma.

$C_0(n) \supset C_1(n) \supset C_2(n) \supset \ldots$ complex Bott $\nu-$ sequence,

$R_0(n) \supset R_1(n) \supset R_2(n) \supset \ldots$ real Bott $\nu-$ sequence.

More precisely,

s	$C_s(8r)$	$R_s(8r)$
0	$\cup_{p+q=16r} \mathrm{U}_{16r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\mathrm{O}_{16r}/\mathrm{U}_{8r}$
1	$(\mathrm{U}_{8r} \times \mathrm{U}_{8r})/\mathrm{U}_{8r}$	$\mathrm{U}_{8r}/\mathrm{Sp}_{8r}$
2	$\cup_{p+q=8r} \mathrm{U}_{8r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\cup_{p+q=4r} \mathrm{Sp}_{8r}/(\mathrm{Sp}_{2p} \times \mathrm{Sp}_{2q})$
3	$(\mathrm{U}_{4r} \times \mathrm{U}_{4r})/\mathrm{U}_{4r}$	$(\mathrm{Sp}_{4r} \times \mathrm{Sp}_{4r})/\mathrm{Sp}_{4r}$
4	$\cup_{p+q=4r} \mathrm{U}_{4r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\mathrm{Sp}_{4r}/\mathrm{U}_{2r}$
5	$(\mathrm{U}_{2r} \times \mathrm{U}_{2r})/\mathrm{U}_{2r}$	$\mathrm{U}_{2r}/\mathrm{O}_{2r}$
6	$\cup_{p+q=2r} \mathrm{U}_{2r}/(\mathrm{U}_p \times \mathrm{U}_q)$	$\cup_{p+q=2r} \mathrm{O}_{2r}/(\mathrm{O}_p \times \mathrm{O}_q)$
7	$(\mathrm{U}_r \times \mathrm{U}_r)/\mathrm{U}_r$	$(\mathrm{O}_r \times \mathrm{O}_r)/\mathrm{O}_r$

Sketch of proof for $R_1(n)$. J. Milnor, "Morse Theory" (Princeton, 1963)

$$R_1(n) = \{ A \mid \{A, A\} = 0 \, ; \, A \overset{J_1}{\leftrightarrow} A^c \}.$$

Let $A_{(0)} = \mathrm{span}_{\mathbb{C}} \{ c_\alpha \}$, $A^c_{(0)} = \mathrm{span}_{\mathbb{C}} \{ c^\dagger_\alpha \}$.

Then $R_1(n) = \mathcal{U} \cdot A_{(0)}$ where $\mathcal{U} = \left(\mathcal{U}(E_{+i}(J_1)) \hookrightarrow O(\mathbb{R}^{2n}) \right)$.

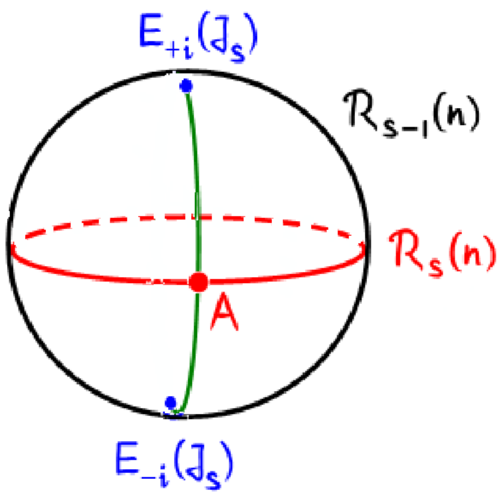
$A_{(0)}, J_1$ determine "quaternion" structure, invariant under $Sp \subset \mathcal{U}$.

Bott Periodicity (VI): Bott Map

Make the identifications $X \equiv R_{s-1}(n)$, $X^\nu \equiv R_s(n)$;
and $\Omega^\nu X \equiv$ paths from $E_{+\mathrm{i}}(J_s)$ to $E_{-\mathrm{i}}(J_s)$.

Given $A \in R_s(n)$,
assign to A a minimal geodesic $\beta : [-\pi/2, \pi/2] \to R_{s-1}(n)$
by $t \mapsto \beta_t(A) = \mathrm{e}^{(t/2)J_s J(A)} \cdot A, \qquad J(A) = \mathrm{i}(\Pi_A - \Pi_{A^c})$.

This is the Bott Map ...



Comment. Same for $C_s(n)$ instead of $R_s(n)$.

Bott Periodicity (II')

ν–sequences: complex and real

		π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
C_{even}	$\mathrm{U}/\mathrm{U}\times\mathrm{U}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
C_{odd}	U	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R_0	O/U	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
R_1	U/Sp	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
R_2	$\mathrm{Sp}/\mathrm{Sp}\times\mathrm{Sp}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
R_3	Sp	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0
R_4	Sp/U	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
R_5	U/O	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
R_6	$\mathrm{O}/\mathrm{O}\times\mathrm{O}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R_7	O	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Bott Map

Table 1. Bott Periodic Table of the stable homotopy groups of symmetric spaces

Kitaev Sequence

From symmetries to pseudo-symmetries.

Example 1. Time-reversal symmetry for spin 1/2 :

$$TA = A, \quad T^2 = -1, \quad T \text{ complex anti-linear.}$$

Consider $J_1 := \gamma \circ T$.

J_1 unitary, orthogonal, $J_1^2 = T^2 = -1$,

$J_1 A = \gamma(TA) = \gamma A = A^c$.

Def. A free-fermion ground state of symmetry class s is a polarization $W = A \oplus A^c$ where the complex vector space A is subject to

- Fermi constraint : $\{A, A\} = 0$
- pseudo-symmetries : $J_1 A = \ldots = J_s A = A^c$.

(Clifford algebra : $J_l J_m + J_m J_l = -2\delta_{lm} \mathrm{Id}_W$, $(J_l^* = J_l^{-1} = J_l^T)$.)

Example 2. Time-reversal symmetry and charge conservation

$$TA = A = QA, \quad Q = +1_V - 1_{V^*}.$$

Consider $J_2 := iQJ_1 = i\gamma TQ$.

J_2 unitary, orthogonal,

$J_2^2 = -(QJ_1)^2 = Q^2 J_1^2 = -1$,

$J_2 A = iQA^c = A^c$,

$J_1 J_2 + J_2 J_1 = 0$ since J_1 and Q anti-commute.

Q (alone) kills the Fermi constraint.

$QA = A \rightsquigarrow A = V_h \oplus V_p^*$, $A^c = V_p \oplus V_h^*$

$\rightsquigarrow \{A, A\} = 0$ is automatically satisfied.

Without loss one can work with Fermi projection on V_h.

Kitaev Sequence ("real" and "complex" classes)

class	symmetries	s	pseudo-syms
D	none	0	**Fermi constraint**
DIII	T (time reversal)	1	$J_1 = \gamma T$
AII	T, Q (charge)	2	$J_2 = \mathrm{i}\gamma T Q$
CII	T, Q, C (ph-conj)	3	$J_3 = \mathrm{i}\gamma C Q$
C	S_1, S_2, S_3 (spin rot)	4	see below
CI	S_1, S_2, S_3, T	5	
AI	S_1, S_2, S_3, T, Q	6	
BDI	S_1, S_2, S_3, T, Q, C	7	

class	symmetries	s	pseudo-syms
A	Q	0	**Fermi constraint**
AIII	Q, C	1	$J_1 = \mathrm{i}\gamma C$

Q: Why do 3 spin generators amount to 4 pseudo-symmetries?

Tool: (1,1) periodicity.

$$C_s(n) := \{A \in \mathrm{Gr}_n(\mathbb{C}^{2n}) \mid J_1 A = \ldots = J_s A = A^{\mathrm{c}}\}$$
$$R_s(n) := \{A \in C_s(n) \mid \{A, A\} = 0\}$$

Double the dimension ($\mathbb{C}^{2n}$ to $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$) and let

$$I = \begin{pmatrix} 0 & \mathbf{1}_{2n} \\ -\mathbf{1}_{2n} & 0 \end{pmatrix}, \quad K = \mathrm{i} \begin{pmatrix} \mathbf{1}_{2n} & 0 \\ 0 & -\mathbf{1}_{2n} \end{pmatrix}, \quad \widetilde{J}_l = \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \quad (l = 1, \ldots, s)$$

Note: K is "imaginary": $\{Kw, Kw'\} = -\{w, w'\}$

Lemma. $C_s(n) \simeq C_{s+2}(2n), \quad R_s(n) \simeq R_{s+1,1}(2n)$

Proof.
$$A \mapsto \widetilde{A} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes a + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes a' \;\middle|\; a \in A,\, a' \in A^{\mathrm{c}} \right\}$$

Corollary. Let (note $K = \mathrm{i}\widetilde{J}_1\widetilde{J}_2\widetilde{J}_3$)

$$\widetilde{J}_l := \begin{pmatrix} \mathrm{i}S_l & 0 \\ 0 & -\mathrm{i}S_l \end{pmatrix} \quad (l \leq 3), \quad \widetilde{J}_4 := I, \quad \widetilde{J}_l := \begin{pmatrix} 0 & J_l \\ J_l & 0 \end{pmatrix} \quad (l \geq 5)$$

Then (1,1) periodicity isomorphism $\curvearrowright s$ pseudo-symmetries equivalent to $s-4$ pseudo-syms. plus 3 spin rotation symmetries.

Diagonal Map

Bott-Kitaev Periodic Table

Question: does there exist a ``Diagonal Map''?

from Hasan & Kane, Rev. Mod. Phys. (2011):

	Symmetry				d							
	AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
$U/U\times U$	A	0	0	0	0	$\boxed{\mathbb{Z}}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
U	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$O/O\times O$	AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
O	BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
O/U	D	0	1	0	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
U/Sp	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\boxed{\mathbb{Z}}$	0	0	0	$\mathbb{Z}$	0
$Sp/Sp\times Sp$	AII	-1	0	0	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
Sp	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
Sp/U	C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
U/O	CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

?

Quantum Hall Effect

He-3 (B phase)

QSHI: HgTe

Majorana

Bi_2 Se_3

?

Diagonal Map: heuristic

Clean limit: translations are symmetries

Conserved momentum $k \in M \curvearrowright$ decomposition $A = \bigoplus_{k\in M} A_k$

Gapped system (insulator) $\curvearrowright$ $\{A_k\}_{k\in M}$ vector bundle

Under $A = \bigoplus_{k\in M} A_k$ the Fermi constraint $\{A, A\} = 0$ refines to

$$\{A_k, A_{-k}\} = 0 \quad \text{(for all } k \in M\text{).}$$

Thus our free-fermion ground states are vector bundles $\pi : \mathscr{A} \to M$ subject to a $\mathbb{Z}_2$-equivariance condition

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\tau_{\mathscr{A}}"] \arrow[d, "\pi"'] & \mathscr{A} \arrow[d, "\pi"] \\ M \arrow[r, "\tau_M"] & M \end{tikzcd}$$

with non-trivial involution $\tau_M : M \to M, \quad k \mapsto -k.$

Diagonal Map $(d,s) \to (d+1,s+1)$

Starting point: $\widetilde{J}_1,\ldots,\widetilde{J}_s$ and $\{\widetilde{A}_k\}_{k\in M_d}$

Preparatory step: jack up by (1,1) periodicity

$\curvearrowright$ New starting point: $J_1\,,\ldots,J_s\,;I,K$ and $\{A_k\}_{k\in M_d}$

Define: $\boxed{A_{k,t} := \mathrm{e}^{(t/2)KJ(A_k)}\cdot A_k}$ $\qquad J(A) = \mathrm{i}(\Pi_A - \Pi_{A^{\mathrm{c}}})$

Note:

1. Fermi constraint: $\{A_{k,t}\,,A_{-k,-t}\} = 0$ ✓
2. Pseudo-syms: $J_1 A_{k,t} = \ldots = J_s A_{k,t} = A^{\mathrm{c}}_{k,t} = I A_{k,t}$ ✓
3. Degeneration: $A_{k,t=\pm\pi/2} = E_{\mp\mathrm{i}}(K)$ ✓

Outcome: V.B. $\{A_{k,t}\}_{(k,t)\in M_{d+1}}$ in class $s+1$ on $M_{d+1} = \widetilde{S}(M_d)$

D (none) **DIII (*T*)**

Example 1. $(d,s)=(0,0)$ to $(d,s)=(1,1)$.

$$n=1:\ R_0(1)=\{\mathbb{C}\cdot c,\,\mathbb{C}\cdot c^\dagger\}$$

(1,1) doubling $\curvearrowright$ tensor with $(\mathbb{C}^2)_{\rm spin}$ and let $K=\mathrm{i}(\sigma_1)_{\rm BdG}\otimes(\sigma_1)_{\rm spin}$,

$I\equiv J_1=\gamma T=(\sigma_1)_{\rm BdG}\otimes(\mathrm{i}\sigma_2)_{\rm spin}$, $A=\mathrm{span}_{\mathbb{C}}\{c_\uparrow^\dagger,c_\downarrow^\dagger\}\cong|\mathrm{full}\rangle$.

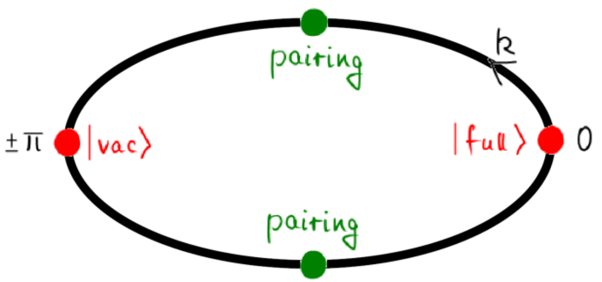
$$A_k=\mathrm{e}^{(k/2)KJ(A)}\cdot A=\mathrm{span}_{\mathbb{C}}\left\{c_\sigma^\dagger(-k)\cos(k/2)-c_{-\sigma}(k)\sin(k/2)\right\}_{\sigma=\uparrow,\downarrow}$$

In BCS form: $|\mathrm{g.s.}\rangle=\mathrm{e}^{\sum_k\cot(k/2)P_k}|\mathrm{vac}\rangle$ where $P_k=c_\uparrow^\dagger(k)c_\downarrow^\dagger(-k)$.

For more general $K=K(\alpha)$:

$$P_k=c_\uparrow^\dagger(k)c_\downarrow^\dagger(-k)\,\cos\alpha+\left(c_\uparrow^\dagger(k)c_\uparrow^\dagger(-k)-c_\downarrow^\dagger(k)c_\downarrow^\dagger(-k)\right)\,\sin\alpha$$

Topological 1d superconductor with spin-triplet pairing and *T*-invariance



Example 2. $(d,s) = (1,1)$ to $(d,s) = (2,2)$ — *DIII (T)* to *AII (T, Q)*

(1,1) doubling $\curvearrowright$ tensor with band space $(\mathbb{C}^2)_{\mathrm{ph}}$

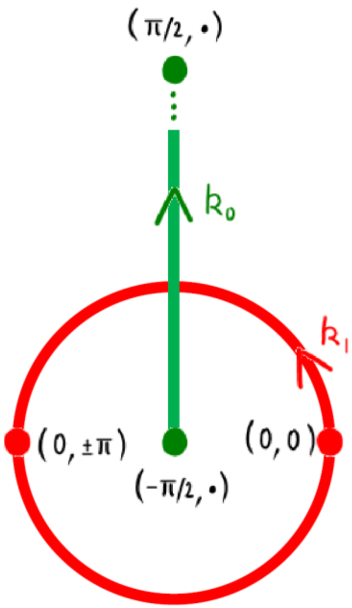
Topological 1d class-$DIII$ superconductor (after ph-transformation):

$$A_{-k_1} = \mathrm{span}_{\mathbb{C}} \left\{ c^\dagger_{h\uparrow}(k_1)\cos(k_1/2) + \mathrm{i} c^\dagger_{p\downarrow}(k_1)\sin(k_1/2);\ \&\uparrow\leftrightarrow\downarrow, \mathrm{i}\to -\mathrm{i} \right\}$$

Apply 1-par. group: $A_{-k} = \mathrm{e}^{-(k_0/2)KJ(A_{-k_1})} \cdot A_{-k_1} = \mathrm{span}_{\mathbb{C}} \Big\{ \&\uparrow\leftrightarrow\downarrow, \mathrm{i}\to -\mathrm{i};$

$$\left(c^\dagger_{h\uparrow}(k)\cos(k_1/2) + \mathrm{i} c^\dagger_{p\downarrow}(k)\sin(k_1/2) \right)\cos(k_0/2)$$

$$+ \left(c^\dagger_{p\downarrow}(k)\cos(k_1/2) + \mathrm{i} c^\dagger_{h\uparrow}(k)\sin(k_1/2) \right)\sin(k_0/2) \Big\}$$



Note: $A_{\pm\pi/2,k_1} = \mathrm{span}_{\mathbb{C}} \left\{ c^\dagger_{h\uparrow} \pm c^\dagger_{p\downarrow},\ c^\dagger_{h\downarrow} \pm c^\dagger_{p\uparrow} \right\}$

$\curvearrowright$ $k_0 = \pm\pi/2$ are isolated zeros of Kane-Mele Pfaffian

$\curvearrowright$ Kane-Mele invariant non-trivial (QSHI)

Next SPT phases in line: 3d insulator (*C*II), 4d superconductor (*C*), etc.

Disordered Systems:

Non-commutative Geometry Approach

Non-commutative Geometry.

A. Connes: spectral triple $(\mathcal{A}, \mathcal{H}, D)$

Integer Quantum Hall Effect.

NCG approach developed by J. Bellissard et al. (following Connes).

C*-algebra $\mathcal{A}$ of bounded operators.

Pairing between $K_0(\mathcal{A})$ and cyclic cohomology.

$\rightsquigarrow$ Hall conductance = non-commutative Chern number.

Prodan, Schulz-Baldes (2013), Thiang (2014), Kellendonk;
Boersema, Loring (2015), Carey et al. (2016) ...

Topological Invariants – a classification tool

1. Topological invariants of a manifold M

- DeRham cohomology

  $H^q(M) = \text{closed } q\text{-forms on } M \big/ \text{exact } q\text{-forms on } M$

- Homotopy groups

  $\pi_q(M) = \text{set (group) of homotopy classes of maps } S^q \longrightarrow M$

Submanifold $N \subset M$

- Relative cohomology, relative homotopy

- Invariants in low/special dimensions ...

2. Invariants for vector bundles $E \xrightarrow{\pi} M = \bigcup_\alpha \mathcal{U}_\alpha$, $E\big|_{\mathcal{U}_\alpha \subset M} = \mathcal{U}_\alpha \times \mathbb{C}^n$

- Homotopy classes

Consider sub-bundles $E \subset M \times \mathbb{C}^N$

Associate with $E \xrightarrow{\pi} M$ the map $\psi_E : M \to Gr_n(\mathbb{C}^N)$

$$x \mapsto \pi^{-1}(x) = E_x$$

Equivalence relation: $E \sim E'$ iff $\psi_E \sim \psi_{E'}$ (via homotopy)

- Isomorphism classes

Local trivialization $\phi_\alpha : E\big|_{\mathcal{U}_\alpha \subset M} \longrightarrow \mathcal{U}_\alpha \times \mathbb{C}^n$

Transition functions $g_{\alpha\beta}^{(E)} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathcal{U}(n) \subset GL(\mathbb{C}^n)$

Equivalence relation $E \sim E'$ iff $g_{\alpha\beta}^{(E)} = \lambda_\alpha \, g_{\alpha\beta}^{(E')} \lambda_\beta^{-1}$ with $\lambda_\alpha : \mathcal{U}_\alpha \to \mathcal{U}(n)$.

- K-theory classes ($\longrightarrow$ disorder)

Grothendieck Functor

turns an Abelian monoid $\mathcal{M}$ (= commutative semigroup with neutral element)
into an Abelian group K (the Grothendieck group of $\mathcal{M}$).

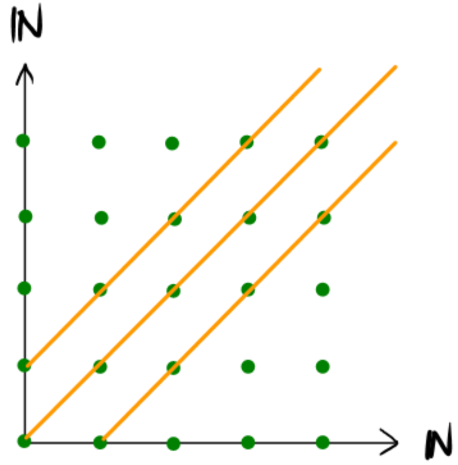
Example. $\mathcal{M} = \mathbb{N}$ (natural numbers) $\rightsquigarrow$ $K = \mathbb{Z}$ (integers)

Pictorially,

$$i(n) = (n, 0)$$

$$\mathbb{N} \xrightarrow{i} \mathbb{N} \times \mathbb{N} \xrightarrow{j} \mathbb{Z}$$

$$j(m, n) = m - n$$



In general: $\mathcal{M} \times \mathcal{M} \ni (m_1, m_2) \sim (n_1, n_2)$ iff $m_1 + n_2 + k = m_2 + n_1 + k$
(for some $k \in \mathcal{M}$)

Topological K-theory.

$\mathbb{K}$-vector bundles $E \longrightarrow M$ $(\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H})$.

Direct sum $E \oplus F$ of vector bundles (Whitney).

Isomorphism classes: $[E] \oplus [F] = [E \oplus F]$.

$\mathrm{Vect}(M)$ = Abelian monoid of isomorphism classes of vector bundles over M.

Topological K-theory:

$K(M)$ = the Grothendieck group of the Abelian monoid $\mathrm{Vect}(M)$.

Ring structure by tensor product $E \otimes F$ of vector bundles

Idea of the NCG Approach

<u>Clean system</u>		<u>Disordered system</u>
Fourier/Bloch theory: $H(k)$	Hamiltonians	C^*-algebra $\mathcal{A}$ (bounded ops.)
$\downarrow$		$\downarrow$
Vector bundle $E \longrightarrow M$	ground states	(Fermi) projection P
$\downarrow$		$\downarrow$
Isomorphism class $[E]$	topological phases	K-theory class $[P]$

NCG setting for statistical topological insulators

Statistical ensemble of Hamiltonians with symmetry G (translations, reflections, rotations) on average over the disorder.
G is represented on Hilbert space $\mathcal{H}$ by $g \mapsto u(g)$.

Probability space $\Omega \ni \omega$ (random variables)
G acts on Ω by $g \mapsto D(g)$.

Random Hamiltonian $H : \Omega \to L(\mathcal{H})$, $\omega \mapsto H_\omega$, must satisfy the covariance relation

$$\boxed{u(g)\, H_\omega\, u(g)^{-1} = H_{D(g)\cdot\omega}}$$

G-invt prob. measure: $\displaystyle\int_\Omega f(\omega)\, d\mu(\omega) = \int_\Omega f(D(g)\cdot\omega)\, d\mu(\omega)$

Operator-theoretic setting.

Defn. A C^*-algebra $\mathcal{A}$ is

a complex vector space $\mathcal{A}$ with an associative product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$

and a $\mathbb{C}$-antilinear map $\mathcal{A} \to \mathcal{A}$, $a \mapsto a^\dagger$ with $(ab)^\dagger = b^\dagger a^\dagger$

and a norm $\mathcal{A} \to \mathbb{R}_+$, $a \mapsto \|a\|$ ($\mathcal{A}$ is Banach, i.e. $\|\cdot\|$-complete)

and $\|a^\dagger a\| = \|a\|^2$ (the norm is the spectral radius).

The flexibility in realizing the product on Hilbert space allows to mimick Fourier transform and still have a (non-commutative) Brillouin zone.

Definition of $K_0(\mathcal{A})$.

Projectors $p^2 = p = p^\dagger$.

Addition $p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ in $\mathcal{A} \otimes \mathcal{M}_{n,n}(\mathbb{C})$, $n \to \infty$. ("stabilization")

Equivalence relation: $\mathcal{A} \otimes \mathcal{M}_{m,m}(\mathbb{C}) \ni p \sim q \in \mathcal{A} \otimes \mathcal{M}_{n,n}(\mathbb{C})$

$$\Longleftrightarrow \quad p = vv^\dagger, \quad q = v^\dagger v \text{ for } v \in \mathcal{A} \otimes \mathcal{M}_{m,n}(\mathbb{C}).$$

$K_0(\mathcal{A})$ = the Grothendieck group of the Abelian monoid generated by equivalence classes of projections $p \in \mathcal{A}$.

Modification needed!

Build the theory on complex structures instead of projections:

Addition: $I \oplus I' = \begin{pmatrix} I & 0 \\ 0 & I' \end{pmatrix}$.

Equivalence relation: $\mathcal{C}_{CAR}(W_{\mathbb{R}} \otimes \mathbb{R}^m) \ni I \sim I' \in \mathcal{C}_{CAR}(W_{\mathbb{R}} \otimes \mathbb{R}^n)$

$$\Longleftrightarrow \quad I \oplus J_{vac}^{N-m} \sim_h I' \oplus J_{vac}^{N-n} \text{ for some } N \geq \max(m,n)$$

by homotopy in $\mathcal{C}_{CAR}(W_{\mathbb{R}} \otimes \mathbb{R}^N)$

Numerical Invariants.

Example. DeRham theory:

Pairing
$$H^q(M) \otimes H_q(M) \longrightarrow \mathbb{R}$$
$$[\omega] \otimes [c] \longmapsto \int_c \omega$$

Algebraic K-Theory.

- Pairing between K-Theory and cyclic cohomology (A. Connes)

  Example: Hall conductance $= \mathrm{Tr}\big(P\,[[X_1,P],[X_2,P]]\big)$

- Index pairing (with K-homology):

  operator-theoretic version of Laughlin-type argument

Bulk-boundary correspondence.

short exact sequence:

$$0 \longrightarrow \mathcal{E} \xrightarrow{\;i\;} \widehat{\mathcal{A}} \xrightarrow{\;\mathrm{ev}\;} \mathcal{A} \longrightarrow 0$$

boundary half-space bulk

six-term exact sequence (complex case):

$$\begin{tikzcd} K_0(\mathcal{E}) \arrow[r, "i_*"] & K_0(\widehat{\mathcal{A}}) \arrow[r, "\mathrm{ev}_*"] & K_0(\mathcal{A}) \arrow[d, "\mathrm{Exp}"] \\ K_1(\mathcal{A}) \arrow[u, "\mathrm{Ind}"] & K_1(\widehat{\mathcal{A}}) \arrow[l, "\mathrm{ev}_*"] & K_1(\mathcal{E}) \arrow[l, "i_*"] \end{tikzcd}$$

projection ops.

unitary ops.

See E. Prodan, H. Schulz-Baldes:
Bulk and Boundary Invariants for Complex Topological Insulators (Springer, 2016)